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ABSTRACT

This document contains papers presented at the 19th annual conference of the Mathematics Education Research Group of Australasia. Topics of the presentations include learning research, mathematical representations, problem solving, strategic learning behaviors, algebraic thinking and learning environments, teaching and learning of algebra, assessment, disabilities, calculators, collective argumentation, teachers' beliefs and practice, primary mathematics, differential calculus, teachers' knowledge, trigonometry and geometry, professional development, issues in teaching, standardizing the curriculum, team writing, statistics, Newman error analysis, gender issues, Internet, transition to secondary mathematics, computers and technology, negative numbers, subtraction, aboriginal educators' views, graphics calculators, language, area, probability, word problems, classroom communication, mathematical investigations, ethics and morality, integrating science and mathematics concepts, students' attitudes, instructional computing, expository writing, mathematical autobiographies, problem posing, misconceptions, discussion-based teaching, the Riemann integral, diagrams for solving word problems, fairness and fractions in early childhood, children's probability judgments, phenomenology of writing-to-learn, teachers' beliefs about teaching behaviors, and linear programming. An author index and a subject index are also included. (JRH)

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MATHEMATICS EDUCATION

Edited by

Philip C Clarkson

THE UNIVERSITY
OF MELBOURNE

**Mathematics
Education Research
Group of Australasia**

AUSTRALIAN CATHOLIC
UNIVERSITY

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Learning Research

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Over the past decade we have been learning how to research learning - in classroom settings and elsewhere. Research into learning in classroom settings is widely practised. Increasingly, research is addressing teaching and learning events in the workplace. Historically, the research methods employed to investigate learning have been refined and replaced as our image of learning and, consequently, the goal of our research has changed. The availability of new technologies offers researchers the capability to compile rich data sets with a level of complexity not previously possible. The analysis of such complex data sets poses a significant challenge. While new technological tools also offer the possibility of new forms of analysis (particularly of qualitative data), it is the sophistication of our theories and analytic frameworks that will determine whether we are able to utilise the potential of the new techniques. Central to the realization of this sophistication is the operationalization of the constructs with which our theories are expressed. This paper explores contemporary approaches to researching learning and some of the questions associated with this type of research.

Introduction: Operationalized Constructs

Within mathematics and science education, there are several contenders for the title of central construct: ability, learning, teaching, competence, mathematical (or scientific) meaning, problem solving, understanding, reflection, abstraction, negotiation, task, and activity, come readily to mind. The difficulty with such central constructs lies in their observation and recognition within overt social activity. In order to recognize "Learning", for instance, we must develop an operational definition, whether formally articulated by a researcher, or implicitly enacted by a teacher. To whom are the operational definitions of our constructs most important? Those responsible for research and for assessment and evaluation are the most obvious users of operational definitions, since these are the groups responsible for the investigation, the monitoring and the refinement of the constructs which drive the domain. It is the research perspective that provides the focus for this paper.

The observation of any social activity: (i) involves an act of selection on the part of the observer, and may omit elements (actions or contextual details) which the participants or another observer would see as significant (*selectivity*); (ii) distorts the phenomena being observed,through the presence of the observer or through the act of observation (*indeterminancy*); (iii) cannot claim to represent the meaning which the activity has for the observed participant(s), but only the meaning which the researcher has constructed for the activity (*subjectivity*); (iv) misrepresents the construct by dissociating it from other related constructs (*reductionism*) when the meaning of the activity resides in the dynamic interplay of the combination of constituent constructs (for example, ability, affect, achievement, and understanding); (v) misrepresents the construct by dissociating it from the contextual setting of its occurrence (*decontextualization*); (vi) accords a reality to the construct as "observable" which misrepresents the inferential chain by which the researcher builds a representation of the construct as "observed" from overt social actions, even though the construct may relate to an individual's cognitive processes (*reification*); and, (vii) focuses only on the "observed activity" without recognizing that this is itself only a part of a larger social event (the research activity) which has its own participants (observer and observed), context (immediate social setting and enfolding research setting), purpose (observed task and overarching research aims), actions (including those of the researcher), and consequences (*dissociation*).

In summary: the apparent utility of operational definitions masks several concerns about their use in research. In particular, the commitment by the researcher to a specific operational definition of a construct being studied might render the research insensitive to some of the associations or nuances of meaning which actually characterize the construct from the perspective of the respondent/subject/student. Recent developments in educational research (and in learning theory) have led to acceptance of the idiosyncratic and legitimate subjectivity of the research subjects in comparison with the spurious claims regarding the objectivity of the researcher, which characterised earlier research. As a consequence, the goal of research into learning undertaken in 'naturalistic' social settings is not the establishment of the legitimacy of competing claims regarding "what really happened", but rather the consideration of what can be learned from the comparison of the multiple stories compiled from the accounts of the various participants (including the researcher's account) in the social setting.

The need for the operationalization of constructs is not diminished by this change in goal. Instead, the acknowledgement of the relativist character of any account of social activity commits us to defining explicitly those terms we employ in the analysis and comparison of the various accounts, since we are no longer able to presume any specific (absolute) meaning, by implicit appeal to the authority of culture or the conventions of use.

> The question mark that postmodernism puts on the 'truth' of all discourse means only that all knowledge is contextual, not that all knowledge is false, nor that one cannot support the validity of one claim over another in a specific circumstance.
>
> (Scholle, 1992, p. 276)

The operational definition of the terms we employ in comparing the various accounts derive from our purpose and help to make that purpose explicit for the reader of our research account. Commitment to a particular operational definition should not be construed as a claim to truth, but rather as delimiting the field of discourse - the boundaries of our analysis - within which the viability of various accounts can be compared.

Approaches to Learning Research

The word "community" is much in vogue amongst educational theorists. Mathematical activity, like scientific activity, is engaged in by individuals as members of intersecting communities of discourse and practice. It is membership of a community of practice that defines the context and the purpose that an individual associates with a task. It is as a member of a community of discourse that an individual construes the meaning of a task and frames an appropriate response. Classrooms may be conceived as communities of learners, but these are only one of the many types of communities in which teaching and learning may occur. It may be that it is the community that defines 'meaningful' learning. And a community may be a family, a clerical team, a class, a maintenance crew, a congregation, or a nation.

It has been acknowledged for some time that teaching and learning are organized quite differently in school and workplace setttings (Becker, 1986; Collins, Brown & Newman, 1989; Hall & Stevens, 1995; Lave & Wenger, 1991; Resnick, 1987; Rogoff, 1990). Yet, the means by which teaching and learning are organized as ongoing practical activities within workplaces has received very little study. Studies of ethnomathematics and 'street mathematics' (Carraher, 1988) have documented the character of the mathematics of daily economic transaction, in ways that distinguish it from 'school mathematics' without necessarily contributing to our evolving model of learning, other than to underscore the situated character of all learning.

What form does learning take in classroom settings? This is perhaps the most obvious of the questions which might be the subject of educational research, yet it remains one of the least well answered. Recent attempts to study learning have taken two common forms: Clinical studies of one or two children in situations described as "Teaching Experiments" (for example, Schoenfeld, Smith & Arcavi, 1993; Steffe, 1993); and descriptive studies of mathematics classrooms, in which videotapes of classroom interactions have provided data for later analysis (for example, Cobb, Wood & Yackel, 1992). Such studies have as their focus the construction of a model of the process of learning mathematics. The work of Steffe, Cobb and their co-workers presume a constructivist model of learning and seek to both validate and elaborate such a model through their research. Schoenfeld's approach is more concerned with theory construction.

Steffe and Schoenfeld have restricted their study to the learning of individuals in quite tightly controlled situations involving few participants. The challenge confronted by Cobb and his colleagues and other such classroom-based research is the challenge presented by the complexity of the classroom as a research site; the interaction of social and personal variables in a setting over which the researcher can exert no control. While studies of the first type might be characterized as "experimental" or, at least, "clinical", studies of the second type have much in common with ethnographic studies of community behaviours. However, this second category of studies retains a focus on the learning of participant individuals, and depends to a large extent on inferences concerning cognitive process drawn from videotape data of overt social actions.

There are advantages and disadvantages to both approaches. Either experimental control is retained by creating a research context which does not resemble the classroom contexts to which the research will ultimately be applied, or commitment to a realistic social context leads to a lack of control over variables and a lack of opportunity for interaction with the learners while learning and consequently demands a high level of inference on the part of the researcher as to the thinking that actually occurred during an observed lesson.

Each approach is also characterized by a particular theoretical position and a particular purpose. The 'Teaching Experiment' has been variously interpreted and applied, and in its most clearly articulated form offers a distinct methodological alternative to either workplace or classroom research, since it accepts no obligation to mimic or to investigate society's 'naturally occurring' educational settings.

> The teaching experiment methodology is based on the assumption that students have a mathematical reality that is distinct from that of the researchers, and that the researchers, through interacting with the students, can come to understand that mathematics and how to engender changes in it in ways that seem appropriate and productive.
>
> (Steffe & Thompson, 1996, p. 23)

In identifying the purpose for their research, Steffe and Thompson articulate a reform agenda to be informed by Teaching Experiment research.

> It is our intention that the mathematics of students replace contemporary school mathematics. This is the main thrust of our work, and we want it to be understood in this way. We strive to specify the mathematical concepts and operations of students and to make those the conceptual foundations of school mathematics.
>
> (Steffe & Thompson, 1996, p. 2)

Schoenfeld and Clarke and their co-workers (Schoenfeld, Smith, & Arcavi, 1993; Clarke, 1996), while employing very different research techniques are similarly committed to the development of models by which the learning of mathematics might be represented in order to inform instruction. An exhaustive discussion of different approaches to researching learning is beyond the scope of this paper. The research methods and analytical techniques of Schoenfeld and Steffe exemplify the richness and diversity of the research endeavour in mathematics education. The remainder of this paper examines research in classrooms.

Classroom Research

Learning has been studied in many ways and in many settings. Classrooms are distinguished by being environments with the specific purpose of serving as sites for learning. Learning in classroom settings is no less situated than learning in other settings. This may seem an obvious assertion, but one consequence of the current enthusiasm for workplace learning could be an eroding of the status of the classroom as a setting for legitimate, meaningful mathematical activity. If the community is to continue to employ schools as the principal means by which young people are educated and enculturated into academic disciplines, then educational research must attend to the characteristics of the learning environments in such settings and to the processes by which learning occurs in classrooms. Many researchers have made the classroom the focus of their efforts (for example, Yackel & Cobb, 1993). There are certain characteristics of structure and purpose shared by all classrooms. Equally, each classroom presents a unique social setting, which facilitates and constrains the actions of all participants and the form of the consequent learning. Classroom research typically attempts to capture the characteristics of learning in one classroom in a form that will bear translation to other classrooms.

Early attempts to research learning in classrooms made extensive use of non-participant observation. The following excerpt from Bourke (1984) is representative of the reporting of data collection by observation.

> The observers used were all qualified teachers and were trained to use a low-inference classroom observation schedule . . . The observer was required to record, approximately every five seconds, the class grouping being used, the participants in each classroom interaction, and the nature of the interaction for a series of five minute periods during the lesson.
>
> (Bourke, 1984, 6)

Among the noteworthy observational techniques are Flanders' work on *Interaction Analysis*, (Amidon & Hough, 1967), and the *Systematic Classroom Analysis Notation (SCAN)* (Beeby, Burkhardt & Fraser, 1980). These techniques are typical of the observation schedules employed by non-participant observers. Researchers seeking naturalistic or ethnographic data have typically involved participant observers. That is, observational data is collected by researchers who are participant in the actions, events and contexts being studied.

Much of contemporary classroom research employs videotapes of classroom activity. The use of videotapes allows the exhaustive analysis of classroom activity at a level of detail which a classroom observer applying an observational code in real time could not hope to emulate. Videotape data provide a permanent record amenable to multiple analyses. However, to infer student thought processes and the significance of classroom events on the basis only of videotape data has been viewed as an unjustified extrapolation, and many researchers sought to supplement their videotapes with other sources of data. The resultant composite data sets were rich, complex and multi-faceted.

> Observational procedures included video recording, audio recordings of public and private student talk, continuous observations of individual case study student behaviours, and photographic records of student work. Our purpose was to create the fullest possible continuous record of each individual case study student's experience of the unit.
>
> (Alton-Lee & Nuthall, 1992, p.4)

Other researchers (Anthony, 1994; Clarke & Kessel, 1995) sought to minimize the need for researcher inference regarding student thought processes. Clarifying and corroborative data were provided by the students themselves in interview situations, where their accounts of the significance of classroom events and their associated thought processes could be reconstructed with the assisting prompt of the classroom video record. More detail regarding the rationale for this research technique can be found elsewhere (Clarke, in press; Clarke & Kessel, 1995).

The technique employed by Clarke and his co-workers provides an illustration of the use of currently available technology and can be outlined quite simply. One camera is directed at the teacher, while the other camera is focussed on a group of about four students. The teacher's utterances are recorded through a radio microphone and a single microphone was used to record the conversations of all four students. The two video images are mixed to produce a composite picture in whch the students occupy most of the screen with the teacher image superimposed in a corner of the screen. This combined image is recorded onto video-8 tape using a compact videorecorder attached to a laptop computer. The researcher, seated at the rear of the classrom, is able to listen simultaneously to both student conversations and teacher utterances and recorded field notes onto a word processing document on the computer using *CVideo* software (Rochelle, 1992). The field notes can be "time-tagged" to corresponding events in the video record. The field notes enable the researcher to document impressions of significant classroom episodes and learning events as well as provide reference markers for the subsequent interviewing of student subjects.

Students from the groups which had been videotaped are interviewed immediately after the lesson. The videotape record is used in the interview to stimulate student recall of classroom events. The use of the *CVideo* software enables the researcher to locate within the field notes reference to actions of the student which seem to be of significance either to the researcher or to the student. Having found a particular item in the word document, the software can be used to very quickly find the corresponding moment on the video record. This is then played back and discussed. The audio record of the interview provides a third source of data. A more detailed account of this research technique can be found in Clarke (in press) and Clarke and Kessell (1995).

Other researchers, notably Ball (1993, for example), have compiled sets of related data, including classroom videotapes, interviews and student work samples, and utilized available computer technology to synthesise a single complex interconnected data file. Such complex data files are amenable to multiple analyses, facilitating equally the investigation of particular behaviour types over many students or many lessons and the longitudinal study of an individual's learning. It is the analysis of such complex data sets that now presents the most significant challenge to learning researchers.

Analyzing Complex Data Sets

Research techniques, such as those outlined above, offer researchers richer, more complex sets of data than have ever been available before. The analysis of such complex data sets poses a significant challenge. One aspect of this challenge is the search for effective theoretical lenses through which to view the data.

Gore (1994) employed the work of Foucault (1977, 1980, for example) to carry out "a systematic multi-site study of the micro-level functioning of power relations" in classroom settings. Gore (1994) explicitly sought to "systematically examine the minute practices of classrooms." In this case, Foucault's writing on the subject of power relations formed the basis of a coding system. Gore stated explicitly that her research was not an ethnography, but, instead, applied a complex coding system to detailed descriptions of classroom events in order "to illuminate the mechanisms of schooling."

By way of contrast, Jungwirth (1991) identified a theoretical position which synthesised symbolic interactionism and ethnomethodology in her use of "microsociological approaches."

> These are based on the following assumptions: People construct subjective meanings for things but nevertheless arrive at a common reality and at a knowledge which is experienced as universally valid. The commonly accepted valid meanings are the result of social negotiation processes. As a consequence, social interaction establishes the reality to be analyzed.
>
> (Jungwirth, 1991, p. 266)

Another approach is the development of new methods for the schematic representation of data. Such schematic systems carry implicit theoretical meaning, since certain spatial associations and arrangements are allowed within the system while others are precluded. One example is Schoenfeld's recent use of such a schematic system to represent the shifts in teacher goals in the course of instruction (Schoenfeld, 1996).

> The basic idea is that a lesson can be parsed into large units corresponding to large-scale action plans, that these in turn can be parsed into smaller units (corresponding to smaller-scale action plans), and so on - down to the level of small linguistic chunks, on the order of a line or two of transcript.
>
> (Schoenfeld, 1996, p. 6)

As Feynman demonstrated in physics, a schematic system can be a powerful analytical tool; modelling relationships between key variables, embodying specific algorithms and suggesting further research questions and lines of investigation. Schoenfeld's scheme offers this sort of analytical power and could be applied with equal legitimacy to the analysis of shifts in student goals. An example of the application of this scheme for "A full parsing representing action plans and goals" is given in Figure 1.

> If you trace across horizontally, you can see that each chunk on the left (an action plan) contributes to at least one of the goals on the right. By looking at the "goal traces" on the right, we get a sense of which goals were addressed, how frequently and how intensively.
>
> (Schoenfeld, 1996, p. 9)

Clarke, Helme and Kessel (1996) have used the *NUD•IST* computer package to analyze the complex text files compiled from the synthesis of classroom videotape transcripts, observational field notes, and student interview transcripts. For example, text analysis can identify points at which the student's state of knowing demonstrably changes from uncertainty to comparative certainty or conviction. It may be that a student recounts a situation in which they came to "know" or to "understand" something related to the topic dealt with in the observed lesson. These accounts are also taken to constitute data regarding the process of coming to know.

In particular, any student use of the verbs, "know", "understand" or "learn" can be analysed in detail with regard to the subject and object of each verb's use. That is, what sort of things can be "known", "understood" or "learned"; what sort of experiences, events, images, or people appear to be associated with "learning", "understanding" or "coming to know"; and, who is it that "knows", "understands" or "learns" things? In this analysis, the software package *NUD•IST* (QSR, 1994) is employed to undertake a textual analysis of the frequency of association of terms such as those above in student classroom discourse and in student discussion of video-recorded classroom situations.

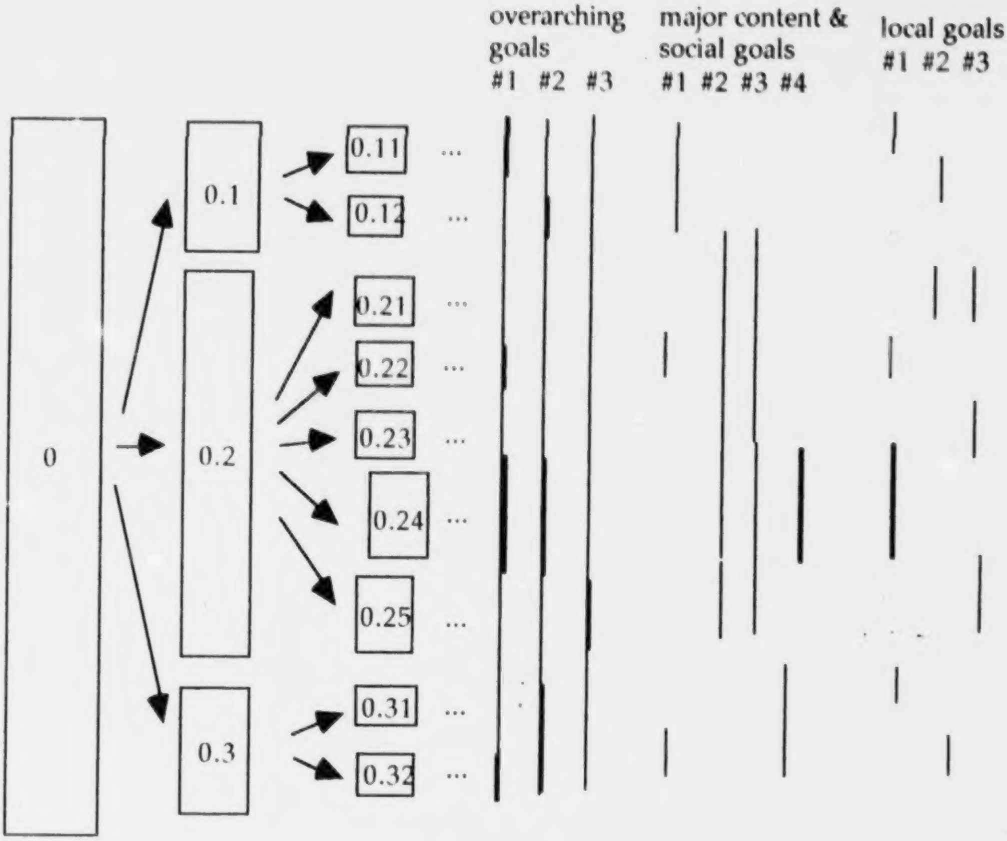

Figure 1

NUD•IST is fundamentally an indexing tool, which allows the cross referencing of particular terms or phrases. Central to its use in this type of research is the capability of *NUD•IST* to reveal clusters of related terms: related by common situation (textual or social situation), or by association with a common key term or referent. In addition, relationships can be identified between clusters of terms. *NUD•IST* facilitates the identification of the sort of "grounded key" discussed by Glaser and Strauss (1967). The search for the associations between textual elements, which appear to structure the process of coming to know, is a form of exploratory content analysis. Use of *NUD•IST* is intended to reveal structure within the learning process by identifying "nodes" of associated terms such as "know" and "understand". Structures emerge through the associations identified between these nodes. The use of an indexing tool such as *NUD•IST* offers a form of replicability not previously associated with conventional content analysis. As with any content analysis, the identification of "key terms" must derive from a clearly articulated theoretical position. Technological aids like *NUD•IST* offer the researcher a new capability to identify associative links within complex, qualitative data sets. The search for these associative links must be purposeful and theory-driven, otherwise we will witness the sort of random exploratory analysis that characterized some early applications of statistical techniques to educational data.

Conclusions

We continue to learn how to research learning. Our efforts to date have led to the development of a fascinating variety of research techniques and some distinct differences in methodology. The mathematics education research community is all the richer for this diversity. Technology has increased our ability to study the complexity of classroom and workplace learning, offering us the tools to synthesize different data types into data files of substantial complexity and interconnectedness. This capability is matched by the existence of analytical tools capable of demonstrating associative links in qualitative data with a level of replicability and rigour that was previously only associated with quantitative data.

Among the factors on which our success depends, one of the most important is our ability to define what it is that we are studying. And these definitions must be framed in terms that inform and structure our research. Education abounds with ill-defined terms. We must scrutinize the articulation of our theories to ensure that each construct has been appropriately situated both theoretically and operationally. The real advances in learning research will continue to come from the ingenuity and insight of individual researchers, who find new theories and new perspectives on the constructs at the heart of our discipline, and which inform our research efforts from design and data collection to analysis and interpretation. New representational schemes facilitate the interpretation of data, and new perspectives on research methodology provide a correspondence of subtlety and sophistication between our theories, our research techniques, our analytical tools, and the interpretation of our findings.

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Good Use of Technology Changes the Nature of Classroom Mathematics

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Powerful use of technology has the potential to radically alter the nature of classroom mathematics. Fundamental issues related to the use of technology can transcend the boundaries of the type of technology used and the level of schooling. Examples from the *Calculators in Primary Mathematics* project will be linked here with parallel examples and discussion from the research literature in order to explore some of the effects of technology on the nature of mathematical activity, classroom practice and the curriculum.

Introduction

> Technology is not just about tools — good use of technology shifts the very focus of mathematical activity in the classroom. (Jones & Lipson, 1995)

Research into the use of technology in mathematics education — the theme of this year's MERGA conference — has been extensive (for surveys of such research see, for example, Dunham, 1992; Dunham & Dick, 1994; Fey, 1989; Hembree & Dessart, 1986, 1992; Ruthven, 1995).

It would be both impossible and foolish for me to attempt such a survey here — impossible because of the limitations of time and space and foolish because another of the keynote speakers, Jim Kaput, is the author of an encyclopedic survey (Kaput, 1992) and is infinitely more knowledgeable on this topic than I am.

It is my belief that many of the issues provoked by the use of technology transcend the boundaries of the type of technology used and the level of schooling at which it is used. It is therefore my intention today, on Teachers' Day at this conference, to attempt to link my own research — into the use of calculators with young children — with the research of others, in order to explore some of the effects of technology on the nature of mathematical activity, classroom practice and the curriculum.

I will not attempt to provide a comprehensive coverage of major work in the area, or even of the different major uses of technology. Rather, I will select examples from my own experience with the *Calculators in Primary Mathematics* project, which I believe are relevant to these issues, and attempt to draw together common threads from the research literature.

The *Calculators in Primary Mathematics* Project

The *Calculators in Primary Mathematics* project was a long-term investigation into the effects of the introduction of calculators on the learning and teaching of mathematics in the lower primary grades. The project, which commenced with prep and grade 1 classes in 1990, involved approximately 1000 children and eighty teachers in six Melbourne schools. Children were given their own calculator to use whenever they wished. The project moved up through the schools, with the children, to grade 4 in 1993. New children joined the project each year on entry to school.

The project investigated four main questions: How can calculators be used in lower primary grades? What effect will calculators have on children's learning of number concepts? What changes will there be in teachers' expectations and the curriculum? What effect will their presence have on teachers' beliefs and classroom practice?

Teachers were not provided with classroom activities or a program to follow. Instead they were regarded as part of the research team investigating the ways in which calculators could be used in their mathematics classes. Feedback and support were provided through regular classroom visits by members of the project team and through the sharing of activities and the discussion of issues at regular teacher meetings and through the project newsletter.

Data was collected through regular classroom observations, teacher interviews, written teacher evaluation sheets and questionnaires, as well as large scale written testing and interviewing of children at grade 3 and 4 levels.

The Nature of Mathematical Activity in a Technology Rich Environment

> [The questions] mathematics educators and researchers ... are posing [are]: What are the *new* things you can do with technologies that you could not do before or that weren't *practical* to do?" Once you begin to use the technology, what totally new things do you realize might be *possible* to do? (Pea, 1987, p. 95)

It is obvious that technology opens up a huge potential tool kit which can be used to carry out many of the mathematical tasks which we have, in the past, regarded as an integral part of the mathematics curriculum. However, while first attempts at technology use have often been directed towards using these tools to alleviate some of the drudgery associated with finding answers, mathematics educators believe that the arrival of calculators will have a much greater effect — they will "change, permanently, what it is to do mathematics and what mathematics we ought to do" (Kissane, 1993).

One of the aims of the *Calculators in Primary Mathematics* project was to document classroom use of calculators with young children in order to demonstrate to teachers what new things become practical and possible when children have access to calculators. Project teachers, who were often initially uncertain about how they could use calculators at all with such young children, found that they were being used in their classrooms in diverse and unexpected ways (see, for example, Groves & Cheeseman, 1993; Groves, Cheeseman, Dale & Dornau, 1994; Stacey, 1994a; Stacey & Groves, in press). More importantly, one of the overwhelming responses from project teachers was to report that the nature of children's mathematical activity changed — children were much more likely to be engaged in open-ended, exploratory activities, where they were in control of the level of mathematics at which they were working. One infant teacher described it this way:

> I'm a lot happier to go where the children want ... it's a lot less teacher directed... The calculators have enabled us to do that ... You don't have to structure things so that you know the answer will be within their reach... If the children want to find out stuff, I say "go for it"... because I'm not so worried about them finding out things they won't understand anymore... I think I'm being a lot more open-ended with their activities... I'm putting more on them to do more finding out. I'm just sort of starting them off ... The activities I try and do with them are the ones they can take themselves where they want to go ... You never know what's going to happen.

Numerous examples of very young children exploring highly sophisticated number concepts were recorded. For example, in many prep classes, the calculator was used extensively for counting, by using the built-in constant function for addition and, often, recording the numbers obtained on long vertical strips. One child, Alex, stated that he was going to count by 10's to 1050. How did he know, at age 5, that 1050 would be in the sequence? Another child looked at her long "number roll" and observed that counting by 9's usually leads to the units digit decreasing by one each time, while the tens digit increases by one. Many children made conscious predictions about the next number in their sequence — even when they could not read the numbers aloud. Simon, also in prep, decided to count by ones on his calculator without doing any recording. He started at 1 000 000 — he knew it was a million because his mum had told him that a million has six zeros. Another child challenged him to reach "one million one hundred". At first, he said that there was no such number, but as he got to 1 000 079 he began to think that perhaps there was. When he finally reached 1 000 102 he was thrilled to see that he had "gone right past it".

Corbitt (1985, p.16) warned against making *a priori* assumptions about the appropriateness of mathematical topics for primary children. With calculators, large numbers, decimals and negatives all became available through counting. Counting backwards lead some children to encounter negative numbers. In one prep class where children had been discussing and drawing "what lives underground?", Alistair said "minus means you are going underground". When questioned what would be the first number above the ground, he said "zero". Other grade 1 and 2 children began using their calculators to count by decimal numbers. For example, one girl counted aloud as she used the constant function, saying "point one, point two, ... point nine, point ten". She then noticed that the display on her calculator showed something different and worked out why this should be so.

Having the support of a calculator enabled counting to assume a much more prominent place in mathematics lessons. Although all of the teachers continued to use concrete materials as the basis for their mathematics teaching, counting no longer needed to be tied to the rather small range of numbers which can feasibly be displayed with such materials.

Some children, of course, were still struggling with number recognition and some were very uncertain about all aspects of the calculator. The wide range of skills and understandings present in any classroom appears to be highlighted by the presence of the calculator. Corbitt (1985, p. 17) comments that "student-directed learning suggests that students may become dispersed throughout the learning environment". One teacher commented that the calculator "provided a tool for me to understand a lot better the way they are thinking and processing than I have been able to do before". Another commented "I don't think I've ever been as concerned as to what the children know ... before. I think it has opened things up a little bit more and I'm more interested in them telling me what they know".

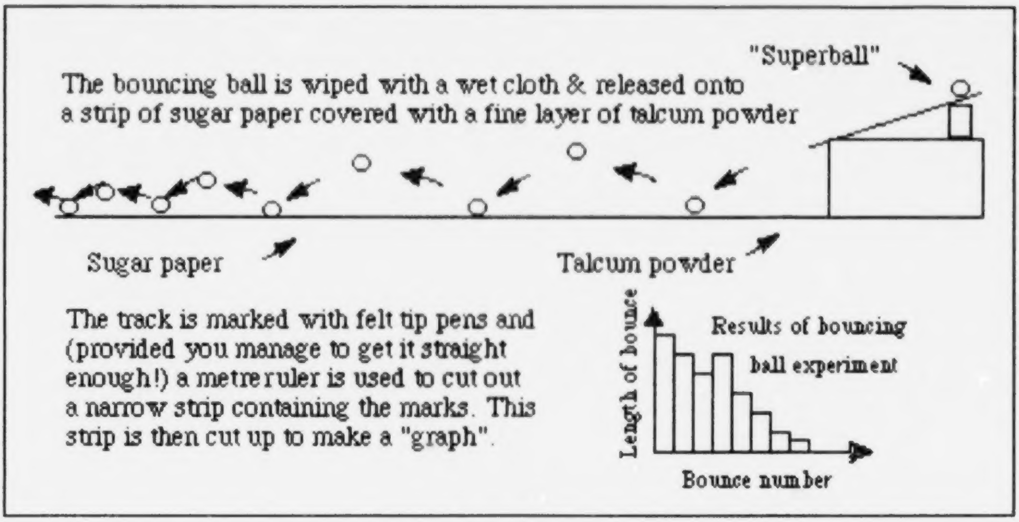
The fact that a technology rich environment supports more exploratory activity on the part of the learner has been confirmed at all levels of schooling for a wide range of technological applications. For example, Fey (1989) regards software such as *Cabri* (Laborde, 1990) and *Geometric Supposer* as aiming to

> restore the act of discovery and conjecturing to a geometry course that has become a deadly routine of proving things that have been well known for centuries, and to facilitate inductive reasoning by making multiple tests of conjectures easy to execute. (p. 246)

Fey cites Yerushalmy and Houde (1986, pp. 421–422) as stating that when using *Geometric Supposer* "students spent the majority of class time discussing and doing geometry rather than listening to a teacher talk about it."

Jones and Lipson (1995) vividly demonstrate how the use of technology can transform the modes of enquiry available for students. With access to the World Wide Web a reality in many schools, secondary data for classroom use can change from being merely numbers invented by textbook writers to fit chosen examples, to data obtained by other people for real purposes, which can then be used to answer quite different real questions posed by teachers and their students. The potential is now available to pose a question and then search the web to find data which might provide an answer. In their conference presentation, they illustrated this by asking the question "What is the 'lifetime', in test matches, of the captain of an Australian cricket team?" A quick tour of web sites allowed appropriate data to be captured into a word processor and transported into a graphic calculator for analysis — a stunning feat in itself! A box plot of the data showed its skewed nature, with a clear outlier — Alan Border.

Also in Victoria, the *Technology Enriched Algebra Project* (see, for example, Asp, Dowsey, & Stacey, 1993; Asp, Dowsey, Stacey & Tynan, 1995) addressed the issue of providing concept and process rich learning experiences in algebra through the use of appropriate computer software. Graphing software enables students to experiment with a variety of examples which promote intuitive understanding of the bahaviour of various functions. Preliminary results from this project suggest that not only will the emergence of technology lead to new directions in school mathematics, but that differences in tools will promote the development of different concepts.


The bouncing ball is wiped with a wet cloth & released onto a strip of sugar paper covered with a fine layer of talcum powder

The track is marked with felt tip pens and (provided you manage to get it straight enough!) a metre ruler is used to cut out a narrow strip containing the marks. This strip is then cut up to make a "graph".

Figure 1. The bouncing ball experiment

The bouncing ball example, illustrated in Figure 1, originated from the *Mechanics in Action Project* (see Williams, 1989). The data produced can be plotted using a graphic calculator, in order to attempt to fit a curve (see Groves, 1989). This activity, which appears in a modified form in Asp, Dowsey, Stacey and Tynan (1995), leads to a very different type of mathematical activity than that experienced by students when plotting curves manually. I remember vividly my own attempts at fitting a curve to the data from this activity. While I feel I fully understand the functions involved in this and similar data fitting exercises, the actual *experience* of using relatively sophisticated technology, which nevertheless required you to fit the curves *bare handed* , provided me with a very clear view of the limits of my real intuitive understanding. I believe there is a great difference in the quality of these experiences and those where you are required to fit curves to *clean* data.

Flavel (1995, p.256), working in the upper secondary school, argues that while interactive computer software which provides multiple-linked dynamic representations "can enrich mathematical experiences in a way which is not possible with static pen and paper representations … for the software to have mathematical meaning to the learner, a supportive teaching learning model [is needed]". Such a model needs to "enable a community of learners to form where members can freely conjecture, justify and celebrate their learning".

The Nature of the Mathematics Classroom

> While technology provoke[s] innovation and in a sense makes it possible, the important changes are those that follow from changes in the teacher's beliefs about mathematics, teaching, learning, students, and the appropriate use of classroom time. Teachers need … to make wise pedagogical decisions about how to proceed as students generate ideas and conjectures, many of which may be entirely new to the teacher. Lampert … likens the experience to leading students on a field trip to a large city where each student is equipped with a motor scooter. (Kaput, 1992, pp. 548-9)

Widespread change in mathematics learning and teaching has not yet come about with the advent of calculators because it is no trivial matter to change these. Teachers need to rethink the nature and content of mathematics, the mathematics curriculum, mathematics teaching and mathematics learning, as well as develop new and substantially different skills for teaching and assessment (Willis & Kissane, 1989, p.74).

In the UK, the *Calculator-Aware Number* project (CAN), which commenced in 1986 under the direction of the late Hilary Shuard, was entered into as a curriculum development project. The CAN project, however, found that the calculator's full potential could not be realised without a change in teaching style, which, in turn, required substantial work on the part of teachers and constant support (Duffin, 1989, p.15).

In two of the six *Calculators in Primary Mathematics* project schools, one of the foci for research was the effect of the introduction of calculators on teachers' beliefs and practice. One of the original hypotheses of the project was that teachers would adopt a more open-ended teaching style as a result of the increased opportunities for exploration of number presented by the calculator.

Interviews with the seven teachers, over a three year period, were used to investigate their perceptions of changes in their mathematics teaching. Groves (1993a) found that most of these teachers claimed to have made substantial changes to their teaching of mathematics, with all seven commenting that their mathematics teaching had become more open-ended, while four teachers described their mathematics teaching as having become more like their teaching of language. At the same time, teachers saw calculators as encouraging them to talk more with the children, as well as providing opportunities for children to share and discuss what they have done — five of the teachers specifically mentioned more sharing and discussion as a change in the mathematics teaching. For example:

> I've had to really encourage them to share what they've done. I think I've always done that in language. I haven't really done that very much at all in maths before. What I see in the use of calculators and how it's changed my maths teaching is that I think I'm teaching maths now more in a way that I've been teaching language for a while, and I've always taught maths much more formally … . It certainly encouraged me to talk to the children much more in maths, and discuss how did they do this, why did they do that, and getting them to justify what they're doing, which I guess previously I haven't done in maths. Much more discussion and sharing.

In a similar vein, Kennedy (1994) — in an article which likens the effect of graphic calculators on mathematics teaching to the effect of compact discs on the record industry — describes how deciding to allow students to use graphic calculators in his own calculus class changed his whole mode of teaching:

> The first thing I did was let them use their graphing calculators all the time. The next thing I did was to start every class with a problem, which the students would talk out until a solution emerged that they could explain to each other. What I discovered, of course, was how useless my crisp set of lecture notes had been all these years. The students were discovering the

> results *without me*, and then *showing each other how to solve the problems.* Every so often I still tie things together or generalise, but for the most part I let the course evolve through what the students are doing, and I provide the direction by the problems I select. I'm still not sure what the heck I'm doing, but I do know this: There is more mathematics going on in my classroom these days than there ever has been before. Now there are more people doing it. (p. 606)

Guskey (1986, pp. 7-10) presents a model of teacher change which assumes that the major motivation for teachers to change is the desire for improvement in student learning outcomes and that changes in teachers' classroom practice need to precede changes in their beliefs and attitudes. Clarke and Peter (1993) present a dynamic model of teacher professional growth, which traces its origins to Guskey's model. In terms of these models, technology can be seen as an external stimulus for change in classroom practice, which, through an iterative process of reflection and enaction, leads to long-term change in teachers' beliefs and practice.

The Nature of the Curriculum

Impact of Technology on the Mathematics Curriculum
Some mathematics becomes *more important* because technology *requires* it.
Some mathematics becomes *less important* because technology *replaces* it.
Some mathematics becomes *possible* because technology *allows* it.
(National Council of Teachers of Mathematics, 1991, Transparency 34)

The 1990 report of the National Research Council, *Reshaping School Mathematics*, identified technology as one of several pressures to reshape mathematics education and argued for a "zero-based" approach to curriculum development — one which makes no *a priori* assumptions about the content of the curriculum, but instead "starts from scratch". In such a curriculum, the inclusion of any topic needs to be justified on its own merits with no area being immune from scrutiny (p.38).

The potential role of technology in redefining the school mathematics curriculum has long been recognised. For example, in 1976, the Swedish National Board of Education, set up the ARK project (Hedren, 1985; Brolin, 1990) to investigate three aspects of calculator use in grades 4 to 12. One of these was the possibility of altering the curriculum due to the calculator. While outcomes of the project were generally positive, Brolin (1990, p. 196) comments that many grade 4–6 teachers "see the teaching of algorithms as the primary function of mathematics courses. Consequently, they regard the use of calculators as a threat to their role as teachers".

In Britain, the *Calculator Aware Number Project* (CAN) project began to explore what a curriculum might look like if it takes seriously the implications of the availability of calculators. Participating schools had to agree not to teach formal written algorithms. Although there was no formal evaluation of the learning outcomes for students, the CAN project reported that children developed a wide range of strategies for carrying out calculations and reached a high level of numeracy for their age (Duffin, 1989; Shuard, Walsh, Goodwin & Worcester, 1991; Shuard, 1992).

The *Calculators in Primary Mathematics* project, like the CAN project, was concerned with a "bottom-up" approach to curriculum, as opposed to a "top-down" one which would dictate what was to be taught in classrooms. Teachers and schools were not required to abandon the teaching of standard written algorithms. Instead one research question was to establish the future role of pencil-and-paper calculations in classrooms where there was free access to calculators.

For many teachers one of the frightening aspects of calculator use is the possibility that children may encounter very large numbers, negative numbers and decimals "before they are ready". Teachers in the *Calculators in Primary Mathematics* project, who became comfortable with calculators in their classroom, took the opposite view. They saw their previous curriculum constraints as having imposed artificial boundaries on the children. For example, one prep and grade 1 teacher commented:

> Once ... in grade 1 we didn't extend beyond 50 at the most when we were counting. The understanding was thought to be limited for children of that age and, once they ran out of fingers and toes, that was as much as we expected them to do. But they really do have an understanding now, and they can translate what they are finding out on their calculator into concrete materials.

Many teachers — especially those teaching the youngest children — found that the presence of the calculator was forcing them to redefine their actual curriculum — whether or not this then conformed to the school's stated curriculum. As the project progressed, curriculum goals were being further challenged by the notion that the children were covering topics from higher grade levels.

The project hypothesised that as teachers became aware of the possibilities they would change their expectations of children's mathematical performance and, as a consequence, the curriculum would also change. In practice, while there appeared to be considerable change in the classrooms, it was impossible to discern any change in the stated curriculum. Perhaps it was an unrealistic expectation on our part that such complex issues could be resolved at the classroom or even school level, even in Victoria, which at least at the time of the commencement of the project claimed to have a "school-based curriculum". While it is likely that a longer time-frame may be necessary to produce discernible and lasting change is school curricula, it also needs to be acknowledged that the timing of the project was such that it was caught up in massive institutional changes which were happening throughout the Victorian school system — changes which would almost certainly have worked against curriculum change at that time.

The potential for curriculum change is by no means limited to the primary school. For example, Ruthven (1995, p. 232) likens the advent of the graphic calculator to that of its arithmetic predecessor, stating that "speculative writing has suggested that, in its emerging form, the graphic calculator subverts much of the traditional secondary mathematics curriculum".

Nevertheless, there is no question that curriculum change in general is lagging behind the potential created by the advent of technology. As well as the difficulties related to achieving curriculum change, there are some other factors which also need to be considered.

Firstly, while it is clear that technology increasingly has the capacity to *replace* much of the mathematics in the traditional curriculum, the question of which skills are necessary for achieving understanding remain (Fey, 1989, 241; Kaput, 1992, 533). Fey regards the question of an appropriate balance between skills, concepts and problem solving as "probably the most important issue for research in technology applied to our field" (p. 259).

A second, and related, issue is that of seeing how curriculum goals can be achieved in new ways — seeing what is made *possible* because technology *allows* it. Plunkett (1979, p. 5) argues that the advent of calculators provides the opportunity to abandon the standard algorithms, but notes that the "quite acceptable premise that children should learn how to calculate … [often leads to] the conclusion that they should be taught the standard algorithms." The *Calculators in Primary Mathematics* project team felt it important to document the types of activities engaged in by children because we believe that one reason why curriculum change has been so slow is the fact that many teachers with a commitment to children's development of number concepts resort to the teaching of standard written algorithms because they see no other way to systematically involve children in activities related to number.

A third issue relates to learning outcomes. Almost twenty years ago, Bell, Burkhardt, McIntosh and Moore (1978, p. 1) reported that "far from undermining basic arithmetic, the calculator seems to encourage and help children in developing their own number skills". Nevertheless, despite research evidence to the contrary (Hembree & Dessart, 1986, 1992), there is still a widespread fear that calculators will undermine mathematics learning (Wheatley & Shumway, 1992).

In order to investigate the long-term effect of calculator use on children's learning of number, the *Calculators in Primary Mathematics* project conducted an extensive program of testing and interviews, with and without calculators, at the grade 3 and 4 levels from 1991 to 1993. The last cohort of children at each year level who had not taken part in the project acted as the control group. Despite fears expressed by some parents, there was no evidence that children became reliant on calculators at the expense of their ability to use other forms of computation. The interviews showed that children with long-term experience of calculators performed better overall on both sets of computation items than children without such experience — in one set they could use any tool of their choice, while the other required mental computation only. These children also performed better on a wide range of items involving place value for large numbers, negative numbers and, more particularly, decimals. They also made more appropriate choices of calculating device and were better able to interpret their answers when using a calculator, particularly where decimal answers were involved. No detrimental effects were observed in either the interviews or the written tests (see Groves, 1993b; 1994; Stacey, 1994b).

Conclusion

> These examples … show how … calculators and computers can be used not simply … to implement traditional practices within school mathematics; but that they can become central components within novel approaches to thinking and teaching, in which interaction with the computational tool is the key to its use as a thinking support and a teaching aid. It is important now that these more radical uses of computational tools should now be evaluated. (Ruthven, 1994, p. 32)

Research of the type reported here has established that technology can alter the nature of classroom mathematics. A substantial body of research also appears to show that the use of technology leads to positive learning outcomes. However, such research is fraught with difficulties. In particular, should assessment of learning outcomes include the use of the same technology which is used in the teaching? How can this be the case when one group has had exposure to the technology and the other has not? If the technology is not used in assessment, how can this fairly assess the group who have been using it throughout the teaching experiment? Dunham and Dick (1994, p. 441), speaking about the use of graphic calculators, describe this as the "Catch-22 situation in attempting to perform an experimental-control group study of achievement". They go on to state that "even if the researcher were able to match exactly the content and instruction … one would immediately want to know *how* students used the graphing calculator and investigate *why* the differences appeared".

Thus research into evaluating the "radical uses" of technology now needs to move beyond the effectiveness studies and address the more fundamental questions about the actual processes of teaching, learning and thinking (Ruthven, 1995, p. 239).

Furthermore, for genuine curriculum reform to take place, it will be necessary to re-examine the curriculum in terms of the "big ideas" of its mathematical content and the pedagogy which can be employed to achieve understanding of these ideas. These notions will need to be separated from the existing web of skills and techniques which have been built up and set into concrete before the ready availability of technology as we know it at present.

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The Role of Physical and Cybernetic Phenomena in Building Intimacy with Mathematical Representations

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We examine the educational possibilities afforded by new connections between physical and simulation-based data to build intimacy with function representations. Historically, technology was first used to facilitate actions within notations, then to link them, eventually bi-directionally. Yet recent data strongly indicate that students' difficulties with interpreting and productively using mathematical notations continue. We suggest that students need phenomena expressive linked to the notations, and secondly, that students themselves should be immersed in generating such phenomena.

General Introduction

Simulations and computer-based models are proving to be the most powerful resources for the advancement and application of mathematics and science since the origins of mathematical modeling in the Renaissance which led to the development of algebra, calculus, and the historical explosion of knowledge and technology that followed. The move from static models built in inert media using traditional notations to dynamic models and associated visualization and analytic tools built in interactive media are changing profoundly not only the nature of mathematics and science but the nature of inquiry in these disciplines (Glass & Mackey, 1988; Haken, 1981). These changes in inquiry involve both a change in the kinds of phenomena that must be considered, and a change in the nature of argumentation and acceptable evidence (Holland, 1995). Furthermore, these cybernetic tools are finally beginning to appear in education. While their appearance is recent, their educational presence will accelerate into the next century as cybernetic experience begins to infiltrate all aspects of human activity, and as the move from static, inert models to dynamic, interactive ones is followed by moves to ever more immersive simulations and eventually to virtual reality (Heim, 1993). This evolution in modes of experience will be accompanied by an evolution from classical mathematics and science to nonlinear science and mathematics, where iterative, dynamical models will extend the range of phenomena subject to systematic modeling and investigation (Pagels, 1988).

However, to introduce these powerful tools and methods, fully formed in the ways that they are used by mature scientists, mathematicians, engineers, or other professionals who apply them is akin to introducing the classical tools of mathematics and science full-blown, to novices (AAAS, 1989; Richards, Barowy, & Levin, 1991). We know that this does not work, and the history of mathematics and science education is, in part, the story of dealing with this fundamental fact. We must now come to terms with the new complications associated with these new tools, especially for younger students, for whom non-cybernetic experience is simultaneously critically important yet under exploited in school learning. At this point, we need basic principles that would help structure productive interactions between physical experience and cybernetic experience in mathematics and science across all age levels (White, 1993).

A Look Back Over Recent Uses of Multiple Representation Computer Systems

Over the past ten years, computer technology has made possible and popular the physical linking of mathematical representations, most especially, graphs, tables and formulas, usually in the context of teaching and learning algebra and calculus. In recent years, the linking has become bi-directional, and much student activity and exploration becomes possible exploiting the interactive nature of the technology. Figure 1 has become a commonplace in descriptions of curricula, texts and learning objectives (Confrey, 1992; Kaput, 1986, Kaput, 1989; Lesh, et al., 1987). You will note that in Figure 1 the "symbolic formulas" vertex is exaggerated, to reflect its dominance in practice. Indeed, character string notations are "more equal" than any of the others, as is evident, for example, if one looks at how functions are input on most devices and how functions are named or identified in activities - as algebraic formulas.

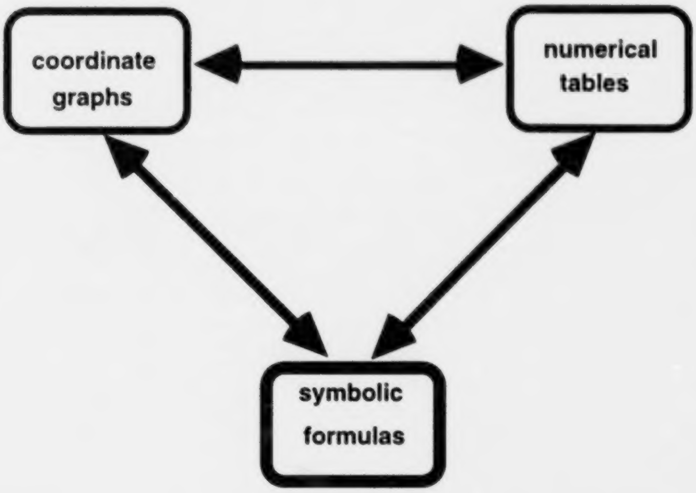

Figure 1: The "Big Three"

However, it is becoming increasingly apparent that student difficulties in learning and using functions have not disappeared (Eisenberg & Dreyfus, 1994; Goldenberg, 1988; Sfard, 1995; Teles, 1989; Thompson, 1994b; Vinner & Dreyfus, 1989; Yerushalmy, 1991). Indeed, the detailed analysis by Schoenfeld, Smith & Arcavi (1994) exhibits the extraordinarily fragile nature of understandings of linear functions when those functions are studied strictly in terms of graphs, formulas and tables apart from referential anchors in students' experience. Whether or not the physical (computational) links become conceptual links, understandings seem inadequate and incomplete. One may ask why this is the case. Our reply heeds the words of Anna Sfard (PME-NA 1994 Plenary Address):

"The emperor is *only* clothes!"

Representations need to represent *something* (other than each other). They need anchoring in student's physical, imagistic, emotional experience. But this, too, is a widely held maxim. "Connections" play a central role in the NCTM Curriculum and Evaluation Standards (NCTM, 1989) and in most reform rhetoric. The connections to the wider world are depicted in Figure 2, where the text descriptions of situations are typically offered in text, as "word problems," and where the connections are in the minds of the problem writers and, presumably at some point, in the mind of the student. Hence the lighter arrows.

As is indicated in Figure 2, the historical approach attempts to link the mathematical notations to the wider world of phenomena and situations in a weak and abstract manner that depends on a *target* student ability to interpret *textual* descriptions of phenomena and situations that are not themselves experienced in any direct form. Further, student ability to link conceptually the textual descriptions of phenomena and situations is typically ill-formed and far from robust. The mathematical side of the curriculum seeks to use the phenomena and situations to help learn mathematics, while the science side of the curriculum seeks to use the mathematics to deepen understanding of the phenomena and situations. Indeed, science instruction, much more frequently than mathematics instruction, attempts to provide some experiential version of the phenomena as part of the instructional context. *Nontheless, we suggest that the mutual constitution of meaning for not only the notations and links among them, but for the phenomena and situations that they may be used to model, is insufficiently rooted in authentic student experience.* Put simply, from the perspective of the students, the phenomena and situations either are missing altogether or are inadequately linked to the mathematics that is to be involved. The instructional strategy offered in this paper is to offer stronger and more direct connections to phenomena and situations, as reflected in Figure 3.

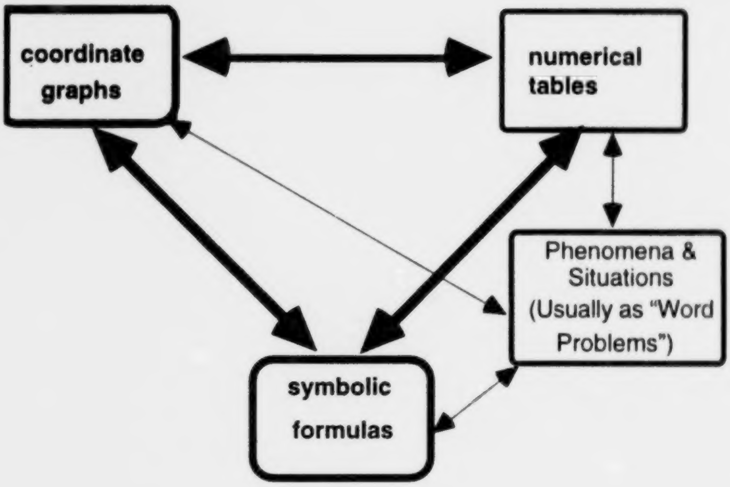
coordinate graphs
numerical tables
Phenomena & Situations (Usually as "Word Problems")
symbolic formulas

Figure 2: Linking the Big Three to Phenomena & Situations - Weakly

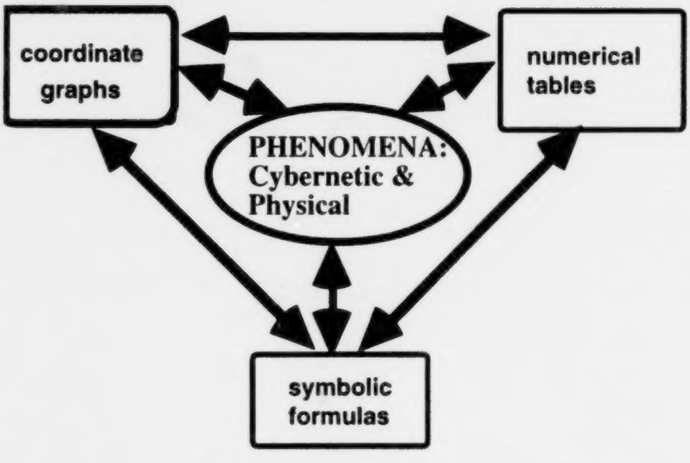
coordinate graphs
numerical tables
PHENOMENA: Cybernetic & Physical
symbolic formulas

Figure 3: Stronger Links to Directly Experienced Phenomena

The kinds of phenomena that may be appropriate vary widely depending on the learning and teaching objectives. Ideally, experience of these phenomena and situations should both embody a measure of generality as well as deliberately tap into the expressive, imagistic, kinesthetic, linguistic, personal, social and other resources that students bring to the learning situation. Since our instructional target is the Mathematics of Change, as described below we begin with motion phenomena, which include many of the desired characteristics, including a readily transferable descriptive vocabulary: words and phrases such as "speeding up," "slower," "catching up," "stopping," "turning around," "back and forth," "going at a constant rate," etc., are widely applicable to describe change phenomena whether or not they involve physical motion. That is, this language applies to many different kinds of quantities other than velocity, position or acceleration. Furthermore, they engage kinesthetic resources based in the first person experience of motion as well as imagistic thinking based in the varieties of graphical representations of motion. Finally, through an appropriate choice of activities, students can become engaged in expressive activities and even aesthetic concerns usually associated with dance and collective motion.

Relating Cybernetic Vs Kinesthetic Experience: Illustrations Involving the Mathematics of Motion

The forms of interaction between computer-based and non-computer-based experience can take many forms depending on the domain and the sophistication or age of the learner. We have been concentrating on the mathematization of motion, an intellectual effort that began with the Scholastics (Clagget, 1968; Kaput, 1994), continued through Galileo and Newton (Edwards, 1979), and is re-emerging as an item of interest in the context of nonlinear phenomena modelable via dynamical systems (Nemirovsky, 1995; Prigogine, & Stengers, 1984; Sandefur, 1990, 1993). Of considerable importance is the fact that our efforts to include this activity in the earlier grades requires us to engage students in characterizing their motion via graphical, narrative, and kinesthetic methods rather than using algebra - as was the case historically well before the time of Newton, because algebraic tools were unavailable (Kaput, 1994). While a large body of research exists regarding student mathematization of motion, it mainly focuses on high school and college age students, and generally deals with "regular" motion that is describable algebraically (McDermott, 1984; Thornton, 1992). Indeed, student modeling of constant-acceleration and harmonic motion is usually a goal of this work. However, we are interested in modeling motion (1) beginning in the early grades, and (2) beginning with the "irregular" motion of children's own bodies and objects controlled by them.

The move to younger children heightens the importance of concepts of rate, another well-studied area (Piaget, 1970, 1976; Piaget & Inhelder, 1974; Thompson, 1994). Indeed, work with speed and velocity provides an excellent arena for the learning of rate ideas, and we are especially interested in tracking the growth of concepts of rate where, unlike the standard approach, students are confronted with variable rates early and develop a sense of constant rates as a means of handling the complexity of these variable rates - via the development of a sense of average speed, or mean value. It may be worth pointing out that mean value is, mathematically, an important concept upon which much theoretical structure depends as is evident from an examination of the proofs of most any university calculus text (e.g., Fleming & Kaput, 1979).

Relating Kinesthetic and Cybernetic Motion: The SimCalc Project Approach

While we often use the phrase "motion simulation" to refer to our computer-based motions, it is important to realize that screen motion of objects amounts to visually real motion to which all the basic perceptual and analytic resources can be applied. They yield visually experienced motion, whether they are subject to first person or third person control (as for example, the difference between a simulated car that is being "driven" in real simulation time from within with a windshield view, etc., and a car observed from the outside whose motion is specified by giving its velocity graph, for example). Nonetheless, they do not yield kinesthetically experienced motion, although it is also possible to create a motion by dragging an object via mouse-control on the screen, which involves a quite direct connection between hand-motion and simulation-motion (Stroup, 1993, 1996).

The physical world provides *kinesthetically rich* experiences of motion, but *quantitatively poor* experiences - where time and position are difficult to measure and use to quantify velocity. This presents an especially difficult challenge for young students who have only the vaguest idea of what one might measure and why. (One need only recall the highly stylized and tightly choreographed "experiments" in college physics laboratories aimed at enabling students to measure acceleration due to gravity.) As a contrast, an appropriate computer simulation can provide quantitatively rich experiences, but, obviously, kinesthetically impoverished ones. For example, one can engage students in graphically controlling and observing motion of a computer-based elevator in a building, where each floor of the building is clearly numbered, where hot-linked position and velocity graphs are labeled, time is visually marked by a sweeping time cursor, and all relevant quantities are directly and easily adjustable. Indeed, one can enable the students to control multiple elevators and compare motions one against the other as they are repeated and/or changed.

The SimCalc Project (Kaput, 1995) has produced a whole series of motion simulations involving objects or characters moving in various simulated "worlds" that students can control in various ways - swimmers in multilane pools, dancers on a ballroom floor, people and animals moving across various scenes (e. g., duckies on a pond, people walking along a path), cars and trains that can be "driven," and even a schematic "dots world," where the student controls the movement of dots across a simple scrollable, zoomable, numerically labeled grid. However, unless the screen-object motions produced by the students can be linked to their personal experience of motion in more than an abstract way, their motion experience is only visual and disconnected, much as in a video game (Kaput, 1994). It does not support connections with knowledge about their movement in space, or the movement of other physical objects. We are studying two ways of building such connections, varying in their level of student interactive intimacy with the motion data:

(1) Connections based on alternating between physical and cybernetic motion, where students engage in activity structures in one realm designed to support learning in the other; and
(2) Connections based on integrating simulation-based and physically-based data collected in real time within the same computer system.

Examples of 1: Connections of the first type can take the following form, beginning with physical motion, partially inspired by the pioneering work of Mary Barnes (Barnes, 1993). Versions of these activities have been piloted with 9 and 10 year-olds (Kaput, 1995) and a larger scale study involving more than 50 students in grades 3-5 is underway as of this writing. Children walk and run along a numerically labeled straight path with speeds characterized qualitatively as "slow," "medium," and "fast" where measurements take the form of timing "trips" along pre-specified distances, noting distances traveled over given time intervals, comparing different students' versions of slow, medium and fast, etc. This limited 3-valued version of velocity matches children's limited ability to discern and discuss velocity differences. The next activities are on-line with, say, an Elevator (vertical) or "Walking World" (horizontal) simulation, that are likewise initially restricted to 3 speed values, but where the values are explicitly given as quantities, say 1 floor/sec for "slow," 2 floors/ sec as "medium," and 4 floors/sec as "fast." These values are represented graphically on a velocity graph as horizontal straight line segments whose student-adjustable length determines the duration of the motion as depicted in Figure 4. In this situation, the student drags a choice of S, M, or F segment onto the velocity graph and stretches it to the desired width (duration). This can be done for two or more elevators to compare motions associated with different graphs.

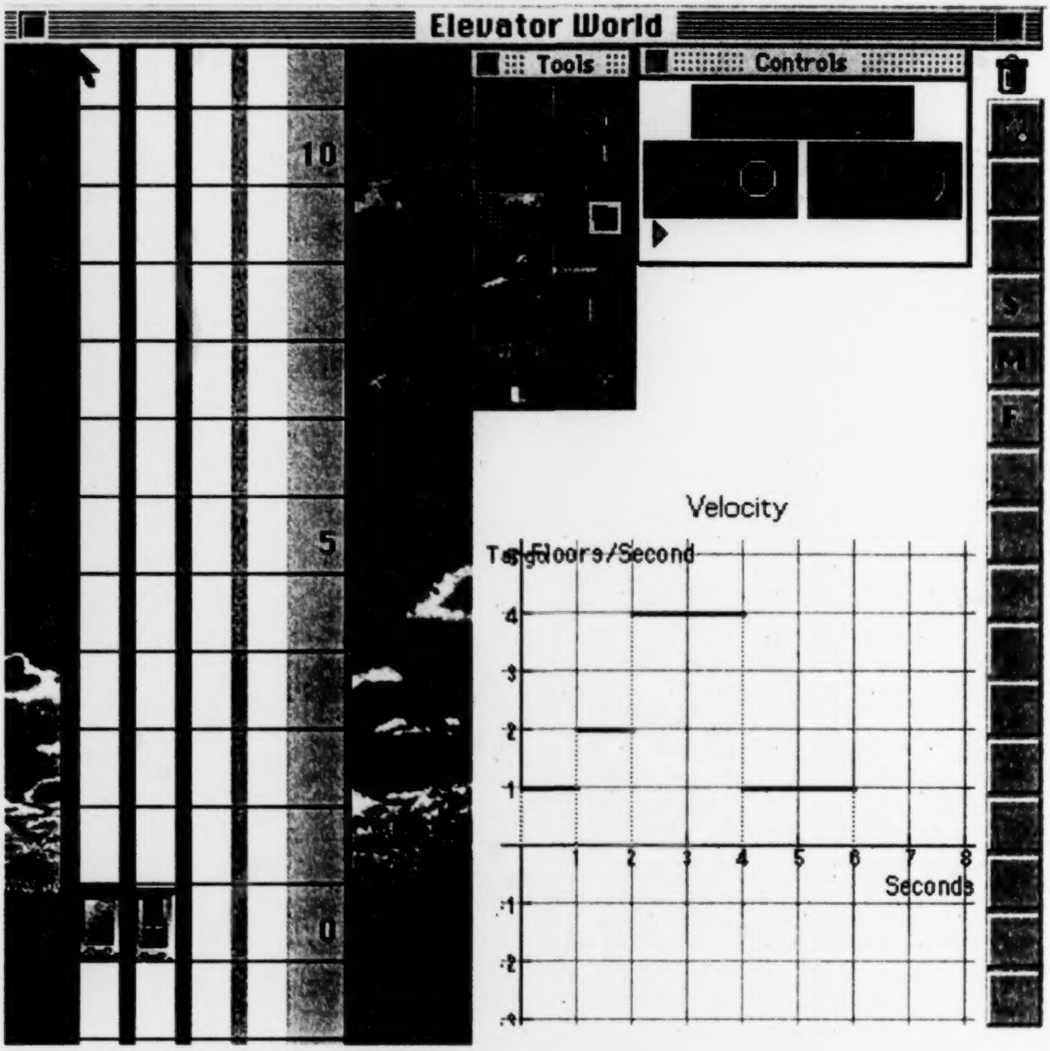

Figure 4: Elevator World

Students examine how far elevators travel over different time intervals at these different speeds. In the simulation, the experience is strongly quantitative, with much computation associated with testing conjectures. ("Snap-to-grid" is turned on so that all time and velocity values are whole numbers, reducing computational complexity.) Gradually, they are introduced to variable speed elevators, mixing slow, medium and fast speeds, and confront such questions as whether one could get to a given floor with a constant medium speed at the same time as an elevator that first goes slow, then fast. This exercise in determining mean values amounts to adjusting the height of the velocity graph (which some students begin to relate to the area under the horizontal velocity graph), but it is devoid of physical meaning until the students return to the physical activity with the same activity-structure - where they now move in pairs. Here their kinesthetic sense of "catching up," for instance, on a slow-then-fast trip simultaneous (side-by-side) with a partner's medium speed trip, often confronts their tendency to race. This type of situation, where maintaining a constant speed is problematic, raises questions of what constant speed means and yields a deeper, more connected meaning for average and constant rates. Later, students return to the computers to deal with more complex versions of the mean-value activity than could be handled physically, where the visually

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editable graphs make this possible. To introduce negative velocities, students again return to the physical realm, where they act out and measure times for various types of "round-trips," for example, trips that go slow in one direction and fast in the other. When they return to the computer, where restrictions to positive velocity values are removed, they confront the question of what kind of velocity graph will make the elevator go downward. Once again, the physical experience takes more precise quantitative form as they compute positive and negative areas (either by multiplying length dimensions of the rectangles involved, or by counting grid-squares).

Examples of 2: Physical/cybernetic connections of the 2nd type are based on newly available functionality that builds on the traditional computer-based laboratory ability to import and graph quantitative data from measuring instruments on a real-time basis.

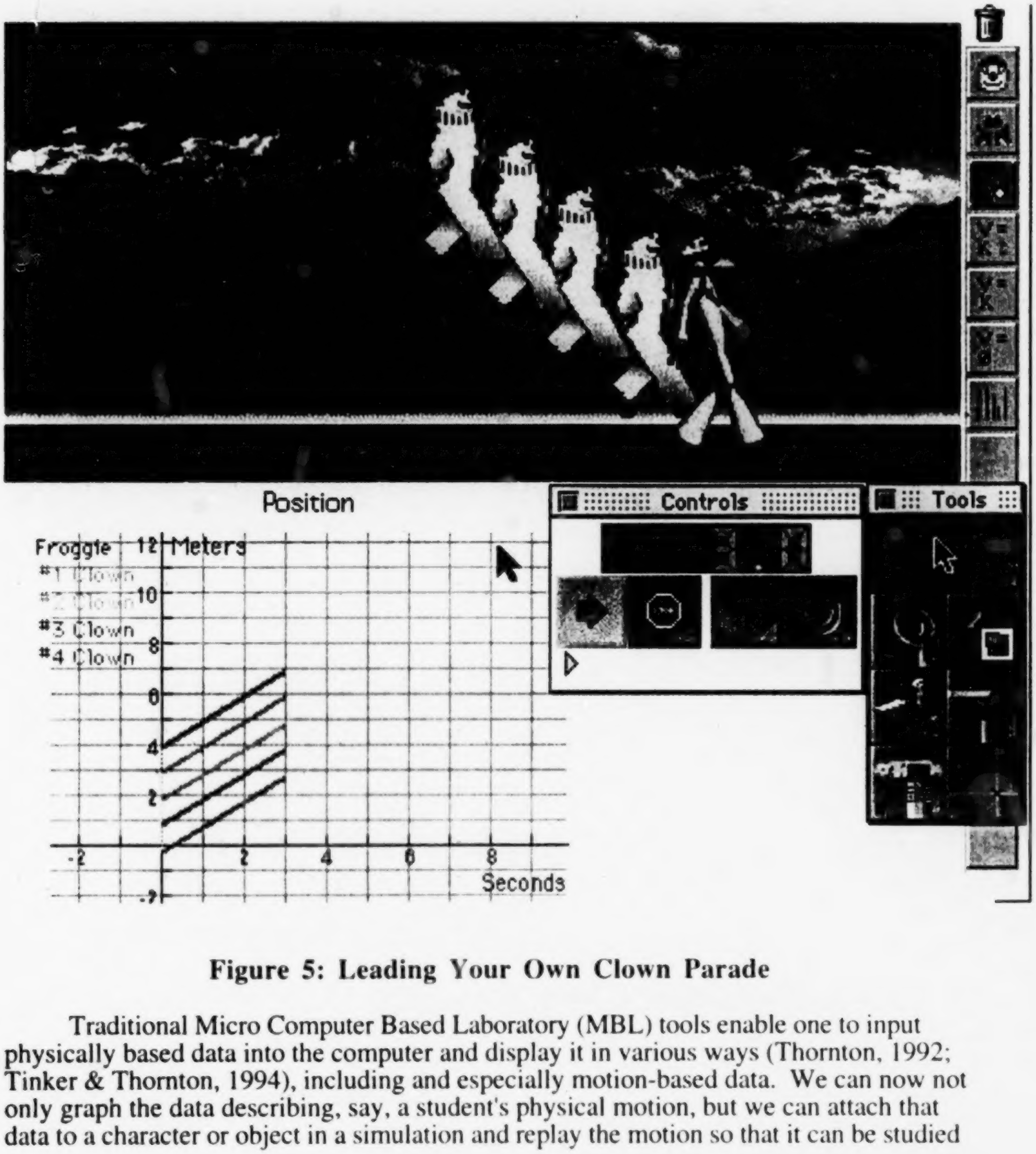

Figure 5: Leading Your Own Clown Parade

Traditional Micro Computer Based Laboratory (MBL) tools enable one to input physically based data into the computer and display it in various ways (Thornton, 1992; Tinker & Thornton, 1994), including and especially motion-based data. We can now not only graph the data describing, say, a student's physical motion, but we can attach that data to a character or object in a simulation and replay the motion so that it can be studied

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for subtlety that may not be evident in its initial transient enactment. But more importantly, this imported data can be mathematically transformed and compared with the original, and students can interact with either the original or transformed data in a variety of ways. For example, they may try to create physically the motion of the transformed data (and then compare their attempts with the "original" transformed data), they might try to approximate the original data with that generated by an algebraic formula, they might try to create physically a motion that matches the algebraic data, they might compare their motion with some particularly interesting aspect of motion of a classmate, they might try graphically to find a mean-value for a physically generated motion, and so on. In all these cases, they can examine all attempts both from multiple perspectives: coordinate graphical, algebraic, numerical, and via actual screen motions. This kind of intimate interaction between the student's physical experience and mathematical experience, while sharing some aspects of what Papert (1993) refers to as "syntonic" experience, is both new and a harbinger of a whole new style of instruction that mixes "first person action" and "third person" observation and adjustment of mathematical objects. One can regard it as yet another step in the broad evolution of integrating cybernetic experience into human activity (Heim, 1993). For example, in Figure 5, the "froggie" could enact your imported physical motion, which is used as the template for a collection of clowns whose motion is graphically edited so that they follow you in a parade (or, alternatively, move in some interesting counterpoint to your motion).

Our aims are broader than simply understanding motion. Ultimately, we want students to exploit their understandings in other settings, with quantities other than velocity, position, etc. In the terms of Figure 2, we want students to make the wider connections to their world that were among the traditional objectives of mathematics instruction. Accounts of students achieving these connections are in preparation.

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Some Teachers' Beliefs and Perceptions of Problem Solving

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This paper reports the results of a preliminary investigation into primary school teachers' beliefs about the role of problem solving in learning mathematics. A survey was used to gather teachers' favourite problems and comments about traditional versus constructivist perspectives on using problems in classrooms. At least half of those surveyed chose problems which they believed promoted learning but most teachers adopted a position that saw problem solving as an end rather than as a means in the learning process.

The problem-solving focus to learning has been advocated for some time and yet many teachers do not seem to have responded to this call for reform. This may be because teachers are not committed to this idea. They may not believe that a problem solving approach is the best way to teach school mathematics. There is a lack of information about what teachers at large believe and understand about problem solving and how they might then try to teach it. In an effort to explore teachers' beliefs about problem solving, this study examines teachers' favourite problems and their beliefs about the role of problem solving in learning mathematics. The data reported in this paper are part of a larger study which focuses on primary school teachers' beliefs about mathematical problem solving and how these beliefs influence decision making in their classrooms. It also aims to explore what factors might promote or hinder the implementation of problem solving approaches.

Different perspectives of mathematics, learning mathematics and teaching mathematics are examined. It is suggested that the perspective adopted by teachers represents their belief system about the nature of mathematics and influences the decisions which are made when teaching mathematics. In this study a survey was used to investigate teachers' favourite problems and to obtain comments about the role of problem solving in learning mathematics. The responses give an indication of teachers' beliefs about characteristics of worthwhile mathematical tasks as well as beliefs about whether a problem solving approach to learning mathematics is acceptable.

Perspectives about Mathematics and Mathematics Learning

Two distinct views or conceptions about what mathematics is and how children learn mathematics can be described. These views represent the end points of a continuum of varied conceptions which may be held by teachers. At one end, mathematics is perceived to be a fixed body of facts and procedures which are memorised and rehearsed so that students can reproduce them when required. This view has been referred to as the instrumentalist view (Ernest, 1989) and may lead to a traditional approach to teaching. At the other end, mathematics is viewed as a "science of patterns" where students are actively engaged in pattern-seeking by exploring this dynamic, evolving discipline (Schoenfeld, 1992). This view has been referred to as the problem-solving view (Ernest, 1989) and is consistent with a constructivist approach. Each of these perspectives has implications for views about how children learn mathematics and also about how mathematics should be taught (Burton, 1993).

Traditional Perspective

The traditional perspective about mathematics leads to a teaching approach which emphasises skill acquisition, speed, accuracy and answers rather than processes. The teacher is an authority and manager of learners who does not need to know a great deal about how children learn mathematics (Ernest, 1989; Battista, 1994). It is sufficient to accept that students passively receive information and if they are able to reproduce procedures then this is evidence that it has been learned. The teacher instructs, often from a prescribed text in which all of the necessary subskills are sequenced with sets of exercises to practise. There is little need for reflection or even awareness that there may be viable alternatives to this method of teaching (Ernest, 1989).

Some teachers who use this traditional approach incorporate problem solving into mathematics lessons but they tend to view problem solving as something which is done at the end of a topic when the appropriate skills and procedures have been mastered (Burton, 1993). Wright (1992) found that many primary school teachers saw problem solving as an 'end' in that mathematics is learned so that students can solve real-life problems. For this approach, problem solving is seen as separate from or supplementary to the traditional curriculum. Authors of many textbooks reinforce this view by placing problems at the end of chapters.

Constructivist Perspective

The constructivist perspective about mathematics has led to an approach to teaching which emphasises active engagement with mathematical tasks, with children constructing their own meanings as they are confronted with learning experiences which build on and challenge existing knowledge. The teacher's role is that of a facilitator and requires considerable reflection as the teacher must observe student responses, challenge student thinking and encourage risk taking within a supportive classroom environment. Teachers need to know about how children learn mathematics (Battista, 1994). They need to constantly review student understanding and to "guide children's constructive activities until they eventually 'find' viable techniques" (Steffe & Cobb, 1988, pvii-viii).

From this perspective, problem solving is viewed both as central to the way children learn and as a means by which they learn as they are confronted with activities which are problematic (Wright, 1992). This approach provides the opportunity for children to reflect on current procedures and develop more sophisticated meanings and strategies. Problems are used as the focus of lessons with mathematical concepts and skills being developed as required.

The two perspectives described above represent opposite ends of a spectrum of views about mathematics and how children learn mathematics but there are many possible views in between. Ernest (1989) described a middle perspective as the Platonist view of mathematics. This view sees mathematics as a "static but unified body of certain knowledge"(p250) where content is consistent, connected and objective. The teacher's role is that of explainer as the textbook approach is modified, adapted and supplemented with problems and activities.

Problems and Problem Solving

Based on the perspective adopted, teachers form beliefs which directly influence their classroom practices including lesson planning and choice of tasks. When preparing to teach mathematics, a significant decision for the teacher is to choose appropriate problems. This can be challenging as questions which are problems for some students in the classroom may not be problems for others. In choosing a worthwhile mathematical task, Lappan (1993) suggests that the teacher must judge

> ...how well the task represents the embedded concepts and procedures, how likely the students are to bump into the mathematics in the course of investigating the problem, how well the task represents what is entailed in doing mathematics, and what skill development the task will or can support (p524).

She warns that problems should not be chosen just because they are fun or use manipulatives but rather that they allow students to engage in significant mathematics. Good tasks capture student curiosity and include opportunities for mathematical thinking as well as developing concepts and skills.

Teachers may have different beliefs about what constitutes a mathematical problem. Defining the terms 'problem solving' and 'problem' has caused considerable concern for mathematics educators and teachers. Teachers give many different definitions

(Funkhouser, 1993) which must lead to differences in classroom practices including variety in the kinds of mathematical tasks which are presented to students. When asked to define problem solving, Funkhouser found that only one-third of teachers were able to give a definition which included specific problem solving terms. The remainder gave very vague definitions. Hembree's (1992) examination of a large number of studies revealed definitions that range from traditional word problems to non standard problems.

Earlier definitions tended to consider questions which were asked in words as problems whereas more recent definitions require the presence of a blockage as a necessary prerequisite for a question to be considered a problem. These are clearly conflicting definitions. Whereas on one hand a problem can be a term for routine exercises, on the other hand it can refer to a question which is difficult or creates a perplexity for the solver. According to House, Wallace and Johnson (1983)

> ...a mathematical problem is a situation that involves a goal to be achieved, has obstacles to reaching that goal, and requires deliberation, since no known algorithm is available to solve it...Problem solving is the process of attacking such a problem...(p10).

In this study one aspect of the investigation is to try to discover what practising teachers believe to be a good problem and another is to ascertain their beliefs about the role of problem solving in learning mathematics.

The Survey

A teacher belief survey was designed to gather background information, favourite mathematical problems and comments about two scenarios or vignettes. The background information included gender, number of years teaching experience, grade currently being taught, role in the school, number of days inservice attended in the last two years as well as the number of these which were devoted to mathematics.

Each teacher was asked to record their two favourite problems and to briefly describe why these were chosen. This question was designed to be an unobtrusive measure of what teachers really think about problem solving. It was anticipated that the kinds of problems chosen would give some indication of each teacher's classification of what constitutes a problem and what characteristics of these chosen problems made them particularly useful.

The scenarios were structured so that two perspectives on the role of problem solving in learning mathematics were presented (Appendix 1). Each included a teacher who believed that problem solving could be used as the focus of learning mathematics and a teacher who believed that children need practice with basic skills with some problem solving when time permits. These two perspectives reflected the problem solving as a 'means' versus problem solving as an 'end' approaches to learning mathematics. Respondents were required to comment on each scenario giving reasons why they preferred a particular approach to teaching. It was anticipated that these responses would reveal espoused beliefs about mathematical problem solving and its role in the teaching and learning of mathematics.

Responses were sought from practising teachers, some of whom were part-time post-graduate students in teacher education courses. A total of 39 teachers were surveyed; 23 were chosen for analysis on the basis that the respondents were responsible for teaching a primary grade class and that they had recorded at least one favourite problem. The NUD*IST software program (QSR, 1994) was used to create an index system to manage and organise the data. Even though this was a small sample of teachers, the process of organising the data into categories highlighted emerging issues which will inform the next stage of the larger study.

Survey Results

Background information indicated that the sample of teachers represented a wide range of teaching experience and included at least one teacher for every grade in the primary school years. Analyses revealed that eleven teachers had been teaching for five years or less, another eleven teachers had been teaching from six to 15 years and one had been teaching for more than 15 years. Five were teaching in the K-2 grades, eight in grades 3 or 4 and ten in grades 5 or 6. There was a large range of inservice days attended with most teachers having attended one or two mathematics inservice days in the last two years. The ratio of females to males was approximately three to one.

Favourite Problems

The favourite problems which teachers recorded were examined to identify content and problem type. Responses ranged from very general statements, for example "money problems", to highly specific questions and were classified according to the main area of mathematical content or strand. A total of 39 problems were provided by the 23 respondents and of these 15 involved number, 12 space, 9 measurement, 2 chance and data and 1 was very general. An initial classification of problem type included the categories of 'real-life investigation', 'mathematical investigation', 'closed, content focused' and 'other' but this classification failed to provide discrete categories. An improved classification involved an open versus closed division. Each of these was then further subdivided into contextualised or decontextualised.

A total of 17 problems (44%) were classified as open as these either had several answers or many ways of approaching them. Seven problems (18%) were classified as open and contextualised since they were presented to the students in a realistic context. Examples of these were :

"Can a dinosaur fit into your classroom?"

"Redesign the classroom."

Ten problems (26%) were classified as open and decontextualised as they were removed from a real context. Examples included:

"Tell me a story from the graph."

"How many different combinations of coins can you find which total 50c."

"Find your stride length, then find objects which are the same, shorter and longer."

The remaining 22 problems (56%) were classified as closed as they usually had one correct answer. Ten problems (26%) were closed and contextualised. Examples included:

"There are 32 children in 4L, 31 in 4T and 30 in 4P, how many children are in year 4?"

"There are chickens and pigs on a farm, given the number of legs and the number of heads, how many of each?"

Twelve problems (30%) were closed and decontextualised. Examples included:

"What is the chance of throwing 3 heads with three coins?"

"Counting in twos via addition."

It was anticipated that teachers' favourite problems would reveal useful information about their definitions of a problem. The examples given did not provide this as problems need to be appropriate for the class or particular students which the teacher has in mind. One teacher recorded the problem:

"Kylie had 37 apples, she ate 12 apples, how many does she have left?"

For many students this would not be a problem but in the lower grades there would be students who would not have a readily available procedure for solving this problem. It is possible that this particular teacher may consider that a problem is a question stated in words. Other responses which did not reveal teacher's definitions of a problem were very vague, for example:

"Measurement - time".

From this information it is not possible to determine exactly what kinds of questions might be posed by this teacher.

It is worth noting that of the 16 survey responses not used in the analysis, seven of these were full-time classroom teachers who did not record a favourite problem. Two of these teachers commented that they "use many" whereas another two stated that they "can't think of any". In a follow-up interview with a voluntary group of eight of those surveyed, a comment was made that it is difficult to think of a problem when you are not in class with the children. Another teacher said that she thought of a text-book question but she did not want to write that down because these are not the best examples of problems. Other comments suggested that problem solving is not planned but just arises out of what happens in the classroom. If this is the case, then it is not surprising that so many of those surveyed were unable to record at least one favourite question.

Recording a favourite problem does not reveal how the problem was posed in the classroom or the way that it was explored or discussed with the students. The problem has been removed from the context of the classroom. A better question would have been to ask how the problem was used in the classroom or to describe a recent problem solving lesson. In an effort to explore what the characteristics of a favourite problem might be, teachers were asked to record why these particular problems were chosen.

Responses to the request to describe why these particular problems were chosen were categorised into four groups - affective responses, problem characteristics, teaching factors and learning factors. The affective responses included statements about enthusing and motivating students and students enjoying the challenge. Desirable problem characteristics were the open-ended nature of the question, the real-world context of a question, that problems could be solved in the context of a game and that some questions could be adapted for use with many classes. Teaching factors included statements about children working in groups which promoted use of mathematical language, lots of activity with hands-on experiences, incorporation of several content strands in one task and possible links with other KLAs. Learning factors were estimation, remediation, visualisation, practise in basic skills, introduction of concepts, challenge and extension. Half of the teachers surveyed mentioned at least one learning factor which indicates that they are aware of the role that problem solving plays in the learning of mathematics. It is unclear whether the other teachers believe that problem solving does promote learning as they focused on the alternative categories.

Teachers' Beliefs about Problem Solving

Responses to the two vignettes were also categorised. The vignettes described possible scenarios with two teachers who had quite different approaches to teaching mathematics. Surveyed teachers were required to comment on the statements by making recommendations with explanations for their decisions. Statements were classified into four groups - class, mathematics, learning and other. The 'class' category was subdivided into ability, behaviour, background and other. 'Mathematics' was subdivided into skills, real-life and other. 'Learning' was subdivided into problem-solving views, non problem-solving views and mixed views for statements which indicated that the teacher was recommending a middle position or combination of the two previous views.

Many comments related to the class or individual students. Several related to the ability of students to cope with a problem solving approach as they may need considerable practice on basic skills. Comments were made about children with learning difficulties, ESL backgrounds and the increased need to cope with language when solving problems. There was concern that children need to be cooperative and that some children were uncooperative, antisocial and immature. One teacher suggested that previous experience with problem solving was necessary before a child would be comfortable with a problem solving approach. Several comments referred to a consideration that children learn in different ways and hence respond to different approaches, this was summarised in the comment:

"All children learn in different ways and I feel we should adopt a variety of approaches to cater for these."

One teacher noted that using a problem solving approach gives students the opportunity to construct their own knowledge.

Comments about mathematics focused on skills and real-life experiences. Most teachers believed that skills need to be developed before problem solving and seemed to support the problem solving as an 'end' approach. This was reflected in the comment:

"Basic skills have to have time to be taught - has to be a priority! Teach these first, then apply problem solving to them."

Seven teachers wrote statements which supported problem solving as a 'means'. This was reflected in the following:

"Because the children are learning skills through problem solving they have a need to use basic skills - a real need, rather than simply the completion of sums in a text."

Ten teachers wrote comments relating to the relevance and importance of children developing problem solving skills and that problem solving provided an opportunity to learn mathematics in a real-life context. Other comments about mathematics included opportunities for discovery learning, investigation from open-ended questions and integration with all areas of the curriculum.

Statements about learning were organised according to their support of a problem-solving approach, a non problem-solving approach or a middle position. Almost every teacher surveyed made a statement which supported the use of problems in the mathematics classroom. It was clear that teachers were aware that problems provide students with the opportunity to use valuable skills and that this was more interesting for them than practising algorithms from a textbook. Most teachers also recommended that skills must not be ignored and that extra practice may be needed. It became clear that isolating comments into these three categories did not really give an indication of each teacher's beliefs as many teachers provided comments which could have been placed in all three categories. There were only four cases where comments were placed in only one category and these teachers each wrote very little. The comments which teachers wrote in responding to these scenarios really needed to be read as a whole.

For some teachers, overall beliefs became clear when the responses to both scenarios were read together. By selecting individual ideas or statements it seemed that the teacher supported a problem solving approach and yet accompanying comments revealed reservations. One case is particularly noteworthy. In response to the first scenario this teacher wrote:

> "Ms Smith would probably be the teacher most favoured by myself...because the children are learning skills through problem solving."

For the second scenario she wrote:

> "Yes I would probably agree that you can teach skills through a problem solving approach in some schools and with some students. I honestly would not adopt such an approach to teach maths.
> (1) The problem solving arena is very overwhelming.
> (2) Usually involves too much language.
> (3) I am yet to find suitable problems for all sub-strands."

This teacher was prepared to be very honest. Even though she was aware of current thinking involving the role of problem solving in learning mathematics she was not prepared to teach that way. She confessed to feeling overwhelmed and she raised two important issues which caused her concern. It is possible that other teachers may have held similar beliefs but these were not revealed on the surveys. Interviews may provide a better opportunity to reveal espoused beliefs.

Another issue relates to what teachers believe to be a problem solving approach to teaching mathematics. Many respondents agreed with Bill in the second scenario and yet their comments suggested that their views of a problem solving approach were different to what was intended. One teacher responded as follows:

> "Agree with Bill - what does John do??? Short groups of problems or tasks can cover all mathematics areas - break them up into small manageable components."

This response seems to suggest sets of textbook word questions rather than using a problem as the focus of a lesson from which skills and understanding are developed. Another teacher cautioned that there may be different definitions of a problem solving approach. She stated:

> "It depends what's meant by a problem solving approach. My
> maths uses plenty of problem solving with open-ended problems.
> This is the way to go I believe."

Given that the term was meant to suggest that problem solving was used as a 'means' rather than an 'end' in teaching mathematics, responses from a majority of teachers seemed to support the problem solving as an 'end' approach. Eight teachers made direct statements indicating that they did not support the view that all mathematics could be taught through problem solving. Many teachers supported a middle position with their comments suggesting that you need to have children practising skills as well as doing problems in mathematics classrooms. This indicated that they interpreted the scenarios to mean that skills alone was one position, and problem solving alone was the other position.

Implications

The survey has revealed that most teachers support the use of problem solving as an end in teaching mathematics but it has also identified further issues for exploration. Teachers do have different beliefs about the role of problem solving in learning mathematics but in order to decipher exactly what these are requires clarification of what each teacher understands by the terms 'problem' and 'a problem solving approach to teaching'. It may be difficult to identify teachers' beliefs using only a survey although Thompson (1992) cautions that verbal responses in interviews can be equally unreliable because teachers may readily give a commitment to ideas about teaching which they know to be currently desirable.

The data about favourite problems highlighted the need to have teachers describe how the problem was used in the classroom. Teachers make decisions about appropriate tasks based on their beliefs, the class and individual students as well as the skills and concepts they are trying to develop. These were not revealed in the surveys and it would have been more useful if teachers had been asked to describe a recent problem solving lesson. Another strategy would be to present teachers with a set of problems and to have them rank these according to their usefulness with a particular grade as well as their value for promoting understanding of mathematical concepts.

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Appendix 1

Scenario 1

A close friend tells you about two Year 5 teachers at her child's school. Ms Jones is concerned that too many children finish primary school without a solid background in the basics of mathematics. She ensures that plenty of basic skills in the 4 operations are practised in her class with the occasional use of problem solving. Ms Smith, on the other hand, does not practise these skills as much but prefers her children to develop their skills by drawing on them as they need them when solving problems. Your friend wants to know which teacher she should request to teach her child. What advice would you give her and why?

Scenario 2

Two teachers are having a conversation in the staffroom. John is clearly frustrated by the number of things he has to fit into the curriculum and barely finds time to cover the essential basic skills in mathematics, let alone investigate problems. Bill responds by saying that you can teach all mathematics by using a problem solving approach. They ask you for your opinion - what do you think?

Classroom Instructional Factors Affecting Mathematics Students' Strategic Learning Behaviours[1]

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The emergence of learning strategies as a critical variable in the active learning process is reflected in the promotion of a range of cognitive and metacognitive learning strategies in recent mathematics curriculum documents. This paper discusses the influence of instructional factors on senior mathematics students' development and use of learning strategies. While acknowledging that there are many causes for students' failure to use appropriate learning strategies reported findings suggest that compensatory instructional practices may limit the development and effective use of appropriate learning strategies.

Mathematics curriculum documents, both in New Zealand and Australia, promote the learner as an active constructor of knowledge. *A National Statement on Mathematics for Australian Schools* claims that "learning is best thought of as an active and productive process on the part of the learner" (Australian Education Council, 1990:16). Likewise, *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992) states that "mathematics is most effectively learned through students' active participation in mathematical situations, rather than through passive acceptance and repetition of knowledge" (p. 18). Within the constructivist framework the self-regulated learner must "appropriately control his or her learning processes by selecting and reorganising relevant information and building connections from relevant existing knowledge" (Mayer, 1992:409).

The processes by which the learner selects, organises and integrates new knowledge and the behaviours and thoughts affecting the learners' motivation or affective state are referred to as learning strategies (Weinstein & Mayer, 1986). Current research suggests that students' availability, selection and use of learning strategies are critical variables in the active learning process (Anthony, in press; Wang, Haertel, & Walberg, 1993). By using various learning strategies, students can intentionally influence the form and quality of the knowledge they acquire. Recent studies in mathematics education (Anthony, 1994a) have found that the range and quality of many students' strategic learning behaviours are limited, and in many cases ineffective in assisting students' knowledge construction processes.

An individual student's strategy selection is linked to numerous factors such as prior knowledge, the nature of the task, motivation, and the availability of resources. Although interrelated, these factors can be broadly grouped into student, context and instructional factors. Student factors would include prior domain knowledge, metacognitive knowledge, learning goals and affective issues related to mood, feelings and interest. Contextual factors would include the physical learning environment, the social nature of learning (peer interactions, discussion, etc.), and the availability of resources. While not dismissing the influence of both student and contextual factors on the development and use of strategic learning behaviours (Anthony, 1996), the present discussion is limited to a consideration of the impact of instructional factors which interact in the students' learning environment. By increasing our knowledge of those factors which are most easily influenced by teacher input, we may be in a better position to encourage and develop appropriate and effective use of such learning strategies as elaboration, organisation, planning, monitoring, checking and reflection.

Method

Findings reported in this paper are from a larger study (Anthony, 1994a) which examined senior mathematics students' awareness and application of learning strategies. An ethnographic approach employed techniques of classroom observation, interviews, student diaries and questionnaires. In addition, four students participated in stimulated recall interviews, based on classroom lessons, which increased the likelihood of access to students' thoughts and covert learning behaviours (Anthony, 1994b).

Setting

Because learning strategies are not applied in a vacuum, but are influenced by a multitude of variables, the use of the authentic learning setting was seen as crucial to the research process. A year 12 mathematics class of twelve students from a large New Zealand coeducational school was the focus of the study. The students represented a cross-section of interests, motivation, achievement levels, gender and ethnicity. The class followed a pre-determined syllabus and assessment structure. Lessons typically began with homework review, followed by whole class instruction through teacher exposition and teacher-student interaction. During seatwork the teacher moved about the classroom encouraging individual students and checking whether they had any difficulties. Most students worked independently, although in two groupings there was considerable peer interaction.

Data analysis

Students' learning behaviours from all the interviews and observations were coded according to their cognitive, metacognitive, affective or resource management goal. Learning behaviours in each of these four broad classifications were then grouped into 36 representative learning strategies under a classificatory scheme. Stimulated recall data were coded according to this classification scheme to provide a quantitative comparison of strategy use.

To present a more holistic view of the students' learning processes stimulated recall interviews provided contextual descriptions of learning strategies in use. Analysis of the effectiveness of each student's strategic learning processes in specific learning episodes such as homework, discussions and seatwork, as well as an examination of factors in the students' learning environment which influenced the development and use of learning strategies, provides the basis for the following discussion.

Time at the research site (a full academic year), triangulated findings, video and audio recordings, and low inference descriptions from student reports increased the "trustworthiness" (Glesne & Peshkin, 1992) of the research interpretations.

Instructional Factors

How does instruction affect the students' use and development of strategic learning behaviours? Campione, Brown and Connell's (1989) commentary on mathematics instruction suggests an air of foreboding.

> Students are not made aware of the reasons for the skills and procedures they are taught. They are seldom given explicit teaching regarding the orchestration, management, and opportunistic and appropriate use of those skills. And they are seldom required to reflect on their own learning activities. These factors help to induce in students a flawed understanding of themselves as learners and of the academic domains they are called upon to master. (p. 111)

Instructional Orientation

Ames and Archer (1988) found that the goal orientation of classrooms, as perceived by the students, affects strategic learning. Students constantly construct interpretations of their teacher's behaviour and expectations, and the nature and purpose of classroom activity. In classrooms that emphasise task mastery goals and understanding, success is seen as dependent on effort and strategic behaviours. In classrooms that emphasise ego, ability and performance orientations, students are socialised with the goal of getting good grades, being judged able, and feel success is dependent on ability.

In a survey, adapted from *High School Science* (Nolen & Haladyna, 1990), all of the students expressed positive perceptions about cooperative work, teaching for understanding, learning from mistakes, independent thinking, and questioning. There were however, some indicators of performance orientation expressed simultaneously. The majority of students felt that they moved onto a new topic before they had really understood the old one, that you had to compete to get good grades, that it was difficult to 'keep up', and that you had to memorise lots of material. Performance orientations were also reflected in some students' negative views of public help-seeking. Avoidance of public help-seeking, related to student feelings of personal inadequacy and a wish to avoid comparison with other students, was particularly common with female students.

Karen: I honestly thought it was called a pictograph. I don't want to say anything in case it is so far wrong I embarrass myself.

Lucy: You feel a bit dumb asking questions. I sometimes ask, but if I got one wrong and the rest right I wouldn't really worry.

Brooks and Brooks (1993:7) suggest that students' unwillingness to answer teacher's questions, unless they are confident that they already know the sought after response, is a direct consequence of teachers' use of questions: "When asking students questions, most teachers seek not to enable students to think through intricate issues, but to discover whether students know the "right" answers."

Jane: Some of the time I don't understand the stuff enough in mathematics to answer questions 'cause I'll probably get it wrong. I only answer questions if I know the answers.

Emphasis on direct instruction, emphasis on skills before understanding, lack of on-line diagnosis, absence of explicit strategic instruction, assessment pressures, and the required coverage of the course may have increased the likelihood of students adopting a performance orientation. The teacher, faced with keeping the class in parallel

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Instructional Orientation

Ames and Archer (1988) found that the goal orientation of classrooms, as perceived by the students, affects strategic learning. Students constantly construct interpretations of their teacher's behaviour and expectations, and the nature and purpose of classroom activity. In classrooms that emphasise task mastery goals and understanding, success is seen as dependent on effort and strategic behaviours. In classrooms that emphasise ego, ability and performance orientations, students are socialised with the goal of getting good grades, being judged able, and feel success is dependent on ability.

In a survey, adapted from *High School Science* (Nolen & Haladyna, 1990), all of the students expressed positive perceptions about cooperative work, teaching for understanding, learning from mistakes, independent thinking, and questioning. There were however, some indicators of performance orientation expressed simultaneously. The majority of students felt that they moved onto a new topic before they had really understood the old one, that you had to compete to get good grades, that it was difficult to 'keep up', and that you had to memorise lots of material. Performance orientations were also reflected in some students' negative views of public help-seeking. Avoidance of public help-seeking, related to student feelings of personal inadequacy and a wish to avoid comparison with other students, was particularly common with female students.

Karen: I honestly thought it was called a pictograph. I don't want to say anything in case it is so far wrong I embarrass myself.

Lucy: You feel a bit dumb asking questions. I sometimes ask, but if I got one wrong and the rest right I wouldn't really worry.

Brooks and Brooks (1993:7) suggest that students' unwillingness to answer teacher's questions, unless they are confident that they already know the sought after response, is a direct consequence of teachers' use of questions: "When asking students questions, most teachers seek not to enable students to think through intricate issues, but to discover whether students know the "right" answers."

Jane: Some of the time I don't understand the stuff enough in mathematics to answer questions 'cause I'll probably get it wrong. I only answer questions if I know the answers.

Emphasis on direct instruction, emphasis on skills before understanding, lack of on-line diagnosis, absence of explicit strategic instruction, assessment pressures, and the required coverage of the course may have increased the likelihood of students adopting a performance orientation. The teacher, faced with keeping the class in parallel with other classes, sometimes looked for shortcuts, or side-stepped the more demanding parts of the course. For example, when introducing differentiation the teacher spent time on the concept of the derivative, including calculations from first principles. However, the learning goal became confused when the teacher stated, "*Now we are going to forget about how it is found and just use the formula*".

Task Demands

The majority of the seatwork and homework time was spent on exercises which provided practice of teacher-provided examples. The focus was on computational procedures and accuracy. Students knew in advance which computational procedures were required to solve the exercises and rarely worked on problems requiring the integration of information across several topics. The teacher often reminded students that practice was the key to learning.

Mrs H: Try them all. People who have attempted lots of work tend to do better in exams.

Mrs H: The reason some of you are not doing very well is that you are not doing your homework. You need to practice until you can say, 'I have met this question before'.

While practice is important when learning procedural skills, reliance on this type of exercise limits the need for active, intentional learning. Without opportunities to do problems requiring higher-order skills, students may come to view practice as a way of memorising set examples and procedures that are to be tested in exams: the means to success is not to think through the problem and integrate information to form new ideas, but rather to recall how the teacher (or oneself) did a similar problem.

When tasks became difficult, involving high-level cognitive processing, or when the answers were not readily available, there was a tendency for students to resist task engagement or negotiate the task demands downwards.

Jake: I didn't do any homework. I don't know how to do it in class, so I couldn't do it at home.

Mrs H: Which of these (Cosine or Sine Rule) would be the easiest to prove?

Lucy: You're the teacher, you tell us.

Students mostly expected the teacher to present 'official' algorithms for solving problems step-by-step, without their needing to reflect on the process.

Mrs H: How do you think we could solve sin x = cos x?

Dean: If we haven't done it, how can we tell you?

The desire to have the information supplied, rather than to be actively involved in the generation and integration of information, was clearly demonstrated by students' preference for 'teacher-given' summaries.

Brent: Can you (teacher) write a glossary of terms we need to know?

Faye: She's a very good teacher, she writes down the answers for you.

Thus, the teacher's activities are constrained by obligations: the "students are not only 'victims' of this classroom culture but also are the 'culprits' " (Voigt, 1994:287).

In other instances, students completed tasks in ways that circumvented the intended learning demands. When tasks were high in procedural complexity (e.g., practical trigonometric investigation) most students spent more time focusing on the procedures, such as measurement, locating a suitable building and recording the information, than on the content. This was reinforced by the teacher's instructions which focused on what students needed 'to do' to complete the task, rather than the learning outcome. In other instances, to complete tasks, students copied work from others, answered questions using prompts from other students or the textbook answers, or offered provisional answers (guesswork) to indicate engagement in the task.

Another major factor affecting strategic learning behaviours is the balance between task demands, instructional support, and compensatory behaviour (Thomas & Rohwer, 1993). Supports are teacher or text provided aids that serve to prompt or sustain student engagement in the learning activity, such as information aids, opportunity for practice, or psychological support. Compensations, on the other hand, reduce or eliminate the demands, thus reducing the need for students to engage in autonomous learning activities. In this study, cognitive loads were reduced by subdividing tasks, setting short term learning goals, and providing informational products, such as a list of specific items in a test, graphs, tables, and summaries that students would otherwise need to generate. An alternative would be to provide orienting information (e.g., a list of content areas to be responsible for), a model for a process (e.g. a table to complete), or a concept map or flow diagram from which the student would form a summary.

Keywords were frequently used by both teacher and students. For example, when the teacher quickly reviewed the students' exam papers (with the emphasis on the teacher reviewing rather than the students) the following comments were included:

Mrs H: When you hear 'gradient' or 'tangent' what should you think about?

Mrs H: What are the keywords, what should the words 'rate of change' tell you?

Mrs H: Look at the paper, the most important thing is to find the keywords.

It is assumed that the recognition of keywords will help students recall the appropriate sequence of actions necessary to solve a problem. However, while a keyword may help some students complete a problem it does little to help students construct meaningful mathematical knowledge. By compensating for problem-solving demands, keywords enable students to complete a problem without necessarily understanding the problem situation, without modelling the problem mathematically, and without acquiring the intended procedural knowledge. Low achieving students, looking for ways to remember problem methods, are particularly vulnerable to misusing the keyword strategy. For example, when trying to solve the problem: *"Find the equation of the line, given $m = 2$ and the x intercept is 8."* Gareth first writes $8 = 2x + c$, looks puzzled, then refers to his text for a worked example. He then writes the answer as $y = 2x + 8$ using the keyword 'intercept' to identify the (incorrect) solution method!

The use of teacher supplied summaries was also intended to support the students' learning; but in reality compensated for student learning activity. On the occasions when the teacher encouraged students to participate in providing summary statements she was met with a reluctance to offer suggestions. Often negative student feedback precipitated the teacher to eventually supply all the summary material or provide an option or way out of the process: *"If you need notes on what we have done today and you don't trust your own, use Chapter 22."* With no external requirement to read for meaning, or to be selective, and with no expectation that students will be responsible for demonstrating their knowledge of the main ideas in a lesson, students have little opportunity to develop the learning strategies of selective attention, paraphrasing and organisation that are needed for autonomous, self-regulated learning.

Presentation of Worked Examples

The presentation and student interpretation of worked examples did not always promote effective strategic learning. Rarely did students ask questions related to the conceptual nature of the problem - preferring to direct their attention to the acquisition of specific information needed for the algorithmic activity. In effect, they sabotaged the instruction by selecting from it only the minimum necessary to answer the teacher's questions and achieve correct performance. The more successful students, however, did attend to the underlying structure of the worked example and used discussion as an opportunity to self-question and generate self-explanations. These self-explanations have the characteristic of adding tacit knowledge about the actions of the example solution, thus inducing greater understanding of the principles involved (Anthony, 1994c).

Opportunity to direct their own learning

To use learning strategies effectively students need time to clarify what has been happening in the lesson: "to engage in such processes as are required to evaluate the adequacy of specific knowledge, make connections, clarify, elaborate, build alternatives, and speculate" (Tobin & Imwold, 1992:21). In reality this was not the case. Teacher prompting, self-answering of questions, and limited wait time were evidenced in most lessons. Low achieving students, in particular, were often interrupted with a prompt or the answer, rather than guidance, when they hesitated or responded incorrectly.

Mrs H: What is the thing inside the square root called? Can anyone remember - it begins with v?

Dean: (calls out) Velocity.

Mrs H: Variance, not velocity. You may be asked to find the variance in the test.

If the teacher regularly answers her own questions she abrogates the need for students to engage in cognitive processing and self-management. Students learn that non-answers quickly generate teacher prompting and willingly accept a passive role in class discussion. Stimulated recall interviews did, however, reveal that students were sometimes answering questions privately; perhaps because of the expectation that others or the teacher would answer, or an unwillingness to engage in public discussion.

Adam: I noticed that she (teacher) forgot to times by n/2 but I didn't really want to speak out because I feel like, because I thought someone else might pick it up as well.

The teacher often used instructional stimuli such as questions related to comprehension and brainstorming to encourage students to make judgements about their knowledge. There was an unstated, but mainly unfulfilled, requirement that if the student judged that personal understanding was inadequate, the student would request help.

Many instructional demands were very structured. While the intent may have been to support and guide learning, students were in fact given little encouragement or opportunity to preview material, explore the text, or generally take responsibility for directing their own learning. On one occasion Adam had completed the required task and was working independently; when the teacher checked his progress she immediately set an alternative task, disregarding Adam's self-selected work.

Adam: She (teacher) said read estimation. I thought she would just come to see what I was doing. I didn't know she would tell me to read something else (surprised tone). It doesn't matter. I can do that at home sometime - it doesn't worry me. I've already done the work on estimation, but I didn't tell her, so it will be like revision anyway.

Instructional cues which allow students to anticipate learning activities also influence strategic behaviours. The teacher sometimes used a technique of going around the class for answers to a set of problems. The intention was to encourage all students to participate. However, some students reported concentrating only on thinking of an answer for 'their turn' - this practice interfered with the process of evaluating other students' answers. Likewise, the predictability of the homework review suggested to some students that there was a limited need to check work, to complete homework, or to persevere when homework became difficult.

Abe: Homework is important, but for some reason I just don't do it!...She (teacher) goes over homework most days, I can pick up things there. I tune in, have the page ready.

Lucy: I try to sort out the problem from the answer, but usually I just give up; we'll go through it in class anyway.

However, for some, the review provided an incentive to complete the homework.

Kane: It's (homework review) a good idea. I supposed it's the sort of thing like, when you're at home you think, I've got to get this done because she'll (teacher) be going over it and sort of getting into trouble type of thing.

Assessment

Performance feedback plays a crucial role in learning. Specifically, in the acquisition of procedural knowledge, attention to feedback when one has made a mistake is critical for shaping accurate metacognitive knowledge. In the present study, most students relied on the textbook answer, or the teacher, as the source for revealing correctness. Such feedback, which is simply given to the learners, without clear reference to the way they tackled the task, is unlikely to lead to effective metacognitive knowledge development. Similarly, when tests were returned the focus of both instruction and the student was on the product, rather than the learning process. Short and Weissberg-Benchell (1989) suggest that students should be explicitly taught to recognise the multiple causes responsible for learning outcomes: "Success experiences would provide information regarding task-appropriate strategies, whereas failure would provide feedback regarding task-inappropriate strategies" (p. 50).

Missing from instruction was explicit references to checking procedures and to the value of checking. Regular prompting for students to evaluate the reasonableness of their solutions, the justification of their procedures, the verbalisation of their processes, and the reflection on their thinking, behaviours that lead to opportunities to learn and develop mathematical thinking, were limited. A notable exception involved solving simultaneous equations, in which checking by back-substitution was an integral part of the procedure. Gareth expressed the view that this procedure was called the 'substitution method' because you "*substituted in at the end*"!

Formal assessment was dominated by questions requiring repetition of teacher-given procedures. Examples of questions that required a modicum of original thought either occurred at the end of the test and were awarded few marks, or were accompanied by hints so as to effectively reduce the demand for high-level thinking.

Gareth: It's more of a concern to know how to get the right answers because you don't really get checked much on understanding, all you get is a list of problems in the test.

Continuous exposure to this practice substantially reduces the task demands and the corresponding development of students' metacognitive knowledge and use of effective learning strategies (Thomas & Rohwer, 1993).

The students gained information about the structure and content of each test from teacher cues and revision of previous years' papers. The predictability of the test content and structure encouraged passive learning. Revision involved a quick flip through the class examples and teacher-provided summaries and reliance on teacher-directed study the day before the test. There was little encouragement for students to identify and evaluate their own particular learning needs.

Concluding Comments

This study highlights instructional factors which appear to contribute toward the development and use of passive learning behaviours in the senior mathematics classroom. When instruction and assessment revolve around the end product of the task, students are encouraged to view learning in terms of 'doing' or 'completing' a task and gear their strategies to that end. Bereiter (1990) labels this adaptation to classroom routines, in which students focus on activities as work, on completion and production rates, as the 'school-work module'. Learning is marginalised and becomes no more than coincidental with school-work.

There were occasions when the teacher, in trying to support the student's learning, effectively reduced the learning demands on the student by doing the student's thinking and processing for them. Such practices included the supply of summaries, the use of keywords, teacher prompting and limited wait-time when questioning, and encouragement of elaborations which focused on recall of procedures rather than concepts. As students persuaded the teacher to be more direct, and to lower the ambiguity and risk in classroom tasks, instruction inadvertently mediated against the development and use of appropriate and effective learning strategies.

Additionally, the assessment structure encouraged the use of learning strategies appropriate for rote memorisation and recall of previously seen examples, providing little incentives for students to use metacognitive monitoring and control strategies. Students' knowledge, or hopes, that the teacher would direct their revision a day or so before each test limited the need to plan or schedule revision. If one believes that learning requires independent thinking assessment should include new tasks requiring new applications of general principles or procedures.

While most students wanted to learn with understanding, and valued cooperation and the sharing of ideas, in reality, they lacked the knowledge and control of a range of appropriate learning strategies. Instruction must not only focus on the need for understanding and learning from errors, but must also model a wide range of learning strategies and provide students with feedback on their use of strategies. Furthermore, instruction that engenders appropriate beliefs about mathematics and learning is needed to ensure that students' strategy selection will be directed at meaningful knowledge construction rather than task completion. Until classroom learning environments encourage students to perceive what they are doing as the construction of knowledge many students will fail to construct appropriate metacognitive knowledge, and will continue to use inappropriate and ineffective learning strategies.

NOTE: This is a shortened version of the paper that won Dr Anthony the 1996 'Practical Implications Award' sponsored by Curtin University.

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Algebraic Thinking Within A Technology-Rich Learning Environment

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Research into individual thinking about algebra faces a significant problem in the tacit nature of knowledge in this domain. This paper documents a research design which incorporated Repertory Grid principles within an image-based research model. Finely detailed study of perceptions of algebraic images offered a powerful complement to the more usual verbal approaches, providing insights into both student thinking about key concepts in algebra, and into the network of relations within which such concepts exist for individuals learning algebra.

Interviewer: . ..This person, towards the end of the lesson, nudges you again ... and says, "What's algebra?" We've had this one before, but how would you explain it to someone who didn't know?
J: Letters?
Interviewer: Then he says, "Letters?" Yeah, that's what we do in English.
J: Um ... numbers and letters?
Interviewer: Okay...
J: Um ... a group of letters that mean something, equal something?
Interviewer: Okay, a group of letters that mean something, equal something ... equal what? Equal numbers?
J: Yeah.
Interviewer: Okay, so a letter like "a" can stand for a number?
J: Yeah.
Interviewer: Okay, can it stand for more than one number?
J: Yes.
Interviewer: So it can stand for what? Two numbers?
J: It can stand for ... I don't know ... anything.
Interviewer: Anything?
J: Any numbers.

The informant in this interview, J, is a female Year 10 student studying the Advanced course in mathematics for the New South Wales School Certificate. The transcript is revealing of the tacit nature of knowledge about algebra. It is clearly disjointed and difficult to articulate for the student and yet, upon probing, is essentially sound in its foundations. The important concept of variable as representing a range of values rather than simply a "placeholder" is present here, suggested at the end of the transcript. It was as a consequence of this recognition of the tacit nature of students' knowledge about algebra that alternative ways of gathering research data were explored, leading to the image-based methods described in this paper.

Student facility with key algebraic concepts such as "function" and "variable," in addition to the more traditional focus upon "equations," is increasingly recognised as central to an understanding of algebra across the years of secondary schooling and beyond, particularly within a technology-rich context (Harel & Dubinsky, 1992; Grouws, 1992; Romberg, Fennema & Carpenter, 1993). Such concepts are mathematically rich, capable of being thought of using not only verbal descriptors but a variety of alternative representations. Studies by Vinner and his colleagues over the past decade (Vinner, 1983, Tall & Vinner, 1989, Vinner & Dreyfuss, 1989) distinguish between a "concept image" ("the set of all the mental pictures associated...with the concept name, together with all the properties characterising them" [Vinner & Dreyfuss, 1989, p. 356]) and a "concept definition" ("a verbal definition that accurately explains the concept in a non-circular way" [Vinner, 1983, p. 293]). Such studies reveal, among other things, that concept images may not always be consistent with the formal definition, but that such inconsistencies are often not apparent. Much of the focus in this area has been upon identifying individual images which students prefer to use when thinking about functions, although the verbal descriptions given as definitions of functions frequently comprise multiple images. This was further supported in Arnold (1992), in which the pattern of representation was quite different when students were asked to describe a function "in their own words." In this situation, functions were most likely to be described as a "rule or relationship", which coincides with the common (non-mathematical) idea of "function", or one of several "multiple-image" definitions, such as an "algebraic object which can be graphed", or a "rule which can be expressed algebraically or graphically" (Arnold, 1992; p. 77). The relative richness of the imagery observed in such studies stands in marked contrast to the impoverished verbal descriptions which are commonly given to both teachers and researchers pursuing student understanding of algebraic concepts. A need was perceived to access this apparently significant source of information regarding the thinking of learners about their mathematics.

Method

The author has recently completed doctoral studies on the use of mathematical software tools by individuals for the learning of algebra. This study of learning to use new tools began as a case study of the teacher/researcher's interactions with a single student (referred to here as S) within a tool-rich algebraic learning context. The encounters occurred within individual tutorial situations over a period of almost two years, with some thirty-six hours of interactions recorded and analysed. As the study progressed, it grew to include five other student informants and two groups of preservice teachers as the cyclic nature of the grounded theory method demanded greater variability within the data, and new research questions and priorities became apparent. This paper describes the responses of three of the participants to a set of algebraic imagery tasks: **S**, a Year 12 student, attempting the high level Three Unit course in mathematics for the New South Wales Higher School Certificate; **T**, a Year 9 student, and **P**, a Year 7 student (who had not commenced his study of algebra at the time of this study).

In order to provide rich context for the study, detailed data were gathered concerning the thinking of participants regarding their perceptions of the nature of algebra and the ways in which it is best learned. As a result of the observations above it was decided to supplement verbal data with student responses to algebraic images, using a modified Repertory Grid (or *RepGrid*) technique, based upon Kelly's (1955) Personal Construct theory. The Repertory Grid was developed as a technique for eliciting, not only components of individuals' thinking about complex concepts, but aspects of the relationships between these components. It has been especially popular as a tool for investigating teacher and student thinking in educational research, offering an attractive blend of data which is both detailed and idiosyncratic in its reflection of individual responses, while at the same time potentially generalisable and amenable to statistical analysis (Solas, 1993; p. 209).

A common format for *RepGrid* analysis involves deriving a series of statements or prompts related to the particular construct in question (e.g. "good teaching"), then presenting these three at a time (randomly selected) and having the participants describe "in what way two of these are alike, and different from the third." This forced discrimination generates a new series of constructs, usually in the participant's own words. Finally, these constructs may be applied back to the original prompts, where the participant commonly uses a five-point Likert-style format to describe the extent to which each of the constructs relates to the original statement.

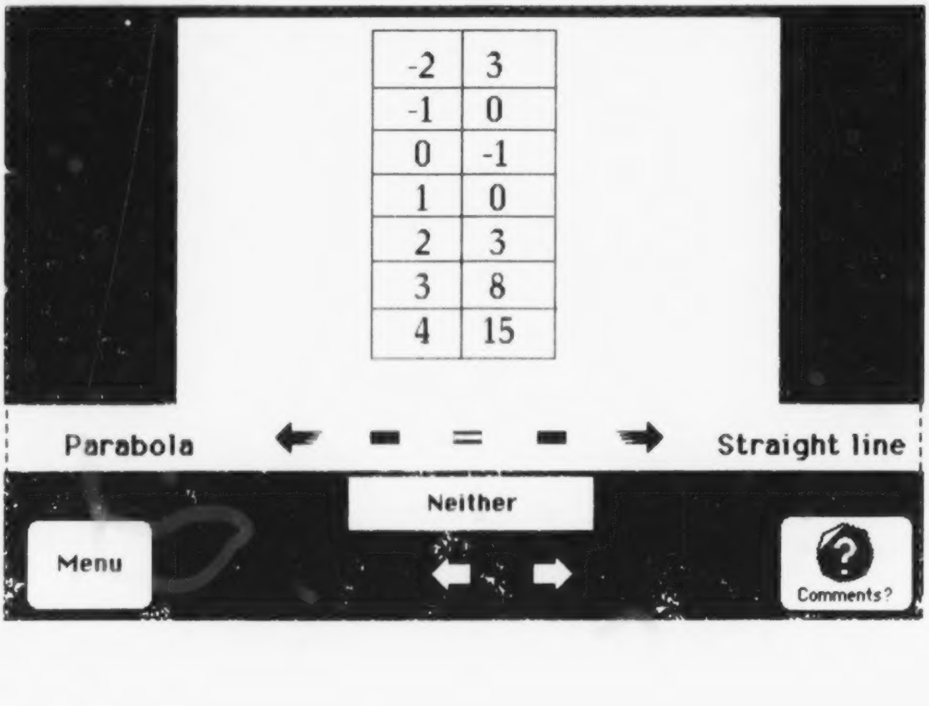
The study described in this paper extended this process to derive data on three distinct levels of increasing complexity. It also made use of the computer as the data-collection device, using *HyperCard* on the *Macintosh*. What are termed here "images of algebra" were elicited through presentation of a series of ten cards, displaying a range of common algebraic visual prompts: the *expression*, $\mathbf{4 - 3x}$, the *equation*, $\mathbf{y = 2x - 1}$, the *graph* of the parabola $\mathbf{y = (x-1)^2}$, the *symbol*, $\mathbf{(x, y)}$, the *symbol*, $\boldsymbol{f(x)}$, the *graph* of the line $\mathbf{y = 2 - x}$, the *table of values* for the relationship $\mathbf{y = x^2 - 1}$, the *expression* $\mathbf{(x - 1)(x+1)}$, the *equation*, $\mathbf{2x - 1 = x + 7}$ and the *graph* of $\mathbf{x = 2}$. Participants responded in three ways:

(1) Respondents were first asked to verbally describe each card, and then to sort them into as many groupings as they could (this may be considered a *first order grouping*).

(2) Participants then engaged in a more detailed discriminatory exercise, in which the ten images were presented three at a time, and they were asked to "choose the odd one out" - to decide in what way one image was different to the other two. This *second order grouping* exercise forced participants to compare and contrast properties of the different images, and so potentially engage in a deeper analysis than the previous sorts.

(3) As a final, in-depth analytical mechanism, participants engaged in a *third order grouping*, a detailed Repertory Grid analysis of the categories which arose from the previous discriminatory activity. Categories identified from the second order grouping were taken in pairs, placed as the end points of a continuum, and then presented with each of the ten original images. For example, after distinguishing between, say, "parabola" and "straight line" in a second-order grouping, a respondent would then be asked to decide the extent to which each of the ten card images displayed these two properties by clicking at points between them (Figure 1). This process attempted to explicitly expose the network of relationships perceived by each individual in their thinking about algebra.

Figure 1: Sample of a *RepGrid* analysis card



-2	3
-1	0
0	-1
1	0
2	3
3	8
4	15

Results

Images were chosen so as to offer the basis for sorts based both upon surface properties (for example, symbol/graph/numbers) and a range of possible other categories, such as functions/non-functions, different representations of the same symbolic form (cards 7 and 8) and even potential errors, such as equating the graph in card 2 with the visually similar symbolic forms of cards 7 and 8. Although the initial verbal/written responses of the participants was revealing of interesting aspects of their thinking, the focus in this paper lies with the three levels of the grouping process.

First Order Groupings.

A useful measure of the complexity of individual thinking about algebra was offered by the first order groupings of the image cards. This process had an immediacy which tended to be revealing of the *signal nature* of the various algebraic forms. Although at times the groupings were idiosyncratic, there were clear patterns of consistency which went beyond the "surface" characteristics of the cards.

S engaged twice in the algebra card sort activity and increased over the intervening period from four categories (*functions, solving equations, straight lines* and *parabolas*) to seven categories (*function, straight line, parabola, equation, coordinates, equation for axis* and *find values for variables*). His first sort was restricted in that he used each card only once, and so sorted them into exclusive categories. He displayed limited cross-representational facility, recognising the equation **y = 2x-1** as a linear graph and the expression **(x-1)(x+1)** as representing a parabola. He also treated the expression, **4 - 3x**, as an "implied equation", capable of solution if "**= 0**" is assumed as a suffix. Functions were included only as symbolic and numerical forms (the table of values implying for S an "input/output" image suggestive of function).

S's increased number of groupings appeared generally consistent but somewhat arbitrary (as in "Equation for axis"), and overall this sort demonstrated little improvement in his cognitive organisation beyond that which was made evident in the first. Although he showed good familiarity with the graphical representation, he appeared unable to interpret the table of values in a meaningful way.

T engaged in three first order sorts over a period of twelve months. Although the number of groupings increased in that period (from three to five), they remained based firmly upon superficial features of the images. In his first sort, **T** distinguished *equations, number plane* and *"things I don't understand."*

Interestingly, **T**'s second sort reduced the number of groups to two: *graphs* and *equations and algebra.* This second sort demonstrated improvement in both the appropriate use of technical terms ("graphs" instead of "number planes") and a clear distinction between what, for **T**, are the two fundamental divisions within algebra: symbols and graphs.

T's third sort (into 5 categories, *graph, equations, coordinates, table* and *expression*) displayed a finer detail and a better grasp of the language of algebra ("coordinates" and correct use of the term "expression") but not necessarily a deeper understanding of the distinctions between the various images. He now had a name for those "things I don't understand" from sort 1, and the expression **4 - 3x** had acquired the signal property of "something to be solved" which led to inclusion with the equations - although, once again, the other expression, **(x-1)(x+1)**, was omitted. **T** had studied expansion of binomials at school by this time, and indicated that he "knew what to do with this one", implying that he saw the expression **(x-1)(x+1)** as something to be expanded. It seems possible that the stronger "expansion" signal served to "swamp" the "equation" signal in this case. Note that he correctly placed the symbol ***f*(x)** as an expression, showing recognition of this form.

Finally, as a novice to algebra, **P**'s sorts would be expected to be based upon superficial cues, since the images possess for him no underlying meaning. His groupings reflect this: *graphical, numbers and letters, mix, loner* and *problems*.

Even at this early stage, **P** recognised that algebraic forms possess a signal nature. His last group, "problems," indicates objects which he saw as requiring some action, although he was unsure of what that action should be. His second sort was much more clearly defined than the first, with three clear and distinct groupings which again reflect the surface features of the algebraic forms: *graphs*, *equals* and *numerals*.

Second Order Groupings

Table 1
Second Order Groupings: "*Pick the odd one out*"

		S (Year 12)	T (Year 9)	P (Year 7)
1	$4-3x$ $y = 2x-1$ Gr ($y=2-x$)	1	1	3
2	Gr (par) $(x-1)(x+1)$ Table	3	2	1
3	(x, y) $2x-1=x+7$ Table	2	2	3
4	$f(x)$ $4-3x$ Gr ($x=2$)	2	2	3
5	$(x-1)(x+1)$ $2x-1=x+7$ $4-3x$	1	1	3
6	Gr ($y=2-x$) Gr (par) Gr ($x=2$)	2	2	3
7	$y = 2x-1$ (x, y) $2x-1 = x+7$	3	2	2
8	$y = 2x-1$ $f(x)$ $2x-1 = x+7$	3	2	2
9	Gr ($y=2-x$) Gr (par) Table	3	1	3
10	$f(x)$ (x, y) $(x-1)(x+1)$	3	3	3

The deliberate comparing and contrasting of algebraic images (picking the "odd one out") offers an added degree of depth to the analysis of participant responses, forcing them to go beyond the often-superficial viewing associated with a verbal description. In particular, respondents who had difficulty in supplying verbal descriptions were provided with a non-verbal means of conveying elements of their thinking about algebra. At the same time, these non-verbal responses were supplemented by comments regarding the choice made, which provided further insight into the reasons for these choices Table 8 summarises the participant responses for this task.

The sorting of the ten images into triads was, in most cases, deliberate rather than random. Each attempted to tease out a distinction which was seen by the researcher as significant, such as that between symbolic and graphical forms (triad 1), expressions and equations (triad 5) and others.

Third Order Groupings

The task which gave rise to the third order groupings was a very time-consuming one, taking up to thirty minutes to complete. For this reason, only the three participants were engaged in this activity - **S** (as the principal informant) and the two junior secondary students, **T** and **P** (whose limited algebraic experience meant that they had been restricted in their access to appropriate language and forms of expression by which their understanding might be examined). The previous tasks in these two cases had furnished limited information regarding their algebraic thinking - it was hoped that this detailed analysis might provide a useful non-verbal vehicle by which their cognitive frameworks could be better assessed.

This task involved three steps:

1. The verbal statements which had accompanied the second order grouping process were examined, and used to give rise to a number of descriptors which appeared to figure prominently in their thinking about the algebraic images. This process of extraction took place in collaboration with the informant, increasing validity for the descriptors (Table 2).

2. The descriptors were entered into the *HyperCard* RepGrid stack and participants again viewed each card individually. This time, however, instead of requiring a verbal descriptor, the descriptors were displayed in pairs, as the ends of a continuum (see Figure 1).

3. Participants chose an appropriate response which situated the given image in relation to the two descriptors.

Table 2:
Descriptors from the second order grouping task

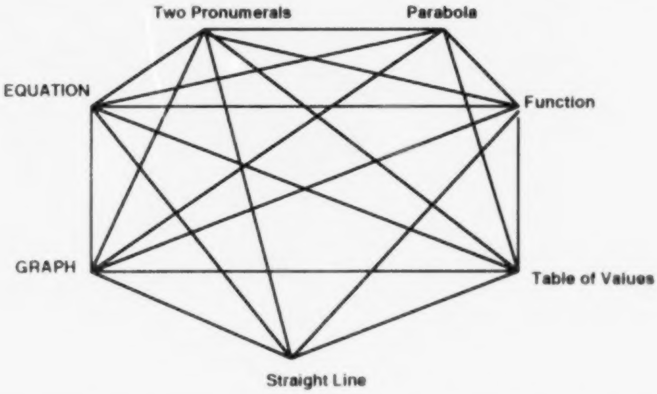
S	**T**	**P**
• two pronumerals	• graph	• numerals
• parabola	• coordinates	• picture
• function	• table	• not sure
• straight line	• x and y	• graph
• graph	• equations	• equals
• table of values	• expansion	• coordinations
• equation	• parabola	• symbol
	• straight line	

The networks of relationships derived from this task proved highly informative regarding the algebraic thinking of the various individuals involved. While the previous sorting tasks had allowed the identification of the various categories by which algebraic objects were conceptualised, this final task allowed these categories to be located within a dimensional space.

Summary diagrams of these concept networks were developed. Heavy lines indicate a strong link between the two descriptors (defined as four or more occasions when these were deliberately related by the respondent). Lighter lines indicate weaker links (less than four occurrences).

S's network of concepts displays well-developed links between all major categories, suggesting good algebraic understanding. His graphical thinking appears to be better developed than that associated with the table of values, with EQUATIONS and TWO PRONUMERALS being most extensively related. **S**'s concept links appear to be of the "all or none" kind, suggesting that he distinguishes less than clearly between them (as in his use of the terms "function" and "equation").

Figure 2: Concept Network for **S**


Two Pronumerals
Parabola
EQUATION
Function
GRAPH
Table of Values
Straight Line

The differences between the concept network for **S** and those for **T** and **P** are immediately clear. While the younger students might have identified as many descriptors, these are poorly developed and associated constructs. Their relationships with other concepts is tenuous at best, illustrative of multistructural understanding at best.

Figure 3: Concept Network for **T**

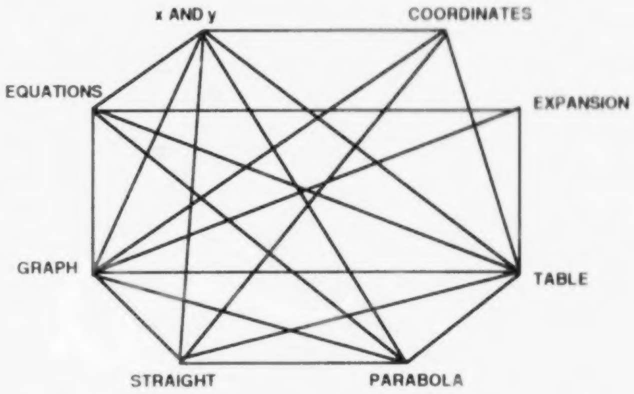
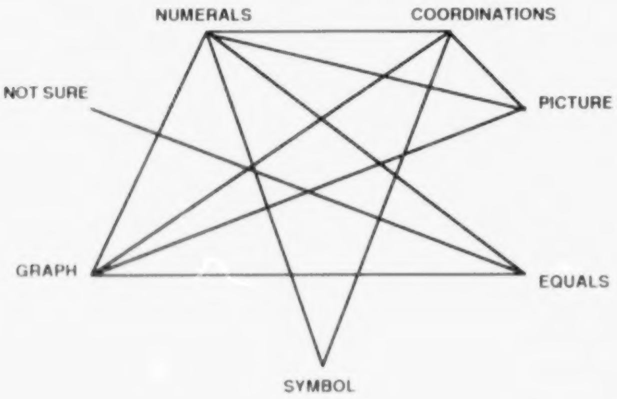

x AND y
COORDINATES
EQUATIONS
EXPANSION
GRAPH
TABLE
STRAIGHT
PARABOLA

Figure 4: Concept Network for **P**


NUMERALS
COORDINATIONS
NOT SURE
PICTURE
GRAPH
EQUALS
SYMBOL

It is hardly surprising to find **P**'s concept network to be even more limited than that of **T**, sure only that algebra involves pictures, graphs, numerals and symbols. Like **T**, repeated descriptors may be recognised: "x and y" and "coordinates" for **T**, "pictures" and "graphs" for **P**. **P**'s few strong links are those between NUMERALS and SYMBOLS, and PICTURES and GRAPHS (demonstrating the symbol nature of algebraic forms). Clearly, for **T**, graphs are more meaningful objects than for **P**, extending even to the recognition of the symbolic connections between equations and expressions and their graphical forms.

Conclusion

The network diagrams provided immediate visual clues as to the cognitive organisation of the participants. They illustrated both the nature and the relative strength of the relationships between the various constructs which made up each individual's cognitive network within the domain of algebra. Across the participants involved in the study, it was now possible to recognise important and detailed features of their algebraic thinking as a result of the intricate examination which has been described. The cognitive profiles which were developed offered valuable insights into the algebraic thinking of these individuals and guidance in the further study of the role of computer tools in the algebra learning process. The matrices which give rise to these networks of relationships are amenable to statistical analysis if desired and potentially the method offers a relatively simple but detailed means of studying the ways in which individuals think about significant mathematical concepts which, all too often, are far more image-laden than they are verbal for learners.

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Future Directions For Studying The Learning And Teaching Of Algebra: Lessons From The Past

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This paper argues for a future direction for studying the learning and teaching of algebra based on collaborative action research between teachers and university researchers. It considers the results of two different types of research undertaken from social and cognitive perspective of the algebra classroom. This research has shown that, regardless of learning, algebra tends to act as a stratification agent for the social reproduction of economic differences, necessitating alternative approaches to existing research.

The 1980s have witnessed a rising interest in research on school algebra culminating in two major publications by the National Council for Teachers of Mathematics (Coxford & Schulte, 1988; Wagner & Kieran, 1989). This research identified major problems in the learning and teaching of algebra (e.g., Booth, 1988; Kieran, 1992). Research in this area has continued into the 1990s (e.g., McGregor & Stacey, 1995) as evidenced by an increase in national and international conferences, competitive grants and publications. There have been new initiatives in instruction (e.g., Quinlan, Low, Sawyer, & White, 1993), but there is little evidence that problems in teaching and learning of algebra have been solved on a large scale. The future of research on, and arguably teaching and learning of, algebra is still open.

From a simple curriculum perspective, algebra could be considered as facing a future similar to computation. Technology, predominantly in the form of calculators, has acted on computation in three ways. First, it has offered itself as a replacement for the algorithms that used to make up the primary computation syllabus. Second, it has provided rapid accurate computation power that has opened up the applications of computation. Third, it has provided powerful teaching aids to assist in the teaching of concepts, principles and thinking strategies that underlie computation. As a consequence to this, the curriculum emphasis in computation is turning from algorithmic procedures to number sense and problem solving. Technology is also providing options for the algorithmic procedures that make up the secondary algebra syllabus (e.g., solving equations, simplifying expressions, drawing graphs, differentiating and integrating functions). It is also opening up new vistas for algebra applications (e.g., the algebra needs of apprentice electronic engineers). It is also providing excellent teaching aids to assist students to discover the concepts and principles behind, e.g., variable, equation and function. These topics appear to offer a rich basis for teaching understanding and the structure of the relationships and operations on these concepts provides a rich field for principles. Thinking of algebra as the language of generalisation seems to complement this approach to algebra and leads to a curriculum emphasis, similar to computation, on algebra sense and applications and modelling. However, the analysis that arrives at this conclusion is artificial in its restriction to curriculum issues and cause-effect argument. It fails to take into account the social consequences of algebra instruction and the proactive nature of learners' cognition.

In this paper, we shall consider the future of research in algebra that is likely to inform effective changes in practice. We argue that in order to discern what type of research is needed in the future, we need to look at what kind of research we have constructed in the past, what questions we have asked and what methodologies we have utilised. Ideas in this paper are grounded in reflection and self-critique about our own involvement in two research projects on algebra as well as other types of research. Our aim is not to present a comprehensive research agenda, but rather to provoke thinking about possible new directions for research that may prove to be beneficial for solving some of the problems in school algebra.

Project One: Social context of algebra teaching

In the first study, we observed four Year 9 classrooms (second year of the secondary school in Queensland) for the duration of a single unit of work which dealt with techniques for solving linear equations (Atweh & Cooper, 1995, Cooper, Atweh, Baturo, & Smith 1993). The classes came from four different private schools with differing socio-economic backgrounds and gender of students. All teachers followed the same chapter from the same textbook.

Our observations and interviews have demonstrated that each of the teachers had constructed certain images of their students' needs and abilities in algebra. These perceptions were in the same direction predicted by previous research findings that formal school subjects are more appropriate for students from high socio-economic backgrounds and for many career aspirations of males. Further, student-teacher interactions in the four classrooms differed in accordance with these constructed perceptions. For example, the emphasis on rigor and formal language was more evident in the high socio-economic boys' school, the emphasis on multiple ways of solving equations and understanding was evidenced in the high socio-economic girls' school, the emphasis on algebra as only accessible to more able students was evidenced in the low socio-economic boys' school, while the low socio-economic girls' school participated in what we have described as a *ritual* of covering the curriculum topic that was not seen as relevant to the current or the future lives of the girls.

In all our publications stemming from this project, we emphasised the significant differences that we observed in the different classes. We also hinted that there were striking similarities in teaching approaches. All classes were quite traditional in their approach to teaching and learning, and dominated by the teacher who solved sample problems and allowed time for student practice of lengthy exercises. A slight exception to this was observed at the high socio-economic girls' schools that utilised group learning to a limited extent. It appeared to us that the differences that we have observed were not related to learning theories adhered to by the different teachers, but to the perceptions that the teachers have developed on the type of students that they have in that school.

This project had its limitations. It did not attempt to investigate why and how these perceptions arose , even less to attempt to change them. Lastly we were never sure that the views that we saw in the four classroom corresponded to those of the teachers involved. We do not believe that the teachers or the students directly benefited from our research.

However, the results of the social context research were in one way unequivocal. Regardless of learning, the algebra acted as a direct agent for social reproduction. The practice of teaching and learning algebra in the low socio-economic girls' class reinforced the position, that they had already accepted, that abstract mathematics such as algebra, and the job opportunities that go with it, was not for them. Like the 'lads' in Willis's study (1977), they built their own classroom subculture as a response to their decision not to take algebra learning seriously. The low socioeconomic boys' learning was much harsher. Their teacher directly used the algebra to separate the class into those who could do it and those that could not. The school's emphasis on abstract learning meant that the only option for most boys was failure. There appeared to be no subculture to maintain self esteem in the face of this failure. For the high socioeconomic boys' class, the algebra reinforced their belief that they were the elite - they were being groomed for professional jobs and that this type of formal and rigorous algebra was for them. The high socioeconomic girls' class was not so straightforward. The teachers' position meant that although the girls learnt that, for them, success in algebra was expected, they also learnt that they could achieve this success in a cooperative manner and that algebra was not the only way they could achieve success in life.

Mathematics as the agent for social stratification is not a new position. Stake and Easley's (1978) research indicated this, as did the research of Anyon (1981). However, it is not a position that curriculum developers seem to have taken into account and it is central to the question of the future of algebra. Algebra may have utility, i.e., be necessary for a variety of vocations, and to improve life chances; it may have elegance as the language of generality; it may be central to the culture of our society and an important component of enculturation. However, in reality, it appears that its major role is to identify the minority who will be offered a chance at structural privilege.

Project Two: Use of manipulative materials and cognitive load

In this study, we observed a Year 8 class and individually interviewed students (7 girls and 14 boys) before and after instruction in linear equations using manipulative materials to develop solutions (Boulton-Lewis, Cooper, Atweh, Pillay, Wilss, & Mutch, 1995) The class was of mixed abilities from a middle socio-economic background school with a reputation for high academic standards and innovative mathematics teaching. The class has completed previous units which used patterns to introduced the concept of variable as generalisation and cups and counters to introduce variable as an unknown number. The unit under consideration used cups and counters to solve linear equations (using the methods of Thompson, 1988, including different coloured cups and counters to represent negative variables and numbers).

Similar to the social context research, the results in this study were unequivocal at one level - the students did not appear able to use the knowledge taught to them about the concrete representations. Only one of the twenty-one students correctly used cups and counters to represent the equation. At the post-interview, no students voluntarily used materials. When directly asked to use them, only four of the twenty-one students could use the materials to generate an answer. When given the choice students preferred a mental approach (based on inverse) which met their needs simply and effectively. As a result most uses of concrete representations were illustrative because mental strategies overrode any knowledge of the generative use of materials. The students did not appear to connect the teacher's material representation to their own mental representations.

The reason for this appears to lie within the processing loads associated with the particular concrete representations used in this class (Boulton-Lewis, 1993) and transferring understandings from arithmetic to algebra (Halford & Boulton-Lewis, 1992). The difficulty for students is that they have to integrate knowledge of laws and relations of arithmetic, knowledge of the mathematical meaning of equals and equation, knowledge of variable, and knowledge of methods to concretely represent variables, numbers and operations by cups and counters to solve the linear equation by materials.

Perhaps the reflections on this study that are relevant to the arguments developed here relate to the context of the research project. The school was carefully selected. We wanted to study the effect of using a concrete approach to algebra in a naturalistic environment. Very few schools in Brisbane use teaching approaches in algebra based on concrete materials. We did not wish to perform a teaching experiment and hence we did not wish to interfere with the normal context of the classroom. Even though the teacher was very eager to get some input for the researchers about his teaching, we were only ready to discuss our observations with him after the study had been completed. At that stage, we faced a great difficulty discussing our observations with him without making him feel that he was being assessed on the success of his teaching. Hence, like project one reported above, this study had its limitations as well. Our own knowledge of some of the problems in teaching algebra may have increased. We are not as confident as to why the teacher decided to implement this particular approach to algebra nor how to empower him to become more reflective practitioner and critical researcher to improve his practice - which he desired. Once again, we doubt that our research had directly benefited the teaching at that school.

Some Characteristics of Past Research on Algebra

The effect of cognitive/psychological constructs to study algebra is well documented. Kieran and Wagner (1989) identified the major factors effecting the teaching of the subject during this century from the task analysis work of Thorndike, to the meaningful learning movement of Brownell, through the developmental theory of Piaget and the more recent constructs of information processing theorists. The authors summarise their short history by observing: "a growing research interest in the factors and processes involved in the learning of school algebra merging from two directions, from cognitive psychologists and from mathematics educators" (p.5).

Kieran and Wagner were reporting on the deliberations of the 1987 conference convened at the University of Georgia as part of the Research Agenda Project which included "mathematicians, mathematics educators, psychologists, technologists, researchers, practitioners, and curriculum developers" (p. 5-6). Presumably "practitioners" meant, or at least included, classroom teachers. No paper presented to the conference was co-authored by school teachers. Likewise, of notable absence are sociological or anthropological perspectives. To be fair, the conference did conclude by raising the need to combine theory and practice and the role of multiple perspectives to study algebra.

> Some of the ideas that emerged from this exchange [during the concluding session of the conference] concerned the two -way relationship between theory and practice and the need to build theories based on the experience of practitioners. It was also suggested that new theory needs to be constructed in order to attempt to tie together the results of research from different traditions and to be able to predict how algebra learning takes place. (p.9)

It seems to us that past researchers in algebra have made certain assumptions about the nature and role of research knowledge and its relation to practice. We argue, that these assumptions are not conducive to the improvement of practice by research findings.

First, using the natural science models of knowledge generation and application, mathematics education seems to have developed the belief that knowledge about algebra learning and teaching should/could best be developed and decontextualised by controlled observations and experimentations by qualified and trained researchers. Further, knowledge developed from different perspectives converges to more comprehensive truths about algebra learning. When this knowledge is verified and validated, it can then be transferred to practitioners by "professional development" and used to improve their practice. Entrenched in the majority of research traditions in mathematics education is the separation of practice of knowledge generation and knowledge utilisation. This separation is graphically represented by Silver in his chapter on the role of research on practice in the 1990 NCTM Yearbook. After challenging the viewing of research as a search for the magic cure for problems of teaching and learning he stated that another metaphor "for thinking about the influence of educational research might be 'osmosis' the general permeation of the field of educational practice by ideas and constructs from the field of educational research and vice versa" (p.1). Silver outlined the main ways in which practitioners can utilise knowledge and methods in educational research. However, in the last paragraph of the chapter Silver did discuss the need for "researcher and practitioners to become collaborators in investigating issues of practical importance for the improvement of teaching and learning of mathematics"(p.9).

Second, behind much of the research in mathematics education is the belief that knowledge in ultimately incremental and convergent. Various studies in algebra education have isolated a number of factors and/or used predetermined constructs to study the teaching and learning of algebra. Viewing the classroom as a holistic complex micro-culture is a new awareness in mathematics education (Bauersfeld, 1992). In Western scientific thinking the study of complexity is believed to be possible by breaking it to small parts, and hence, an expert is seen as one "who knows more and more about less and less". The Gestalt argument that the whole is more than the sum of the parts is often not reflected in educational research. A practitioner, on the other hand, has to function in a complexity. Laboratory knowledge generated by "expert researchers" often has to be balanced by real contexts, resource and structural limitations, teachers' and students' beliefs about the task at hand, and parents' and employers' expectations. Very few research studies have looked at the teaching and learning of algebra from such a complex perspective.

Third, the criteria of objectivity in research findings is being challenged. Much past research in algebra teaching and learning has adopted the stance that the problems that students face in the learning of algebra are due to either mathematical or psychological factors and hence can be overcome by pedagogical changes. Much research seem to concentrate on what the individual student can or cannot do. Students are often said to have not developed understanding even though they were able to demonstrate skill in manipulating algebraic equations to solve familiar problems. Much research reported in the Research agenda includes problems that students have in algebra, mathematical analysis of what algebra is and why is it difficult. Of notable absence from the conference report are discussions of motivation (including values) and algebra as a social construct. Knowledge generated often is a function of the questions raised (Apple, 1981). For example if research does not ask what makes some students successful in pursuing algebra in spite of negative attitudes and/or beliefs towards it, then our knowledge is limited. Evolving paradigms in educational research are raising new possibilities for new questions and methodologies, and are generating new understanding of teaching and learning mathematics. In particular, research from critical theory perspective has raised serious questions about the role of school algebra and everyday life mathematics (Harris, Lave)

In short, past research in algebra can be characterised by being based on the possibility of separation of theory and practice, on the principle that knowledge is additive and by limitation in questions that it raises.

Possible Directions for Research in the Future

The reflection above on our projects and research of others has raised some concern about the gap that exists between knowledge generation and knowledge use in traditional algebra research. The difference between the schools used in the social context project and the school observed for the cognitive study was in their teaching methods. The two studies had demonstrated that this was not sufficient to solve problems of teaching and learning algebra. Both studies have concentrated to look at factors that prevent effective learning. However, because of their design, both studies had limitations in suggesting and empowering change in practice.

We do not claim here to propose a complete research agenda for the future of school algebra. However, we believe that a line of research that may prove useful for improving the teaching and learning of algebra is critical Participatory Action Research (Kemmis & McTaggart, 1988).

> As I see it, action research aims to help people to investigate reality in order to change it (Fals Borda, 1979), and, at the same time, it also aims to help people to change reality in order to investigate it. In particular, action research attempts to help people investigate and change their social and educational realities by changing some of the practices which constitute their lived realities . . . .
>
> Through action research, people can come to understand their social and educational practices more richly, by locating their practices, as concretely and precisely as possible, in the particular material, social and historical circumstances within which their practices were produced, developed, and evolved - so that their real practices become accessible to reflection, discussion and reconstruction as products of past circumstances which are capable of being modified in and for present and future circumstances. While recognising that every practice is transient and evanescent, and that it can only be conceptualised in the inevitably abstract (though comfortingly

> imprecise) terms that language provides, action researchers aim to understand their own particular practices as they emerge in their own particular circumstances, without reducing them to the ghostly status of the general, the abstract, or the ideal - or, perhaps one should say, the unreal. (Kemmis, 1995, pp.)

We believe that through such a collaborative participatory and emancipatory action research approach the following may be possible.

- A question that intrigues us is: with the changes in curriculum emphasis around the world, the advancement of knowledge about algebra, and the technological demands, why is school algebra changing so slowly? An action research approach to study the problems of teaching and learning algebra aims at empowering teachers to critically reflect on their practice and to understand it thus becoming able to change it. Such research rejects the separation between practical and theoretical knowledge.
- The understanding the world in order to change it and changing the world in order to understand it principle advocated here results in the involvement of a more holistic knowledge that is more sensitive to the context in which that knowledge evolves hence more effective in changing the practices involved. It involves questions of cognition, but understands cognition in its wider social, not only personal and psychological, sense.
- Algebra is often used a critical filter for access to the privileges of society particularly to opportunities that do not use algebra or use a different type of algebra to that taught in schools. The approach to research advocated here by necessity has to raise questions of social justice aspects of algebra. Attempts to make the curriculum the same, and to have the same meaning, to all students is not only impossible from a cognitive learning perspective but also undesirable from a social justice agenda that is based on respect for difference rather than the distribution model of equal opportunity.

Such collaborative research takes into account the beliefs and conceptions of students, teachers and researchers (Tait & Boulton-Lewis, 1993). The first study highlighted the different social purposes for teaching algebra, while the second study identified some cognitive processing loads arising from certain teaching strategies. Hence, students resistance to learning algebra could be either to teaching methods or to perceived value of the task. Therefore the proposed approach here teachers, students and researchers may need to articulate their intentions and beliefs in order to understand the practice that they are involved in. In conjunction with the researchers, planned actions may be designed, implemented and evaluated. Perhaps though this process the cycle of disempowerment may be broken.

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The Impact of External Assessment on Teaching Practice: Constraints on Change in the Classroom

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The Victorian "Ripple Effect" study documented the impact of changes in mandated assessment in the final years of schooling on the teaching and assessment of mathematics throughout the secondary school, and showed that assessment can be a powerful catalyst for curriculum reform. This paper reports the final stage of a study which sought to examine the different set of influences on teaching and assessment found in NSW. Interviews with teachers from a broad range of schools explored the constraints which they experience as a consequence of the assessment system, and the extent to which they value, and implement, teaching and assessment practices such as problem solving, investigations, journal writing and self-assessment.

Introduction

It is widely believed that there is a close link between assessment and instruction. Irrespective of whether this link is seen to be desirable or not, the assumption that it exists has motivated attempts internationally to redefine curricula by changing high-stakes assessment. It is not satisfactory, however, to *assume* that a link between assessment and instruction exists; its existence needs to be established by research, and the mechanisms by which assessment is translated into classroom practice identified.

Clarke and Stephens (1996) set out to investigate the purported link between assessment and instruction. During 1993 and 1994, a study was undertaken in Victoria to determine whether the form taken by a "high-stakes" assessment, the Victorian Certificate of Education (VCE), significantly affected classroom instruction at earlier year levels (Clarke & Stephens, 1996; Stephens, Clarke & Pavlou, 1994). The causal link hypothesised between changes to high-stakes mandated assessment and subsequent changes to instruction in earlier year levels has been called the Ripple Effect.

The Victorian study substantiated the Ripple Effect in Years 7 to 10 arising from changed assessment practices in Years 11 and 12. In addition, it provided a more sophisticated characterisation of this Ripple Effect and of the mechanisms by which it occurs. In particular, specific terminology and practices associated with the VCE were found consistently in mathematics instruction at every level of the secondary school. Other practices, such as student self-assessment and the use of student journals, while endorsed by the informed community, did not have the explicit sanction of inclusion in VCE assessment. The uneven occurrence (and widely different interpretations and implementation) of these non-VCE practices in the sample schools was taken as confirmation of the research hypothesis. This study found that teachers are reluctant to embrace new assessment and instructional practices unless these have the endorsement of inclusion in high-stakes assessment. It appears that, where a high-stakes assessment scheme is in place, school policies and practice adapt to accommodate the performances valued in the assessment and to prepare students to engage in such performances.

During 1995, a comparative study was undertaken in New South Wales, where the conjunction of high-stakes assessment in Years 11 and 12 and curriculum advice in Years 7 to 10 is quite different from that in Victoria. Notable differences include: well established (and separate) syllabuses for Years 7–8 and Years 9–10 in NSW; and the presence of a state-wide examination in Year 10. In Victoria, where no comparable syllabus documents were in place, the Ripple Effect study clearly demonstrated a top-down assessment-driven effect on Years 7 to 10. The NSW study examined a different set of influences on teaching and assessment in Years 7 to 10 in the absence of comparable dramatic changes in assessment practices.

The preliminary results of the NSW study (Barnes, Clarke, & Stephens, 1995) were consistent with the hypothesis that mandated external assessment sends strong messages to schools and that the style of assessment and the types of activities assessed play a powerful role in determining which aspects of the mathematics curriculum are valued by teachers. The 1995 questionnaire data established significant differences in the value accorded to "problem solving" and "investigations" by mathematics teachers in Victoria and NSW. While these key terms were employed in the questionnaires for both Victorian and NSW studies, we cannot be sure to what extent they have similar connotations for teachers across the two States. Interviews conducted in the latter half of 1995 sought to determine whether the issue was one of a difference in the valuing of similarly conceived activities or whether these mathematical activities assumed different meanings as a consequence of their different embodiment in high-stakes assessment. This paper reports the consequences of that analysis.

Method

Design of Interview Questions

In both studies, the aim was to determine how external assessment influenced teaching practice at junior secondary level. Because there are two external examinations in NSW, one in Year 10 (the School Certificate Reference Test) and one in Year 12 (the Higher School Certificate (HSC)) both of these were discussed in the interview. Teachers were asked to what extent they saw these examinations as influencing their teaching and assessment practice and how they thought their practice might change if there were no external assessment. In addition, they were asked about a number of specific teaching practices, including some, like problem solving and investigations, which form part of the VCE assessment, and others, such as journal writing and student self-assessment, which are endorsed by the mathematics education community, but do not form part of the mandated assessment in either NSW or Victoria. We sought in the interviews to ascertain the meanings and values which teachers attached to these various practices, and the extent to which they thought these practices might be helpful or otherwise to students preparing for external examinations.

Sample selection

The priority in identifying interviewees was the desirability of obtaining a balance across the school sample and across teacher demographic variables such as gender and length of teaching experience. When completing the questionnaire for Stage 2 of the study, teachers had been asked to indicate their willingness to take part in a short interview, and thirty three had responded positively. From among these, between one and three teachers were selected from each of the twelve schools. As a result, 26 teachers were interviewed during September 1995, either in person or by telephone.

In quoting teacher responses in the results section, each teacher is identified according to length of teaching experience: new (less than 6 years), established (from 6 to 15 years) or veteran (over 15 years) and location and type of school.

Results

Perceived Influence of the Higher School Certificate

The teachers differed in the extent to which they thought the HSC influenced their teaching and assessment practice in Years 7 to 10. As Table 1 shows, the influence of the HSC was stronger for teachers in non-government schools than for their colleagues in government schools.

Teachers who thought that the HSC had only a slight influence on their teaching explained that awareness of the examination made them more content oriented. They mentioned the choice of options in the Year 10 syllabus—for those students likely to undertake the more demanding HSC courses, they selected options which would be good preparation for work in Years 11 and 12. In general, they felt that the HSC influenced their depth of treatment of various topics, because they saw it as important to build a sound foundation for work in Years 11 and 12. This meant emphasising the basics, and perhaps adopting a more theoretical approach than they might otherwise choose. Apart from content issues, they felt that the need to prepare for the HSC made them more examination oriented. Some teachers found it difficult to distinguish between preparation for the HSC and good mathematics teaching. A typical remark was

> Well, what I try and do is basically teach the syllabus, and give them a good grounding in everything, which I consider necessary to do well in the HSC. ... so I suppose ultimately, I'm trying to give them a good grounding in mathematics. (new teacher, metropolitan government school)

Table 1: Perceived influence of the HSC on teaching and assessment in Years 7–10

School type	Strong influence	Slight influence	No influence at all
Government	3	9	3
Non-government	8	3	0

Those teachers who felt that the HSC had a strong influence on their teaching and assessment expressed ideas similar to those above, but indicated a much greater awareness of the importance of the HSC, in many cases right from the beginning of Year 7. In some independent schools, this was very much part of the culture of the school. There was a strong emphasis on training students to perform well in written examinations, but there was also emphasis on developing a deep understanding of mathematics, the ability to reason mathematically, and to express mathematical ideas in written form clearly and correctly. Typical comments from this group included

> The school community, the parents and the students, are so focused on the HSC that everything is focused on it, it's a cultural thing ... right from Year 7. ... Our approach seems to be that this is very real and earnest, that it's important, and the parents are very concerned that the students are moving forward in what they see as traditional maths, so we tend to stay with traditional approaches (veteran teacher metropolitan non-government school)
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> You're very much aware of you're training them, you're training the students, and I always compare it to a sports event, for the students, you're training and training, you put years of training into it. So, yes, especially the Advanced students ... it's preparing for a one-event that is a big event in their lives. ... The way I teach the middle-to-lower ability differs greatly to the way I teach the higher ability students. With the middle-to-lower ability, I'm much more prepared to experiment. ... whereas I'm not prepared to experiment with the higher ability, I think time is too important. (established teacher, rural government school)
>
> I think we have an idea, right from the beginning of Year 7 ... of what it is that we're trying to do in the senior school ... we don't want them to do things by rote, we want them to do things by understanding, and we want them to understand at the highest level that they can. (veteran teacher, metropolitan non-government school)

Perceived Influence of the School Certificate Reference Test

Teachers were next asked about the influence of the Reference Test on their teaching and assessment in Years 7 to 10. The Reference Test is taken by all students in August of Year 10, and is used to moderate school assessments, to determine grades for the award of the Certificate. It consists mainly of multiple-choice items, with a small number of answer-only questions and an even smaller number of free-response questions at the end. Again, opinions about its influence varied, as shown in Table 2.

The four teachers who claimed that the Reference Test had hardly any influence all came from two independent schools, and all had already indicated that they saw the

HSC as the major goal for their students. In these schools all, or nearly all, students continue at school after Year 10, and so the School Certificate has negligible importance as a leaving credential. One teacher said, "The exam itself doesn't suit what we're trying to do. It is a "pick-a-box" exam, and we're trying to train the boys to write coherent answers." (veteran teacher, metropolitan non-government school)

Table 2: Influence of the Reference Test on teaching and assessment in Years 7 to 10

School type	Affects Years 7–10	Affects Years 9–10	No influence
Government	4	12	0
Non-government	2	4	4

For teachers from the other independent schools, and all those from government schools, the Reference Test exerted a strong influence on both teaching and assessment in Years 9 and 10. The main effect on teaching mentioned was the need to complete the syllabus in time for the test. A majority mentioned feeling rushed, being under pressure to complete the course in time. In some cases, this meant teaching a topic in a less-than-optimal way, in order to get through it before the test, and then, perhaps, returning to it, and teaching it "properly" afterwards. Typical comments from these teachers included

> I suppose it cuts out a lot of the activities that we could do, and that would possibly be more interesting for the kids to do, but due to time constraints we can't do because we're forced to condense it down or just teach it in another way so it's done quicker. (new teacher, rural government school)
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> I'm teaching for an exam for the first half of the year, and I find there's a lot of stress trying to get topics completed. ... And then I've come back after the exams and done them properly. (veteran teacher, metropolitan non-government school)

As these comments indicate, many teachers felt constrained by the School Certificate. A few, however, found it helpful: "It's more like a guide and very helpful" (veteran teacher, metropolitan government school) and "[the external examinations] provide a goal for the students." (veteran teacher, metropolitan non-government school)

As for the effect on assessment, all teachers felt it was essential to give students practice in doing multiple-choice questions. In a few schools, this simply meant giving out copies of past Reference Test papers a few weeks before the actual test. Students were encouraged to use these for practice, but were not assessed on them. In all the other schools, the major assessments in Years 9 and 10, and sometimes in Years 7 and 8 also, took the form of examinations modelled on the Reference Test. Teachers felt that, in order to be fair to their students, they had to emphasise preparation for the Reference Test in this way. Several claimed that other schools did it, and their students would be disadvantaged if they did not do so too. Typical comments included

> We make sure that our students always have, in exams, multiple-choice questions to practice, to work on, so that they are accustomed to them and they're not thrown by them and don't treat them as multiple guess! ... And what's more, I'm going to set the Year 7 exam at the end of the year, and I will use some multiple-choice questions from the General School Certificate exam [the "Reference Test" for the lowest level course]. (veteran teacher, metropolitan non-government school)
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> The two major assessments that we have are modelled on the School Certificate in Years 7 to 10, and we have two each year, a mid-year and an end-of-year. (veteran teacher, rural non-government school)
>
> I suppose to a certain extent that you tend to "coach"... I think, to be fair to the students, you have to do that, because I'm quite sure that ... the majority of schools, will be doing the same thing. So we don't want to put them at a disadvantage. (established teacher, rural government school)

Many teachers were not very happy about the situation, but felt forced to go along with it. As one said, "I don't think it's a good teaching method at all, it's just something you have to do."

What If There Were No External Assessment?

As a way of gauging the extent of the influence of the HSC and the Reference Test, teachers were asked "Thinking more about process and teaching style than about content, would you teach differently in Years 7 to 10 if you did not have to bear in mind the requirements of external examinations?"

Six teachers, from both government and non-government schools, said that it would make little difference to what they did. They were satisfied that their current program was effective and met the goals they had established for their students: "you'd still follow some of those procedures, because you know students learn under that system"(established teacher, metropolitan government school). Two other teachers thought that teachers might become slack without the imposed discipline of external examinations, or that the students would lack motivation to work.

> It's good, to keep you honest, and perhaps, I think, stop you taking the easy way out, where … if a kid's having trouble with it, well, maybe it is beyond them. (veteran teacher, metropolitan government school)

> The external examinations keep you honest. I like external examinations. I think they're the only things that are keeping any form of self-discipline within our students. (veteran teacher, metropolitan government school)

The rest of the teachers expressed, in a variety of ways, the idea that removing the constraints of external examinations would free them to be more experimental. It would give them the chance to try more exciting things, to undertake more extension or enrichment work, more "hands-on" practical activities, more investigative activities or group work, or to relate mathematics more to the students' interests. For example

> If we weren't tied to a formal exam, I would hope to do a lot more practical work, a lot more work on estimation, have the sort of ideal maths lab situation, … and to assess the students on what they did in that practical situation, you know, their ability to do practical things, to actually draw a circle with a — good grief! (veteran teacher, metropolitan non-government school)

> You would probably tend to do more interest type activities, more outdoor type activities, that you wouldn't worry about in an exam, and the children might enjoy it more, find it more interesting. (established teacher, rural government school)

> I would like to do more of a thematic approach I suppose … not only in maths, but in other curriculum, other subjects as well. (established teacher, metropolitan non-government school)

The Importance of the Reference Test

To gain some understanding of why so many teachers felt constrained by the requirements of the Reference Test, we asked them how important they thought the test was, and how important the students and their parents perceived it to be.

Answers to this question revealed an interesting contradiction. Most teachers saw the School Certificate as of little relevance nowadays, because the great majority of their students stay on at school and complete the HSC, which then outweighs their School Certificate results. Typical of this attitude was a new teacher in a metropolitan non-government school who said "I don't see it as having any purpose beyond a bit of feedback to the students and their families." Only a few disagreed with this point of view. Some saw the Reference Test results as a guide to standards "an indication as to whether we are being too soft or too hard in the marking of our own assessments" (new teacher, rural government school) or "the only thing that monitors just how we're performing in relation to other schools" (veteran teacher, metropolitan government school). Another thought it was important because "it gives the students the chance to do an external exam" (established teacher, metropolitan government school). In that sense, presumably it helps prepare them for a more important external exam, the HSC.

In spite of the widespread belief that it was not important, however, nearly all the teachers felt it necessary to emphasise the importance of the Reference Test to their students and to put a great deal of effort into preparing them for it. In part, this was simply because of its external status "because it's an external examination, … you can't help but have it influence your teaching" (veteran teacher, metropolitan government school). Other reasons varied from school to school. In some schools, a significant minority of students were likely to enter the workforce without completing the HSC. For such students, the School Certificate results are an important credential. The role of the Reference Test as a moderator means that the grades which can be awarded to these students are determined by the performance of the school as a whole on the test, and therefore it is important for all students to make an effort. Many teachers felt that parents and the local community judged their school's performance by the students' School Certificate results. While this was the case for a government school in a country town, similar views were also expressed by teachers in city schools, and independent schools. "if we didn't get our usual good results … the parents in this school would be, not up in arms, but … I think their concern would become greater" (veteran teacher, metropolitan government school).

One teacher articulated very clearly her awareness of the contradiction in attitudes towards the School Certificate: "Yes, it's strange, isn't it? What's it like? I suppose it's like, maybe, 21st birthday parties. You know, they're not important any more, but we still make a big deal of them. I suppose it's something like that, they used to be important …" (established teacher, rural government school).

Problem Solving and Investigative Activities

Problem solving and extended investigations play an important role in the VCE assessment, so teachers in the parallel Victorian study were asked to articulate their understanding of the terms "problem solving" and "investigations" and the value which they attached to these activities for students in Years 7 to 10 (Clarke & Stephens, 1996; Stephens, Clarke & Pavlou, 1994). To enable comparison with the Victorian sample, the NSW teachers were asked similar questions. They were first asked to describe a problem-solving activity they had used recently with students in Years 7 to 10. If their understanding of "problem solving" had not become clear from the reply, they were then asked to describe the kinds of activities which, in their view, came under the heading of problem solving.

We found great variation in teachers' understanding of what constituted problem solving. At one end of the scale, some thought that any textbook word problem, or any minor change in the statement of a routine question could be called a problem:

> Every question can manage to raise problem solving … just by changing the words you can change the problem for the kids. … Instead of the rote three times two is six, when you say three lots of two, and if you had some aids, say some old cuisenaires or whatever, then it becomes a problem for them, doesn't it?" (veteran teacher, metropolitan government school)

Others saw problem solving as "using their skills from what they've learnt in an unfamiliar situation" (veteran teacher, metropolitan government school). A few teachers mentioned Fermi problems and open-ended questions. Some schools had problem solving task centres for Years 7 and 8. Several teachers spoke of teaching problem-solving strategies, while others doubted the value of this, "when you're solving a problem you're on your own and you're basically bringing out all the things you have in your kit-bag or whatever it is. I think teaching problem solving is a rather dubious idea" (veteran teacher, metropolitan non-government school). It was common for teachers to look for real world applications when selecting problems for less-able students, but to choose more abstract, logical reasoning, or puzzle-type problems for students whom they regarded as having higher ability.

Assessment of problem solving varied greatly, too. In some schools students kept a brief record of their solutions and strategies; in others, solutions were reported orally, either to the teacher or in class. A few teachers regularly placed one or two "problems" on the end of tests, and a number said that they did not formally assess problem solving.

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Teachers were less clear about the meanings which they attached to investigative activities. When asked to describe an investigative activity they had used, or planned to use, with students in Years 7 to 10, many hesitated for a long time, and several mentioned the same activity as they had described in relation to problem solving. One teacher said that she found it hard to differentiate, but simply classed problem solving and several other activities together under the heading of "doing something different" (established teacher, rural government school). Others were stumped, "It's hard to think what to put in that category" (veteran teacher, metropolitan government school).

Several teachers mentioned practical work in geometry (ruler-and-compass constructions, measuring and paper folding to find properties of figures, the area and perimeter of a circle, measuring volumes, using trigonometry to measure heights of objects, outdoor activities to measure areas), statistical investigations involving data collection and graphical representation, and probability experiments. With only a couple of exceptions, teachers appeared to have the idea that investigations involve practical "hands-on" activities. Investigations were commonly seen as a way of introducing a new topic, and only rarely was more than one period devoted to such an activity. Time constraints were the main barrier mentioned. One of the few teachers who saw investigations as requiring a longer period of time, said that he hardly ever used them because of exams and the restricted time available for mathematics.

Teachers seldom asked students to produce written reports of investigations, preferring to obtain feedback from class discussions. They were not explicitly asked about the use of investigations as an assessment tool, but no-one mentioned doing this. Most teachers thought that investigations were important, although few made extensive use of them. Typical comments were "in a way they start to take control of their learning" (veteran teacher, metropolitan government school) "if they're finding things out themselves they tend to remember it better" (new teacher, rural government school) and "a very good motivational thing" (established teacher, metropolitan government school). A few expressed doubts: "Theoretically it sounds great, but I think in practice they don't actually remember much of it. They just remember that ... the day we did this, they spilled the water" (veteran teacher, metropolitan non-government school).

Written Reports

Teachers were asked what sort of activities they might ask students to produce written reports on. Again, there was great divergence of responses. Only one teacher, from a rural government school, mentioned using a draft-redraft process for students' report writing. Seven mentioned projects—either writing a report on a topic in the history of mathematics, or collecting information for a project on personal finance. Eight said that students were asked to write *short* reports of problem solving activities. Four had asked students to write reports of statistical investigations, one asked them to summarise chapters, and six said they did not ask students to produce written reports.

Written comment provided by teachers on such reports was usually brief. Most teachers indicated that they prefer to provide feedback by talking to the student. One commented, "I don't really do it often enough to have much of a set pattern."

Mathematical Journals and Self-assessment

Several teachers had thought about using student mathematical journals, and thought that it would be a good idea, if they had time, but only one had actually done so regularly. This same teacher also used student self-assessment. Apart from this one teacher, none of those interviewed had made regular use of student self-assessment.

Conclusion

The picture in NSW is one of skilled and professional teachers, working hard within the rules prescribed by the education system, to achieve the best outcomes they can for their students. The present form of the HSC examination encourages teachers to emphasise deep understanding of important mathematical concepts, and to encourage correct expression of mathematical ideas. The School Certificate Reference Test creates an interruption to this process, putting pressure on teachers to rush through sections of the syllabus, and to spend time and effort in training students to tackle multiple-choice and short-answer questions. It appears, however, that the NSW system tends to inhibit educational innovation. Teachers are less prepared to experiment with alternative approaches to teaching or alternative forms of assessment, mainly because of pressure of time, but probably also because they do not see a clear pay-off for their students in terms of improved marks in the HSC.

The School Certificate Reference Test is not "high-stakes" assessment in the sense of the Higher School Certificate, but there are still important matters at stake for schools and teachers, which lead them to accord it greater importance than many would wish: their own sense of professionalism, the reputation of the school, and the need for a credential for students who leave school before the HSC. Thus the Reference Test has the effect of shaping the priorities for teaching in Years 9 and 10, except in a few independent schools which can stand back and say to parents, "Our goal is the HSC. We regard the School Certificate as largely irrelevant for our students."

It appears that "problem solving", which has taken on a specific definition for Victorian teachers, is not employed with such consistency of meaning by mathematics teachers in NSW. "Investigation" is not at the moment part of the lexicon of most NSW teachers, and in some cases was assumed to be synonymous with problem solving.

This study suggests that such assessment as the NSW Reference Test may wield curricular power through its mandated, external character, even though it is no longer a truly high-stakes event within the NSW school system. Most teachers acknowledged its lack of significance, whilst simultaneously reporting that they stressed its importance to their students. The importance of the Reference Test was variously associated with its implications for the reputation of the school and with the maintenance of "standards."

The significance of the HSC lies in its character as a timed written examination. The high-stakes nature of the HSC leads teachers and schools to value this form of assessment. Both the Reference Test and the HSC act to inhibit experimentation and innovation in the mathematics curriculum in NSW secondary schools. It was asserted, however, that the HSC examination demanded a depth of understanding that went beyond the replication of procedures learned by rote. In this, the HSC can be taken to represent a vision of "high-quality traditional mathematics." In comparing the Victorian and NSW studies, we see a conflict of paradigms, most clearly evident in relation to the significance accorded to problem solving and investigations. It is the conclusion of this study that these differences in the value systems which govern the professional activity of teachers can be attributed to the form taken by mandated, external assessment.

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Number Comparison Skills of Children With Moderate Intellectual Disabilities

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In this study the authors investigated the ability of children with moderate intellectual disabilities to compare relative numbers and the strategies they used. Ten children, aged 12 to 18 years, with moderate intellectual disabilities were given a sequence of tasks to assess next number determination; understanding of the n+1>n rule; and number comparison skills. The results showed that nine of the children were able to correctly determine the next number in a sequence and had a good understanding of the n+1>n rule tasks, while six of them were successful on the comparison tasks.

Several studies have made significant contributions to the knowledge of how children acquire number understandings (Baroody, 1993, 1992; Becker, 1993; Fuson, 1988, 1992; Gelman, 1993; Gelman & Meck 1992; Michie, 1985; Steffe, 1992; Sophian, 1992, 1995, Sophian, Wood, & Vong, 1995; Wright, 1991; Wynn, 1990, 1992). The results of these studies have provided information to teachers and parents concerning the assessment and teaching of basic number skills and concepts to children. However, the study of how children with intellectual disabilities acquire number skills has been relatively neglected. Baroody and Snyder (1983) and Porter (1993) believe that this neglect is due to the belief that these children are not able to develop numeracy skills, based on reports of their limited capability to acquire basic number skills (Cronwall, 1974; Gelman, 1982; Brown & Deloache, 1978). In this connection, Gelman and Cohen (1988) found that there are qualitative differences between the ways that children with Down Syndrome solve counting tasks, compared with children of normal intelligence. Carr (1995) reported that two thirds of young people with Down Syndrome were only able to recognise numbers and count. However, Baroody (1986, 1988) indicated that these children are capable of acquiring rule-based counting skills, and benefit from a cognitive approach to instruction. His suggestion would be supported by Caycho, Gunn, and Siegal (1991) who reported no significant differences between the counting behaviour of Down syndrome children and normal children of similar mental age.

Baroody and Snyder (1983) found that very few children with moderate mental disabilities were able to make number-neighbour comparisons. As the ability to compare the relative size of numbers is a fundamental number concept, Baroody (1988) suggested rule-based training of number comparison might be effective in promoting retention and transfer of these skills for children with intellectual disabilities. An aspect of relative size is the knowledge that n+1>n, and this has been emphasised by Test, Howell, Burkhart, & Beroth (1993) who used a strategy based on a "one-more-than" technique in teaching money skills to children with intellectual disabilities.

In this study the abilities of children with moderate intellectual disabilities to count number sets and to compare relative number size were assessed and the strategies children used were observed. The extent to which they understood the n+1>n rule and had 'n after' skills were also investigated. This paper presents the preliminary results for ten children; the final sample will comprise thirty children with moderate intellectual disabilities.

Method

Subjects

The sample comprised ten children with moderate intellectual disabilities (seven Down syndrome) aged 12 to 18 years (Mean = 15) from a special school in a northern suburb of Sydney. The study was approved by Macquarie University's "Ethics Review Committee for Human Subjects". The participation and assistance of the School Principal was requested by discussing the importance of the study and written permission letters were obtained from the parents of all children who participated in the study. Intelligence test scores were obtained from school records and the mean intelligence score for the sample measured with the Stanford-Binet test was 40.2 with a range of 36 to 51.

Description of the tasks

Control Tasks: In order to control the effect of linguistic bias (Siegel, 1982) three control tasks were used. In the first task, children's knowledge of the colours used in the comparison tasks was assessed by asking "Show me the blue (or red) dots." (Dots of different colours were glued on separate cards.) The second task was given to ensure that children understood the concepts of "more" and "same". Children were required to select the set with more dots from two sets which were equivalent in length but contained different numbers of dots (Siegel, 1982). In the third task children's understanding of "point to" or "show me" was assessed by asking them to show (or point) to different objects.

Cardinality Task: The "What's-on-this-card?" task of Gelman (1993) was used to evaluate the understanding of cardinality principle. In this task, the experimenter showed the child a card with a picture of one star on it, and asked "What's on this card?" After a response was given ("a star" or "one star") the experimenter responded "That's right, one star." Next a card with two stars was shown and the child was asked "What's on this card?" Additional trials with more items (up to 10 stars) were given.

N+1>N Rule Task: Children's understanding of the n+1>n rule was assessed by showing the child two sets with equal numbers of chips (two, four, six, and eight) in each. After the child had affirmed that the two sets were equal, the experimenter hid each set with a tissue, then added a chip to one of the sets and asked the child, " Which one has more?" The child was required to respond without counting. In each trial the position of the set with more chips was alternated.

N After Task: A modified version of the "n after task" of Baroody and Snyder (1983) was used to examine the skill of next number determination. In the written form of the task the child was given the written sequence to complete (eg., 1,2,3,4,____, or 4,5,6,7,____). However, in order to equate the written and oral conditions, for the oral task, the experimenter asked "What number comes after?" and then gave a sequence of numbers, rather than simply giving a number and asking for the next number, as Baroody and Snyder had done.

Comparison Task: The "Judging Cardinal Equivalence Task" used by Fuson (1988) was modified in this study. Two long, narrow white cardboard strips of equal length, were glued parallel to each other on a 35◊15 cm green cardboard rectangle. One white strip contained a row of red dots, the other a row of blue dots. The strips were placed in front of the child. Five different comparison situations were administered in this task: equal numbers in one to one correspondence; equal numbers of the same length but not in one to one correspondence; equal numbers with one row shorter than the other; each row the same length with one more dot in one row; an additional dot in one row with the "more" row shorter. The number pairs given were: two, two; two, three; three, three; three, four; four, four; four, five; five, five; and five, six.

Because the language of children with moderate intellectual disabilities is limited (Miller, 1988; Bray & Woolnough 1988), the experimenter modified Fuson's instructions. The comparison question was asked as two separate questions. The child was first asked "Do they (the rows) have the same number of dots, or a different number of dots?" If the answer was 'different" the child was asked "Which one has more?" In order to avoid the bias toward responding with the last choice given (Goldstain, 1969, cited in Fuson 1988), the correct answer to the first question ("same" or "different") was always in the initial position. The children were presented with the number comparisons until they gave incorrect responses for three successive trials. The experimenter then returned to the first failed item and gave a verbal prompt, ("You can count them") or if necessary, the experimenter helped the children to count each row and reminded them of the number in each counted set, before asking the next comparison question. The effect of the hint was recorded.

Results

Control tasks

The performance of children in the control tasks showed that all of them knew the colours and the term "point to" while eight children understood the concept of "more", and nine, the concept of "same" and "different".

Cardinality task

Almost all children applied Gelman & Gallistel's (1978) counting principles on the "What's-on-this-card?" task. All children successfully applied the cardinal counting principle and nine of them, the one-one, and stable order principles.

N+1>N Task

In the n+1>n rule, the subjects showed understanding (nine of the ten were correct) for both small and large numbers. While these children did not count the sets five of them mentioned the number in their response, for example; "This one has more, it is five."

N-After Task

Results of the n-after task indicated there was no difference between the written and oral situations, with children achieving high scores in both (all children were correct in the written situation and nine in the oral situation).

Comparison Task

The strategies used for the comparison task are illustrated in Figure 1. The most common successful strategy, used in 36% of the 180 responses, was to count each set. However, many responses (21%) involved a matching and counting by finger strategy. In this strategy children first matched one set on the fingers of one hand, the other set on the fingers of the other hand, then counted the fingers on each hand to compare the numbers. Matching the elements of the two sets by putting them in one to one correspondence was another successful strategy (15% of the responses). Sixteen percent of the strategies were perceptual, based on either length or density. Matching and counting the number of elements in each set was relatively infrequent (4%), and for some responses no identifiable strategy was evident (8%). The strategies used for the comparison task are illustrated in Figure 1.

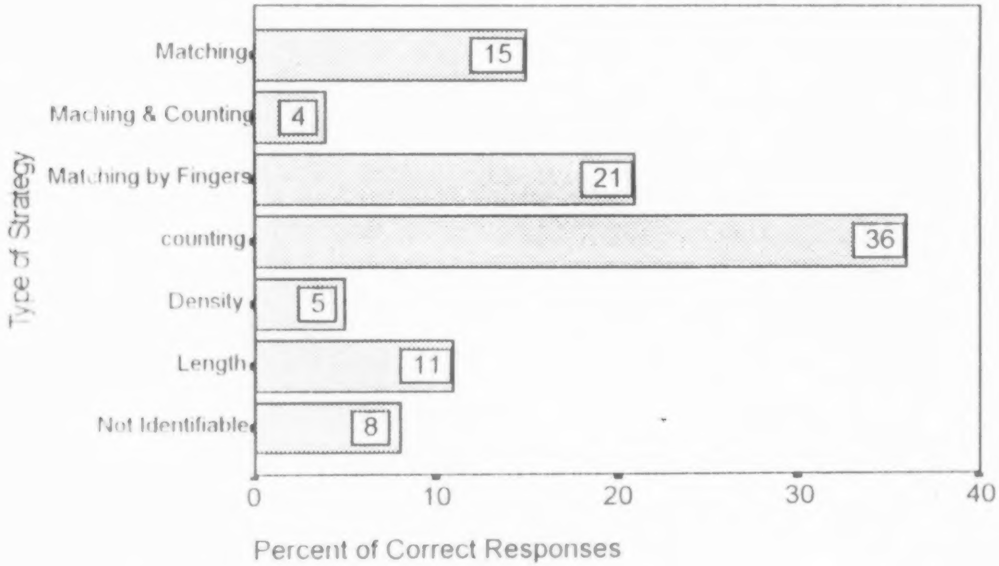

Figure 1: Strategies used on the comparison task.

Of a total of 180 comparison trials, 62% of the responses were correct (45% without a hint and 17% with a hint) and 38% were incorrect. Overall, performance was better for comparisons involving small, rather than large, numbers. Children also gave more correct responses in the trials involving equal numbers in one to one correspondence (seven children were correct) than in trials with the other comparison situations. Table 1 shows that counting was the main strategy used by successful subjects.

Table 1. The performance of the subjects on the comparison task

Subject	Correct response (without hint)	Correct response (with hint)	Not answered or incorrect response	Main strategy
1	18	-	-	counting by fingers
2	16	2	-	counting
3	15	3	-	counting
4	14	4	-	counting & perceptual
5	9	9	-	perceptual
6	10	6	2	counting by fingers
7	-	5	13	not identified
8	1	1	16	matching & perceptual
9	2	-	16	matching
10	0	0	18	

The results presented in Table 1 show that one subject correctly answered all the comparison questions with no hints while one subject had no concept of number comparison. Six of the ten children obtained high scores on the comparison task. The findings showed that giving a hint and then continuing the comparison tasks was most effective for the successful subjects .

Discussion

For children without intellectual disabilities the cardinality principle is acquired before comparison rules (Gelman and Gallistel, 1978). Moreover, Gelman (1993) found that the "What's-on-this-card" task elicited clearer evidence that young children can use verbal counting principles than the other tasks in her study. The results of the present study showed that for these children with moderate intellectual disabilities the cardinality principle was acquired before comparison and that they applied the one-one, stable order, and cardinal count principles in the "what's-on-this-card" task. The findings were also consistent with the results of studies by Baroody (1986) and Caycho et al. (1991) which reported that children with moderate intellectual disabilities have understanding of one-one, stable order, and cardinality principles.

Baroody and Snyder (1983) reported that their subjects were far more successful in determining the next number in a written situation (73%) than in the oral situation (13%). On the contrary, the results of present study showed no difference between the performance of subjects in determining the next number in written and oral situations as nine of the ten children were successful in both. A reason for the differing results might be the different methods of assessing 'n after' skills in the oral situation in the two studies (as mentioned previously).

The results of a study (Baroody and Snyder, 1983) indicated that children with moderate intellectual disabilities had low understanding of the n+1>n rule. Twenty percent of the children in their sample gave correct responses for small numbers and 13% for large numbers. In contrast to their findings, the results of the present study showed that nine subjects understood the n+1>n rule for both small and large numbers. The investigators believe that the difference between the results of these two studies might be a consequence of the way in which children's knowledge of the n+1>n rule was assessed. In this study subjects were not required to count the two sets in order to find out which set had one more. They had to understand the rule that if a single item was added to one of the two equal sets, that set would have more items regardless of the size of the set. Another possible explanation of the high performance of subjects in both the 'n after' and n+1>n tasks might be an increasing emphasis in recent years on teaching mathematics to children with intellectual disabilities (Porter, 1993).

Young and McPherson (1976) found that all of their subjects (children of 4 to 7 years of age with normal intelligence) were able to make correct judgments of relative numerosity when the sets were in one to one correspondence. In the present study seven of the subjects were able to compare relative number judgment in one to one correspondence. A reason for this might be the low intellectual abilities of the subjects. These results are also consistent with Michie's (1984) findings that children made significantly fewer errors when misleading perceptual cues were omitted, or they were assisted to remember before making comparison judgments. Fuson's (1988) claim that when children are reminded of the number of elements in each set counted before the relational questions are asked, children give more correct responses to equivalence tasks was also supported.

Baroody (1986) found that children with moderate intellectual disabilities were more successful in making cardinal sets with their fingers than with other objects and suggested that a reason for this is that children only gradually extend their knowledge of number to objects outside themselves. In the present study use of the strategy in which children used their fingers to match and to count each set in the comparison task was successful for numbers up to five. However, for numbers greater than five this strategy was not successful.

Summary

The results of present study indicated that the children were capable of developing rule-based number skills, such as the n+1>n rule, and of using counting strategies as a means of comparing two sets of numbers. It was also evident that the form of the tasks used to assess number knowledge was crucial for children with moderate intellectual disabilities. Therefore, with well designed teaching strategies children with moderate intellectual disabilities should benefit from an emphasis on cognitive techniques of instruction rather than on rote learning. Such teaching strategies might emphasise the acquisition of number rules to facilitate computational shortcuts (Baroody, 1995).

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Convincing Preservice Teachers about Calculators

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How to convince teachers (other than those who have been involved in calculator projects) of the merits of integrating calculators into primary mathematics programmes is an issue that needs to be addressed. This paper reports effects on the beliefs of a group of preservice teachers who participated in a short but challenging, action-focussed calculator course. The results suggest that preservice teacher beliefs can be changed in a positive direction.

In recent years, several calculator projects (CAN, 1990; Groves and Cheeseman, 1995; Lindale and Biddulph, 1991; Mathematics Education Unit, 1995) have demonstrated the value of calculators in developing children's number ideas. For example, children as young as six and seven years can begin to understand negative and decimal numbers, as well as larger numbers into the hundreds and thousands. Furthermore, teachers involved in such projects have usually come to appreciate the power of calculators to enhance children's learning in mathematics. However, as McChesney (1995, 8) asks,

> Given the overwhelming evidence that appropriate use of calculators can transform the way we teach number in the primary school why are schools and teachers so reluctant to embrace the potential of the calculator?

One likely reason is that the notion that calculators can benefit children's number understanding is counter-intuitive for many teachers and parents. The image of calculator use that springs to mind for these people is that of an electronic number-crunching tool that children are likely to come to rely upon - and thereby become mentally lazy. Visions of dependency, and failing to learn basic number facts, loom large.

Given that recent curriculum documents such as *Mathematics in the New Zealand Curriculum* (Ministry of Education, 1992) assume that calculators will play a central role in primary mathematics education, how can teacher education courses change the understandable mind-set of many practising and preservice teachers? This paper describes one such course and the effects it had on a group of primary student teachers.

Method

The data which inform this paper were collected in 1995 from 24 student primary teachers by means of written reflective statements on a four week calculator component of a 300 level course on Technology in Mathematics Education. They enrolled this course in their third year as an option for their BEd degree.

The Four Week Calculator Component

This comprised two hours per week and consisted of (i) a brief introduction to the research related to calculator use by young children, accompanied by relevant readings, (ii) viewing the Victorian Project video produced by Deakin University entitled *Young Children Using Calculators* (since student teachers in another course the previous year had said that this was an 'eye-opener' for them), (iii) preparing two school mathematics learning sessions incorporating calculators (from calculator resource books provided - see Appendix 1), and (iv) working in schools, for two sessions of approximately an hour each, on two consecutive weeks with a group of approximately three children aged 7-8 years using the prepared calculator learning tasks. Course members had access to sufficient calculators to enable their children to have one each for the learning tasks. The student teachers were expected to collect data on the children's responses to the calculator integrated learning experiences. The in-school, action-research segment was intended to be a miniature version of the calculator projects indicated above.

The Reflective Statement

This was required as part of an assignment on this component of the course. The assignment itself took the form of a research report, with an appropriate literature review, on the school-based segment. Specifically, the students were asked to, "include a reflective statement about the impact of this course and research on your own views about the place of calculators in primary school mathematics programmes."

The data provide some insight into (i) whether the student teachers changed their views, (ii) what their initial views were, (iii) the origin of some of these views, and (iv) what they considered changed their views.

Results and Discussion

Change of Views

Of the 24 student teachers, all appeared to change their ideas about the place of calculators in junior primary school mathematics programmes and now saw them as essential. However, in the case of one student it is doubtful that she is firmly committed to this view. Although she reported that she now considers calculators an important tool for children to use, and that the children's enthusiasm 'rubbed off' onto her, she also remarked that she still believes that children need to learn, understand and know how to perform basic functions through pencil and paper processes.

Initial Views

In the case of four of the student teachers it is not clear what their prior views were but their reflective statements suggest that there was a change in thinking. For example, one wrote, "The calculator unit in particular has shown me just how children's mathematical thinking can be enhanced by using calculators in the classroom." The remainder of the student teachers revealed in their statements what they had thought previously. These initial views can be classified into seven categories.

Naive view: The naive view encompasses the belief that calulators are suitable only for checking answers or quick calculations. Three student teachers held this view initially. For instance, one student reported, "To be completely honest I began this course with a rather naive view of how calculators might be used in classrooms. My ideas for their usage were restricted to checking answers and working things out quickly."

Neutral view: This view was held by two student teachers and largely stemmed from them not having given the matter much previous thought. One of the two wrote, "This unit has really made me stop and realise the importance of employing calculators in primary school mathematics programes. It has been something I have seen little of on teaching practice and, to be honest, I hadn't given it much thought up until now."

Calculators only appropriate for middle school children: This view was held by just one student teacher who observed, "My own ideas about calculators in maths education have changed from believing that calculators are only incorporated into middle school maths."

Basic facts, operations and understanding are needed first: This initial idea was held by five of the student teachers. As one student teacher wrote, "Before undertaking this course, my thoughts on the use of calculators in primary schools were that children should have already grasped an understanding of basic facts before using the calculator." Another commented,

> Research during this course has definitely changed my perspective on the way calculators can be implemented into a mathematics programme. I once held the view that children

> should learn their basic algorithms via the pen and paper method before having the opportunity to use a calculator.

Calculators have a negative effect: The thinking of student teachers here was that calculators can induce intellectual laziness and dependence. Five initially held this view. In the words of one of them, "I must admit that I was one of those people who believed that calculators would mean that children would not learn their basic facts and tables, that they would become lazy as calculators are the easy way out." Another reflected,

> I am afraid to say that I have been forced to radically change my ideas about the role of calculators in the classroom. Previously I believed that using calculators increased children's dependence on technological aides rather than their mental computation and ideas about number.

Calculators represent 'cheating': Two student teachers reported that this was their intial view. For example, one wrote, "Typically, I have held the view that calculators are a form of cheating." It is also possible that this is what a third course participant meant when reporting, "In my own experiences the use of the calculator was taboo up until 3rd form."

Origin of Initial Views

A small number of course members included in their reflective statements data that give some insight into how they came by their intial views about calculator use in primary school mathematics. These views fell into two categories.

Parental view: One student teacher reported that one of her parents influenced her ideas. She wrote, "I recall being told by my mother to 'work it out for yourself' if I asked to use a calculator. Of course initially this gave me the impression that a calculator was a lazy way of doing maths."

Personal school experiences: Six of the student teachers (25% of the course members) were clear that they had developed their initial ideas from personal learning experiences at school. For instance, one course member recalled, "When I attended primary school in the early 1980s I was always made to think that if you used a calculator then it was cheating."

Factors that Changed Views

The data reveal that four factors influenced course members to change their views about the place of calculators in primary mathematics classes.

Introduction of a new curriculum: One student teacher considered that her recent introduction to *Mathematics in the New Zealand Curriculum* (Ministry of Educaiton, 1992) had helped change her attitude that calculators are a means of cheating. She specifically referred to a statement in the curriculum document about providing opportunities for students to develop accuracy, efficiency, and confidence in calculating - mentally, on paper, and with a calculator (Ministry of Education, 1992, p.31). This factor was somewhat surprising because, although I did briefly mention the view about calculators expressed in the curriculum document, I did not give it special emphasis and did not imagine that it would have such an apparently powerful influence.

Viewing the video: In contrast to my anticipated effect of the curriculum document I thought, based on prior experience, that viewing the video would have had a considerable impact on the student teachers' views. Again I was surprised; only two reported that their ideas had been influenced by the video. One wrote, "My attitude towards calculators has changed a great deal from being part of this course, particularly in watching the video presented to us."

Reading the research literature: I always hope that relevant readings provided will have some effect on the thinking of course members but more often than not at the undergraduate level they seem to regard these as part of the 'academic game' one plays when writing course assignments. I was therefore surprised (for a third time) when nine of the student teachers reported that their review of the research literature actually influenced their views about the place of calculators in primary mathematics. For instance, one wrote, "Considering the literature... has somewhat challenged my views of the role of calculators in mathematics education, particularly for younger children. To see the benefits that students can gain from their use is quite enlightening." Another student noted, "I have learnt a great deal about the use of calculators in primary school mathematics. This learning has mainly come from the literature read, and not from the action-research that took place." One of the reasons for this latter comment, she reported, was that she had chosen unsuitable tasks to try with her group of school children.

Working with the children: Apparently the two school sessions on calculator-integrated mathematics activities, developed and taken by the student teachers, had a powerful impact on their views; 22 of the 24 course participants mentioned this. The children's enthusiastic responses, their development of understanding of place value, large, decimal and negative numbers, and their successful efforts at problem solving all convinced most of the student teachers of the value of calculators in junior primary mathematics programmes. For instance, one reflected,

> At this stage of my training I have done many school based lessons similar to this, yet I couldn't help but note the enthusiasm and motivation the children showed in these sessions. I can definitely see the benefit of using calculators right from the age of five.

Another reported, "It was not until I was in the classroom myself with a group of children that my ideas were changed. It was necessary for me to see the children working with the calculators before I could see the relevance they had in maths education." A third wrote,

> I had no idea that calculators had such possibilities for extending pupils' knowledge or that they could be used from such an early age. For example, when some of the students were struggling with the concept of place value the calculator was the ideal tool for developing knowledge. The students seemed to really enjoy the calculator. Maybe it was because they were more in control and everything they needed was in front of them.

Conclusion

The data suggest that the eight hour course component effected a major shift in the thinking of almost all the student teachers with respect to the place of calculators in primary school mathematics programmes. The students' initial range of views come as no surprise, and their comments about their changes of perspective seemed to be genuine. However, it is conceivable that a few of the changed views were constructed to please me and/or gain credit for the assignment of which they formed part.

At first glance the two sessions in school working with a group of children had by far the greatest impact on the student teachers' thinking, but it is probably difficult to separate this element from others in the calculator component of the course. For instance, eight of the nine student teachers who reported that the research literature had influenced their views also mentioned that the school sessions had had an impact as well. The same was true of the two student teachers who mentioned the value of the video in developing their understanding of the potential of calculators for extending mathematics learning, and the one for whom the curriculum document had challenged an existing view.

How can the apparent success of the course be explained? Perhaps humanistic learning theory and control theory is relevant. In terms of humanistic theory (Rogers, 1985) the course provided significant experiential learning that promoted personal involvement. It was both meaningful and pervasive in that the student teachers had to confront their prior ideas about the role of calculators in primary mathematics education. Furthermore their individual views were respected during the course. From a control theory (Glasser, 1986) point of view, the student teachers were largely in control of their own learning as they satisfied their intellectual need to find out how their initial ideas about the value of calculators compared with the research literature and children's actual responses.

Something that this data cannot reveal is whether the student teachers' changed views will translate into classroom practice when the course members become full-time teachers. This is a question which needs to be addressed in further research.

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Appendix 1

Calculator Resources Used for Constructing School Mathematics Activities

Barry, B. and Schmid, V. (1991). *Calculators in the classroom.* Sydney: Harcourt Brace Jovanovich.

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Collective Argumentation in the Primary Mathematics Classroom: Towards a Community of Practice

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This paper explores the nature of Collective Argumentation talk in the primary classroom. Interpersonal, intrapersonal and discursive data collected from three Collective Argumentation classrooms is analysed within a framework which recognises the linguistic, psychological and cultural nature of classroom talk. Findings suggest that Collective Argumentation talk functions to assist students to view the development of mathematical knowledge as occurring within their own community of discourse. Suggestions for employing talk as the basis of a classroom community of practice are provided.

Of recent interest to researchers is the view that students learn within a community of practice, i.e., a community in which participants assist each other to become full participants in the educational practice of the classroom (Forman, Stein, Brown, & Larreamendy-Joerns, 1995). In the domain of History, Seixas (1993) highlighted tensions related to the nature of historical investigation, interpretation and debate that teachers face in carrying knowledge outside of the scholarly community and into the school classroom. Bereiter (1994) in dealing with misconceptions, methodology and authoritative text in science education urged teachers to extend knowledge within the classroom by engaging students in a 'progressive discourse' based on a set of quasi-moral disciplinary commitments. Lampert (1990) argued that 'disciplinary virtues' need to be employed in reconceptualising a classroom mathematical culture where students are assisted in viewing knowledge as emerging from their local 'community of discourse'.

Promoting cognitive development through synthesising scholarly discursive practice and the pragmatic voice of the classroom is a common theme shared by the above authors. Preliminary research reported recently by the present authors (see Brown & Renshaw, 1995) found that positioning Year 6 students' discourse within a Collective Argumentation framework (see Brown, 1994) promoted mathematical understanding and the development of higher mental processes within students.

Collective Argumentation

In the spirit of Mikhail Bakhtin (see Todorov, 1984) Collective Argumentation may be said to be a discursive 'speech genre' within the 'social language' of the academic discipline of mathematics. Incorporating Miller's (1987) principles of cooperation within Vygotsky's (1978) notion of mediated social interaction as the means of facilitating cognitive development, Collective Argumentation gave form and structure to students' collaborative ways of doing and talking about mathematics. A key word structure - Represent, Compare, Explain, Justify, Agree and Validate - created a space within the classroom where personal and disciplinary voices could meet. It also enabled a classroom culture to develop where students' dialogue was influenced by core disciplinary values.

The apparent congruence between the goals of academic discourse and the quality of talk experienced within the Collective Argumentation classroom prompts one to consider such a classroom as a community of practice analogous to the mathematical community of discourse. However, before such a claim can be justified further research into the constitutive elements of Collective Argumentation talk needs to be conducted. It is the aim of this paper to provide insights into what these constitutive elements may be by examining cognitive, interpersonal and discursive evidence taken from three different Collective Argumentation classes over a four year period.

Analysing Classroom Talk

Following Mercer's (1995) paradigm for analysing the quality of talk within a classroom, research into Collective Argumentation talk needs to be focused at three levels.

Firstly, at the linguistic level, classroom talk is examined as text. This analysis is focused at ascertaining what types of 'speech acts' (propositions, explanations, justifications, requests, consensual statements, etc.), 'discourse exchanges' (negotiations, submissions, dominations, frustration's, etc.) and 'topic foci' (content, process, extracurricular, etc.) take place. Secondly, at the psychological level, classroom talk is investigated as thought and action. The focus here is on the dialogic nature of the interaction between participants, that is, the "I"/"other" relationship that is being generated. How is the communicative relationship contributing to students' mutual understanding, group membership, social connectedness and mathematical identity? In other words, what kind of 'ground rules', i.e., implicit norms and expectations (Mercer, 1995), is the interaction displaying and what level of reasoning (concrete, intermediate, abstract) is being made visible?

The third level of analysis is what Mercer (1995) calls the 'cultural' level. The focus here is on those principles of educated discourse that are made evident in classroom talk. For example, the principles of 'reflection', 'generalisability', 'objectivity', 'consistency', 'consensus' and 'recontextualisation' which are core elements of Collective Argumentation. These principles are, to a greater or lesser degree, considered by others such as Bereiter (1994) and Lampert (1990) to be constitutive elements of scientific and mathematical discourse.

Analysing Collective Argumentation Talk

Linguistic Analysis: Interpersonal Plane

At the linguistic level we see the content of Collective Argumentation talk centring on a specific type of speech act which Kruger and Tomasello (1986) refer to as 'transacts'. Transacts are spontaneously produced verbal behaviours that clarify, critique, refine, extend, justify or paraphrase a student's point of view or the point of view of another and may be said to fall into three major categories - statements, questions and responses. In a comparative analysis with conventional collaborative talk employing a correlated *t* test (see Brown & Renshaw, 1995), the content of Collective Argumentation group talk displayed a significantly higher proportion of transacts than did talk generated by conventional groups [33.3% as compared to 23.1%, $t\ (5) = 3.8011$, $p < .006$], with requests for clarification, justification and elaboration of ideas directing the function of group discourse.

The content of Collective Argumentation talk, therefore, resembles what Mercer (1995) refers to as 'exploratory talk', i.e., students' collaborative talk which foregrounds and facilitates the development of reasoning. Requests for clarification, justification and elaboration would also seem to be constitutive elements of talk associated with collaborative inquiry in the Computer Supported Intentional Learning Environments (CSILE) which experts have favourably linked to the scientific community of inquiry (see Hakkarainen, 1995).

Psychological Analysis: Intrapersonal Plane

At the psychological level, two-thirds of independent problem solving protocols of students who had employed Collective Argumentation talk in their groups displayed a mature quality of thinking on a novel problem task, whereas only one-third of the students who had experienced conventional collaborative talk displayed such understanding. Further examination of protocols suggested that the voluntary employment of higher mental functions such as hypothesising, setting and monitoring subgoal achievements and empirically validating outcomes were the features that most distinguished the problem solving efforts of the Collective Argumentation students from the efforts of the conventional collaborative students.

Although no measures of the 'ground rules' that students employ when engaging in Collective Argumentation talk were taken during this particular study, further exploratory research (Brown, 1996) provides a number of insights. A modification of repertory grid technique (Kelly, 1955) was used to investigate the constructs that 10 Year 7 students used to describe doing maths with others (teachers, peers, teacher-aides, etc.) in a Collective Argumentation classroom. The grid was administered and initially analysed on a whole group basis. Table 1 displays the category types generated by the 120 bi-polar constructs elicited from students.

Table 1: Bi-polar constructs by type and as a % of total constructs elicited.

Type	%
Efficacy	30.00
Interactive	22.50
Communication of Meaning	17.50
Competence	16.67
Other	13.33

As can be seen from Table 1, as a group Collective Argumentation students perceived doing maths within the classroom from a perspective that highlighted 'efficacy' (will others help me succeed), 'interactional style' (will others co-operate), the 'communication of meaning' (will an understanding with others be established) and 'competence' (will others have the ability).

These results suggest that students learning within a Collective Argumentation classroom perceive doing maths with others differently to children operating in more conventional classrooms. For example, Stodolsky, Salk and Glaessner (1991) found that fifth grade students' viewed their 'competence' and 'efficacy' in learning mathematics to be framed by 'dependence' (i.e., I can't learn math without teacher instruction). These authors concluded that such views were displayed because these students saw math as an individual performance rather than mastery oriented subject, such as social studies, where the same students evidenced views related to student interaction and elaboration. The fact that Collective Argumentation students viewed maths in terms related to 'communication' and 'interactional style' as well as 'efficacy' and 'competence' would suggest that they view doing math with others (peers, teachers, etc.) as an important feature of mastering mathematical concepts. More specifically, one could say that the quality of communication displayed in classroom interaction was a salient feature of Collective Argumentation talk for these students.

However, suggesting that Collective Argumentation students value quality communication in the classroom provides little insight into the implicit norms and expectations (ground rules) that are necessary for students to take account of in order to successfully participate in such communication. To do this, one must examine classroom discourse itself.

Linguistic and Psychological Analysis: Discursive Plane

A 15 minute segment of a video-taped math lesson in a Collective Argumentation Year 6 classroom was transcribed for analysis. This segment was chosen because it occurred in the Communal Validation part of the lesson where individual students and groups of students interact with the teacher in the presentation and validation of problem solving methods and solutions.

Context and Method: The children had been working in groups representing, comparing, explaining, justifying and agreeing on ideas related to the following problem: 'Jack can clean a room in 10 minutes and Jane can clean the same room in 15 minutes. What fraction of the room will be cleaned in 1 minute if both Jack and Jane work together?' The discourse that followed this group work was analysed according to the framework employed by Forman, et al. (1995) for analysing teacher and student contributions to the function of classroom talk. This framework requires that classroom talk be classified into conversational turns (i.e., student-turn or teacher-turn) with each turn often containing multiple utterances and ending where there is a change in speaker.

Ninety-nine conversational turns were coded according to the following criteria: (a) 'Request-for-answer': a turn in which the speaker asks another speaker to provide specific information without elaboration; (b) 'Request-for-explanation': a turn in which the speaker asks another speaker to elaborate on a statement previously given; (c) 'State-answer': a turn in which the speaker provides a specific answer; (d) 'Explanations': a turn in which justifications or rationales for a point of view are given; (e) 'Restatements': a turn where a speaker repeats what another has spoken; (f) 'Expansions': a turn where a speaker adds to or completes another's statement; (g) 'Rephrasing': a turn where a previous statement is modified but the meaning remains the same; and (h) 'Evaluations': a turn where a speaker makes a statement about the accuracy, conceptual correctness, completeness or relevance of a previous utterance (Forman, et al. 1995).

Results and Discussion: The results of the coding procedure are displayed in Table 2.

Table 2: Conversational turns as a % of total turns by type and agent.

Type	**Teacher %**	**Students %**
Request-for-answer	12	09
Request-for-explanation	05	06
State-answer	00	17
Explanation	00	16
Restatement	04	00
Expansion	04	03
Rephrasing	02	04
Evaluation	03	05
Other	05	05
Total	35	65

As can be seen in Table 2, the students in this Collective Argumentation class generated more conversational turns than did the teacher. They also matched the teacher with requests for answers and requests for explanations. This finding is inconsistent with the Initiation-Response-Evaluation conversational framework that characterises traditional classroom talk, where the teacher mainly asks the questions and provides requests for explanations (Forman, et al. 1995). These results suggest that the students in this Collective Argumentation classroom share control with the teacher of discussion topics and the direction that thought and action take in the discourse of this lesson. They also suggest that the students share with the teacher an interest in the establishment of joint understanding. A striking example of students sharing in the control of the direction of classroom discourse is provided in the following extract.

A group of students (Damien and Lauren) was presenting their thinking about the problem to the class. They had drawn two representations on the blackboard: one organising the problem information into boxes with the problem question occupying a separate box; the other, a drawing of a room complete with bed, window, wardrobe and door. Lauren was presenting her group's ideas by referring to the room diagram:

> Teacher: (To class) Is talking about wardrobes, clothes and beds helping us to work out a solution to the problem? Are there other people who would like to say something? (Expansion)
>
> Student: What fraction of the problem did you get? (Request-for-answer)
>
> Damien: Well we are not working with fractions, sort of. (State-answer)
>
> Student: Why did you set the information in the problem up like that and not use it? Why did you set the information out like that when you didn't use it? (Request-for-answer)
>
> Damien: What? (Request-for-explanation)

> Teacher: Why did you set the information in the problem up like that if you didn't use it? (Restatement)
>
> Damien: Well we did. Well we are. (State-answer)
>
> Teacher: Would anyone like to add to the reasoning process? Damien and Lauren have taken us so far. Is there anyone who can take us the next step forward? The next step forward to a solution. (Expansion)
>
> Greg: You would probably divide the room up into five. (Explanation)

Although it was the teacher who initially cast doubt over the effectiveness of Damien and Lauren's thinking, it was a student who took up this invitation and directed the discussion to the topic of fractions by requesting an answer. Dissatisfied with Damien's answer, again it was a student who directed attention back to the problem information, as represented by Damien, by requesting an answer. The teacher reinforced the direction this discussion was taking by restating the question when Damien sought an explanation. Taking up the teacher's invitation Greg proceeded to provide an explanation of how the room diagram could be viewed in terms of fractions.

An example of students sharing in the teacher's interest in establishing joint understanding is provided in the following extract which came towards the end of the lesson. A group of students (Emma, Pauline and Nicholas) had presented their representation of the problem to the class along with their thinking about the solution method:

> Teacher: Who can understand Emma's and Pauline's and Nicholas's way of doing it? Does it make sense? (Request-for-answer)
>
> Lauren: (To group) So are you saying that together they can clean 5/30's in a minute? (Request-for-answer)
>
> Emma: Yes. (State-answer)
>
> Student: Can you please go over it and explain it right from the beginning? (Request-for-explanation)
>
> Emma: So we divided the room up into 15 parts for Jane because she took 15 minutes to clean the whole room and Jack took..... we divided his into 10 parts and we took 1 minute. (Explanation)
>
> Student: What's that there (pointing to the diagram on the board)? (Request- for-answer)
>
> Teacher: While this group is mapping out the finer points, if you understand the problem can you find someone in the room who doesn't understand the problem and share it with them. (Class breaks up into group discussions with children who understand, explaining to those who don't understand. These discussion groups continue even after the recess bell informs the children that it is time for lunch.)

This extract starts with the teacher emphasising the expectation that the class be able to make sense of explanations. The students take up this obligation by requesting answers and explanations from the group concerning the nature of the solution and the solution method. The teacher lives up to this obligation, not by explaining the solution and the solution method to the class, but by inviting students who understand to share with those who don't understand. The students display their interest in attaining joint understanding by voluntarily continuing their group discussions after the recess bell (a phenomenon rarely observed in conventional classrooms).

In the 15 minute segment, the students provided all the answers which is consistent with the I-R-E framework and all of the explanations which is inconsistent with traditional classroom talk. This suggests that these students view themselves as playing a major role in the development of ideas within the classroom. By providing multiple opportunities for the students to provide answers and explanations the teacher set up the conditions for student ideas to be expressed in increasingly abstract mathematical terms. The following excerpt occurred after the students had been assisted by the teacher to negotiate the meaning of the problem question : 'What fraction of the room will be cleaned in 1 minute if both Jack and Jane work together?'

Joanne: (To group) Could you please explain why you divided (unable to be heard). (Request-for-explanation)
Emma: Yeah, because Jane took 15 minutes to clean the room so we divided the room up into 15 parts and... (Explanation)
Teacher: That will tell you what? (Request-for-answer)
Emma: How many minutes... (State-answer)
Nicholas: She will take 1/15 of the room. (Expansion)
Teacher: She will clean 1/15 of the room in? (Request-for-answer)
Nicholas: One minute. (State-answer)
Teacher: So Jane can clean 1/15 of the room in 1 minute. (Restatement)
Emma: And Jack will clean 1/10 of the room in 1 minute. (Expansion)
Teacher: And Jack can clean 1/10 of the room in 1 minute. (Restatement)
Emma: And so to find out how much that would be altogether, and 1/15 and 1/10 don't add together so we find their closest factor (lowest common denominator) which is 30. So we added them together 2/30 and 3/30 and we got 5/30. That's how much they took 1 minute to clean.... 5/30's of the room. (Explanation)

At Joanne's request, Emma attempts to link the group's representation of the problem (a rectangle divided into 15 parts) with the requirements of the problem question. However, she is unable to see beyond the idea that 15 parts equals 15 minutes. Nicholas provides an expansion on Emma's thinking (She will take 1/15 of the room), but struggles to make the necessary linkage until the teacher rephrases Nicholas's expansion in the form of a request for an answer. Nicholas makes the linkage in his answer and this linkage is restated by the teacher. Emma is then able to make the necessary linkages herself and finishes by explaining the group's solution to the problem in abstract mathematical terms (i.e., 1 minute to clean 5/30 of the room). The class then discusses the answer in terms of equivalent fractions and Emma finally proclaims the group's answer to the problem as being '1/6 in 1 minute'.

As can be seen from Table 2 the teacher provided the majority of restatements, expansions and rephrasings in this segment of the lesson. This finding coupled with the fact that the teacher provided no answers or explanations suggests that the teacher facilitated conceptual development within the classroom by assisting students to reconceptualise their everyday thinking about the problem in the language of mathematics. The teacher expanded, restated and rephrased students' ideas by employing phrases such as 'working out a solution', 'the reasoning process', 'understanding', 'mathematical symbols' and 'sharing ideas'. Such teacher language established the conditions for the emergence of increasingly general mathematical concepts. By making statements such as 'Would anyone like to add to the reasoning process?', 'Does it make sense?', and 'Are there other people who would like to say something?' the teacher invited students to enter the discourse and assisted them to appropriate the disciplinary norms for sharing and evaluating knowledge within the mathematical classroom. The fact that the students engaged in conversational turns classified as expansions and rephrasings is evidence that they were indeed adopting the social norms of mathematical discourse as modelled by the teacher.

It is interesting to note that the students made more evaluations than the teacher. However, teacher evaluations tended to serve a different function. This is illustrated in the following excerpt. A group of students (led by Michelle) had presented their thinking about the problem to the class. This thinking was procedural in nature and centred on summing 10 and 15 and then dividing by 2 (a common strategy used by students when attempting to solve this problem).

Teacher: (Referring the group back to the problem question) What fraction of the room will they both clean if they both work together. Do you think that you have really answered the question? (Evaluation)
Teacher: (To another group of children) Emma does your group want to come out and have a go? (Request-for-answer)
Emma: Yes. (State-answer)

Teacher: (To class) Remember if you want to ask something then ask. (Rephrasing)

Michelle: We would like to revise our answer. Find out..... (Evaluation)

As can be seen in the above excerpt the teacher's evaluation of the group's explanation was phrased in the form of an invitation to Michelle's group to reflect upon the relevance of their thinking rather than in the form of an explicit statement about correctness. This type of evaluation was employed exclusively by the teacher during this segment of the lesson and suggests that the teacher was more interested in having the students conceptualise the problem and apply specific procedures to hypotheses, than in having the students adopt a procedure driven approach to the problem. However, Michelle's evaluation refers explicitly to the answer and a need to revise it. The teacher assisted this group to adopt a different approach to the problem by inviting them to stay at the blackboard and work with Emma's group whose problem solving approach was conceptually driven.

Psychological and Cultural Analysis: Discursive Plane

What are the 'ground rules' that drive Collective Argumentation talk in this classroom? Participants speech indicated that they valued quality talk in social interaction. The above excerpts display that participation in such talk requires from participants (teacher and students) commitments to: (a) shared control of the content of discussion and the direction which thought and action take in discourse; (b) shared establishment of joint understanding; (c) shared agreement about the way that evidence can be brought to bear on ideas; (d) shared willingness to revise ideas when there is good reason to do so; (e) shared interest in progressing knowledge from the everyday to the scientific; (f) shared co-construction of mathematical knowledge; and (g) shared communal evaluation of knowledge. These commitments reflect the quasi-moral commitments that define a mathematical community of discourse as conceptualised by Bereiter (1994) and Lampert (1990).

At the cultural level of analysis, these commitments reflect the principles of educated discourse that invite students to adopt sociocultural ways of reconceptualising the doing of and thinking about mathematics. Through comparing representations, the students were able to *generalise* and *reflect* on their ideas within a context of group membership. Explanations and justifications provided opportunities for students to *objectify* ideas and to incorporate *consistency* into their thinking. Expansions, rephrasings and evaluations provided opportunities for students to attain *consensus* and to attempt *reconceptualisations* within a context that promoted communal attachment.

In summary, the three studies indicate that the principles of Collective Argumentation talk ('reflection', 'generalisability', 'objectivity', 'consistency', 'consensus' and 'recontextualisation) assisted students to display mature thinking when solving a novel problem (Brown & Renshaw,1995), assisted students to value quality talk in social interaction (Brown, 1996) and assisted students in the above classroom discourse to advance their thinking from the everyday to the scientific.

Conclusion

Quality talk in mathematics classrooms need not be a rare event. This paper has shown that the features of Collective Argumentation provide the social context in which students are able to appropriate forms of talking and thinking which are analogous to the broader mathematical community of discourse. Collective Argumentation classrooms, therefore, can be referred to as communities of practice analogous to the mathematical community of discourse. Further research employing linguistic, psychological and cultural levels of analysis along the interpersonal, intrapersonal and discursive planes of operation needs to be conducted to specify the links between the classroom, Collective Argumentation talk and the discourse of the mathematical community of practice.

The insights provided in this paper, however, do offer guidance to teachers interested in transforming their classrooms into communities of practice. Firstly, at the linguistic level, authoritative texts (textbooks and the teacher's voice) need to be viewed as providing a platform for inquiry so that scholarly works may be seen by students as active contributions to the generation of knowledge through metadiscourse. Secondly, at the psychological level, classroom talk needs to reflect scholarly criteria for evaluating discourse if the development of higher forms of conceptual development is to take place. Thirdly, at the cultural level, the social resources of the classroom need to be reconceptualised in terms of assisted participation guided by disciplinary commitments if the search for meaning and understanding is to be facilitated at the local level. Finally, Collective Argumentation offers teachers a pedagogical structure to influence their practice in classrooms so that students may be assisted to view what they learn as having meaning and value in society.

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Teachers' Beliefs and Practice: The Chicken or the Egg?

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The relationship between beliefs and practice is often discussed in research. On the one hand it is perceived that one's beliefs will influence the instructional practice in the classroom. On the other hand professional development focuses on practice, encouraging the reflective practitioner who is prepared to reorganise beliefs as a result of practice. Is this a case of "the chicken and the egg" or does one of the ingredients in this complex recipe carry greater weighting than the other?

This paper explores the relationship between beliefs and practice as perceived by primary classroom teachers implementing a new curriculum document in mathematics. The teachers shared the viewpoint that practice was the ingredient which carried greater weighting. They perceived their beliefs as shifting or being reorganised as a result of practice. They acknowledged, however, that for shifts to take place they did hold one over-riding belief: the belief that educational practice was not fixed and that children's performance could suggest new approaches and ways forward.

Curriculum change is not new to teachers. Over recent years they have come to expect change as an integral component of their profession. In the area of mathematics New Zealand teachers were presented with a new document for implementation in 1992. "Mathematics in the New Zealand Curriculum" (Ministry of Education 1992) emphasises such aspects as:

- posing problems within meaningful contexts
- the use of a variety of approaches
- the use of calculators from Level One upward
- mental calculation
- a variety of assessment methods
- the development of positive attitudes and confidence

The curriculum states that "as new experiences cause students to refine their existing knowledge and ideas, so they construct their new knowledge (p.12)". Mayers and Britt (1995) describe the theory that argues for this view of learning as constructivism and highlight possible implications for the teacher in the classroom.

As the curriculum was introduced into New Zealand school teachers were asked to reflect upon their current practices in teaching mathematics. Many were already using the advocated approaches but others were placed in a position of addressing change. While many of these latter teachers are making shifts in their teaching as they implement the curriculum, others have interpreted written statements in such a way that supports their current practices. In this way they have made little or no change in their approaches or content coverage.

If we accept the curriculum as promoting a constructivist approach to mathematics (Mayers & Britt, 1995) then we must also recognise the possible fear of the unknown faced by some teachers. They do not necessarily understand what is meant by the term "constructivism" and fear that they will not be able to cater for children in their classes pursuing a range of different ideas or approaches. This fear can be caused by a belief that their own mathematics knowledge is insufficient to meet a wider range of demands, or by a desire to retain control over the teaching situation and direction of the lesson.

As part of a longitudinal study exploring the effects of curriculum change in mathematics (Ellis, Pfannkuch, Barton & Thomas, 1994), three primary school teachers were interviewed about their classroom programmes in this curriculum area. Although the focus question was: "What are teachers' classroom practices, and what conditions and teacher beliefs drive these practices?" the teachers interviewed were also asked to present their viewpoints on the relationship between their beliefs and practices and how they worked with other teachers in their school who perhaps held different beliefs or followed different practices. The purpose of the interviews was to clarify direction and content of possible questionnaires to be circulated to primary teachers as part of a larger study.

Beliefs and Practice

What are Some of the Beliefs?

Southwell (1993, p.293) describes a belief as "an idea which, when held, determines the way the individual acts" and acknowledges "there is some evidence to believe that the beliefs which a teacher holds about mathematics affects the way in which mathematics is taught". Van Zoest, Jones and Thornton (1994) see teacher beliefs as belonging to three main categories: "the teacher's view or conception of the nature of mathematics, the teacher's model or view of the nature of mathematics teaching, and the teacher's model or view of the process of learning mathematics". While the survey they have used in their research includes elements related to all three of these categories, other research often places greater emphasis on only one or two of them.

Mayers (1994) expresses the opinion that it is teachers' beliefs about mathematics, rather than mathematics teaching, that have a stronger influence on practice. Frank (1990), however, places emphasis not only on beliefs about mathematics, but also about mathematics learning. She outlines a set of twelve beliefs as identified in a study by Kogelmann and Warren in 1978. Described as "math myths" they are (Frank, 1990, p.11):

> some people have a math mind and some don't; math requires logic, not intuition; you must always know how you got the answer; math requires a good memory; there is a best way to do a math problem; math is done by working intensely until the problem is solved; men are better in math than women; it's always important to get the answer exactly right; mathematicians do problems quickly in their heads; there is a magic key to doing math; math is not creative; and it is bad to count on your fingers.

Frank goes on to state that teachers need to ask themselves which of these myths they hold as beliefs and are conveyed in their classroom practice.

Beliefs Influencing Practice?

At a time of curriculum change the possible impact of beliefs being conveyed into the classroom becomes a key issue for consideration. Much has been written about the influence of teacher beliefs on instructional practice (Thompson, 1992) with relationships between held beliefs and classroom practice being explored.

Examples where beliefs influence practice are described by Southwell and Khamis (1992) who identify similar maths myths to those listed by Frank (1990). They report that teachers who view mathematics as a set of facts and procedures tend to focus on number aspects as they implement any new curriculum in mathematics. They also note that the belief that there is only "one right answer" is strongly opposed to an open-ended problem solving approach in the mathematics classroom, rather the teacher holding this belief is more likely to demonstrate "correct" methods for children to follow.

Howard (1992) highlights curriculum changes which require greater use of calculators and computers in the mathematics classroom. He points out that such changes require many teachers to challenge their beliefs about the importance of learning basic facts and completing pencil and paper computations. This set of beliefs is also viewed as an issue which is often confronted in public forums addressing parental concerns about children's education.

While there is continued discussion on the ways in which beliefs influence practice, studies related to teacher development models focus on ways in which practice can allow a reorganisation of beliefs.

The Reflective Practitioner - Practice Influencing Beliefs?

Malone (1995) supports a reform-minded approach to teacher development which is based on the assumption that if provided with an alternative model teachers may see possibilities for their own practice. The critical factor in this approach is seen to be the need to evaluate and reorganise one's beliefs.

Bush and Kincer (1993, p.323) identify expert teachers as being "more efficient, analytical, flexible, and knowledgeable in their teaching" and support the notion of the reflective practitioner in the classroom. Reflection is seen as a major factor in recognising and perhaps changing one's set of beliefs (Southwell, 1993; Clarke, 1993 and Thompson, 1984). As a sound technique for improving the quality of teaching, reflection should therefore not only focus on the process undertaken by oneself as the teacher but also consider the impact of teaching on the learner.

Studies have been carried out to gauge student-teacher performance following a structured teaching programme which was designed to present a successful model of teaching not necessarily in line with the groups previously held beliefs (Van Zoest, Jones & Thornton, 1994; Mayers, 1994). Findings indicated that exposure to and reflection on changed practice lead to a change in beliefs by the sample groups. It was noted, however, that espoused beliefs were not always immediately in harmony with practice suggesting that time is needed for beliefs to be transferred back into practice.

Clarke (1993) highlights the possibility that teachers may also espouse beliefs that are not in line with actual practice. While some of the factors which inhibit practice may be unavoidable, reflective practice is seen as beneficial in assisting the practitioner to recognise those factors over which they have greater control and therefore potential areas of change.

The Chicken or the Egg?

Thompson (1992)describes the relationship between beliefs and practices as very complex with many factors contributing to its development. Beliefs are viewed as relating not only to a teacher's previous experience but also to their views of control in the teaching process. Beliefs may also be general across a range of curriculum areas rather than being specific to the teaching of mathematics. As beliefs may or may not be consciously held Thompson expresses the need for teachers to face their beliefs.

Thompson has not stated whether it is beliefs which influence practice, or practice which influences beliefs, but such statements as those above tend to suggest a focus on the former relationship. Sullivan (1987), however, notes that student teachers own personal experiences of learning mathematics are known to be the most significant influence on their beliefs about mathematics teaching (cited in Mayers, 1994). While this suggests the relationship in which practice greatly influences beliefs, the example focuses on practice in the form of prior learning rather than teaching.

Other research suggests a more reciprocal or circular relationship between beliefs and practice with each influencing the other. Wood, Cobb and Yackel (cited in Bush & Kincer, 1993, p.323) state that their current work with teachers is based on the assumption that "beliefs and practice are dialectically related ... beliefs are expressed in practice, and problems or surprises encountered in practice give rise to opportunities to reorganise beliefs".

Why do we need to identify the relationship?

Recognition of teacher beliefs is seen as a key factor in teacher education at both pre-service and in-service levels. Southwell (1993, p.293) suggests that:

> If prospective teachers hold certain beliefs about mathematics that mitigate against them teaching mathematics with enthusiasm and real understanding then it is the responsibility of the institution preparing these students for their teaching role to break the cycle which has developed.

Thompson (1984, p.106) expresses a similar need for teacher inservice stating:

> If teachers' characteristic patterns of behaviour are indeed a function of their views, beliefs and preferences ... then any attempt to improve the quality of mathematics teaching must begin with an understanding of the conceptions held by teachers and how these relate to their instructional practice.

Both these statements emphasise the need to establish sets of beliefs which promote effective classroom learning and teaching. This raises the question, however, about how best to establish these beliefs. Do teacher educators focus on the beliefs themselves which can then be transferred to practice, or do they focus on the practice which will influence the beliefs?

Perspectives from Classroom Teachers

The aim of this study was to ascertain perspectives from primary classroom teachers about beliefs and practice as related to their classroom teaching. The data gathered from this study could be used to assist the preparation of questionnaires gathering baseline data as part of the Mathematics Curriculum Change Longitudinal Investigation (Ellis et al, 1994). This latter study aims to identify and understand changes in the New Zealand mathematics curriculum with one area of focus being teacher perceptions.

The Sample Group

Three teachers from Auckland primary schools were interviewed about their classroom practice in mathematics and their opinions about beliefs and practice in relation to their mathematics teaching. The three teachers represent a range of schools which reflects a cross-section of socio-economic areas of Auckland. All three teachers have been teaching for more than fifteen years and are now in senior teaching positions in their schools. Two of the teachers teach in junior classrooms (children 5-7 years of age) and one teaches in the senior part of the primary school (children 8-10 years of age).

Changing Beliefs

While the teachers agreed that they may have come to the teaching profession with a previously-held set of beliefs, they felt these had changed over a period of time. Initially this was seen to be the result of pre-service and in-service instruction and support but an important factor to be taken in to account was that of the confidence gained by positive experiences in teaching. It was on the theme of length of teaching experience, however, where two teachers interviewed highlighted different perspectives. While Teacher N felt that his teaching practice was more open to change

> *"... now that I am more experienced as a classroom teacher and feel confident that I can build on the ideas the children present ..."*

Teacher P pointed out that

> *"... some of the teachers who have been teaching for a long time do not want to put the effort into changing approaches which have worked for them in the past and really need to see the new ideas working before taking them on board."*

An underlying principle identified by all three teachers was that you needed to work at being open in your approach. The following statement by Teacher P was indicative:

> *"...if you do not allow practice to influence belief then you stagnate as you don't try anything new."*

Facing Challenges

Given that the curriculum document outlined content to be covered as well as suggesting possible learning experiences and approaches to present the ideas to children, the three teachers interviewed felt that their beliefs were not the issue. They saw aspects of the suggested practice as sometimes challenging previously held beliefs but necessary in order to realign their beliefs with the demands of the curriculum. The focus of their discussion therefore related to how these challenges were faced.

Teacher D stated:

> *"...with some of the new ideas I like to work with the other teacher at my (class) level. We sit down together and plan out how we will cover the objectives...."*

Teacher P pointed out:

> *"... coming to Dip. Math. Ed. classes and being involved in maths organisations has given me a lot of people to sound ideas out with. I wasn't really confident about maths but now I know that I just need a bit of time to work it out."*

This notion of a support network with school colleagues, professional development or involvement in professional organisations was a continuing theme. Teacher N said:

> *"... the school is really important. You need to know that you have their support. As a staff we have worked together through the issues relating to the new curriculum and we know we have the backing of each other and the principal too. One of the issues we had to face was allowing the children to use calculators and we had parents coming down to school worried that their children wouldn't learn how to add etc. We had talked this all through together so we knew the right (indicating quotation marks with his hands) things to say and we also knew that they would get the same feedback from others as well."*

Teacher P also identified the potential of a school network and discussed the process within her school for parent meetings to go over some of the changes in the various curriculum documents. In this way parents have the opportunity to voice their concerns while being presented with a clear rationale for school decisions that are being implemented.

Influencing Others

Each of the three teachers was asked to move beyond their own classroom and identify any beliefs they perceived to be held by other staff members, particularly those beliefs which appeared to influence curriculum implementation. Within this discussion the teachers identified strategies they had put in place to affect change.

The structure and jargon of the curriculum document was seen to be a valuable asset in providing the opportunity to address needs in other classrooms within the school. Each teacher made reference to approaches by other staff members made to them either in their role as a senior staff member or because of their supposed knowledge of mathematics education. Concerns they identified as being expressed by other staff included:

- difficulties in maintaining control over what was happening if children were left to explore an idea for themselves;
- confusion resulting from a document which they perceived as vague and lacking in direction;
- difficulties in identifying what mathematics (strands) had been covered within a particular unit of work, (this was often related to assessment issues);
- lack of knowledge about some of the content or jargon used (e.g. stem and leaf graphs)
- fear of the words (e.g. "Why do we do algebra in the juniors now?").

The follow-up actions to these approaches were seen as an important factor in modelling practice which could influence a change in beliefs.

Teacher P pointed out:

> *"... it is really important to have the right resource material available. If you leave the old textbooks around then they will be used. I try to show the teachers the new resources ... I always make sure that the first time I suggest they try something for themselves that it is one of those activities that is a sure-fire success. ..."*

Teacher D reiterated the notion that planning together tended to overcome most of the problems, while Teacher N again highlighted the need for a school approach, saying:

> *"... we didn't get involved in the contracts that were happening so as a staff we committed our meeting times to addressing the new syllabus ... we would bring to each meeting those things we didn't understand or just wanted ideas for and everyone pooled their knowledge ... it worked really well - even the more traditional ones on the staff started to try some of the ideas because we would be talking with great enthusiasm about what we had tried in our class ..."*

He went on to describe a scenario where one teacher in the school was opposed to the use of calculators and the children were "used" to influence a change of belief:

> *"... didn't want them in his room and refused to let the children use them at all. There are four of us in the syndicate so we decided to cross-group for a unit of work and the rest of us used calculators as much as possible. The children would go back to ...'s room and during Social Studies or something they would keep asking if they could go next door to borrow the calculators. I suppose it could have backfired but he could hear them saying things like it will be about so-much and he realised that they were still thinking for themselves even with the calculator. He is still thinking it through but he lets them use them a lot more than before..."*

Discussion

The main focus - beliefs or practice?

The initial purpose of the interviews was to explore further the ways in which teachers' beliefs influence their classroom practice. Although it was expected that teachers would focus on such aspects as the use of calculators and the learning of number facts this did not prove to be the case. It soon became apparent that the focus of each teacher's discussion became the practice itself. This could be partly explained by their statements about how they saw the curriculum change affecting their classroom practice, as each teacher highlighted a change in teaching approach rather than changes in content.

The focus of these teachers was clearly on how their mathematics teaching was influencing their beliefs. This differs from much of the literature in two ways. Firstly, the emphasis is on their beliefs about mathematics teaching rather than mathematics itself as suggested by Mayers (1994). Secondly, they clearly defined the practice as influencing beliefs rather than the inverse scenario described by many (Frank, 1990; Howard, 1992; Southwell and Khamis, 1993).

While the teachers were open to change and believed that they could learn from their classroom experiences or practice, they did acknowledge that there may be aspects of their teaching which have been influenced by beliefs that they are unaware they hold. This would support the suggestion that teachers need to be made aware of possible beliefs in order to evaluate and reorganise these beliefs (Clarke, 1993; Frank, 1990; Malone, 1995).

The Teachers' Beliefs

In discussing aspects of the maths myths suggested by Frank (1990) or Southwell and Khamis (1992), the teachers shared the opinion that professional development for teachers both within their own school support structures and from external providers was addressing many of these issues. They agreed that the introduction of a new curriculum had resulted in increased dialogue among staff which, together with sharing of practical experiences, had produced changes in both practice and accompanying beliefs.

Confidence was seen as a key factor in allowing a teacher to focus on their practice and to try out new ideas without fear. While Southwell (1993) would like to see institutions responsible for teacher education taking a lead in breaking possible belief and practice cycles, it becomes apparent that it is only possible to lay a foundation at this stage and that as teachers build their own experiences, so too will they have the confidence to question more openly their beliefs and practice. In this way teachers will adopt a strengthening role as reflective practitioners (Bush and Kincer, 1993).

The Author's Viewpoint - In Summary

Although they were presented with the notion that their beliefs about mathematics and mathematics teaching were likely to influence their practice, the teachers interviewed tended to reject this idea. Rather, they saw effective practice as influencing beliefs and attitudes within their classrooms. They agreed, however, that in this relationship they held an underlying belief which was that:

> *"education was not so closely defined that you could not learn new ideas and approaches from the children you taught."*
>
> *Teacher N*

In this they identified a need for teachers to be open enough to allow their beliefs to be challenged, or confirmed in some instances.

It appears difficult to completely separate one's beliefs and practice, indeed, it may be questioned whether there is any reason for a separation. The complex relationship between beliefs and practice is confirmed in both the literature and the sample of interviews with classroom teachers. If the set of beliefs held by a teacher has been influenced by some form of practice in the past, then research should be focusing more on the practices which give rise to the beliefs than on the beliefs themselves. Is it possible that the focus of a research question on beliefs influencing practice can in reality result in a report which disregards or underestimates the inverse relationship?

If pre-service and in-service approaches are to be effective in promoting practice which supports the new curriculum it is critical to clarify the relationship between beliefs and practice. If one element of this relationship is found to be of greater influence than the other, then professional development work could be focused in that direction. The teachers interviewed suggested that their teaching experience and confidence in their ability to control the teaching situation influenced the extent to which they were prepared to try new ideas and approaches during their teaching career. If this is accepted then in-school professional development will need to recognise and cater for individual needs, providing appropriate guidance and support relative to the individual staff members involved.

The small sample of interviews reported in this paper suggests that the focus of future research should be on practice. It is recognised that a larger sample is required before such conclusions can be confirmed. If beliefs are accepted as greatly influencing practice, then any facilitated pre-service or in-service work would then reflect the beliefs of the person in the facilitation role. Similarly approaches and questions forming a further study in this area will reflect the beliefs of the researcher - a real chicken and egg problem to ponder.

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Teaching Primary Maths
What's Important?
Says Who ?

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Assumptions are made about what primary school teachers need to make them better mathematics teachers. Professional development courses and conferences are held based on these assumptions? Who decides what teachers need? What do teachers want for professional development in teaching primary mathematics? Do they know what's good for them? This paper presents preliminary results of a study of primary school teachers' views of what's important in teaching maths and outlines the inherent difficulties. Teachers views of the influences on their teaching and their ideas of the perfect professional development for maths teaching are also included.

The Mathematics Curriculum and Standards Framework (1995) takes the guesswork out of deciding what's important in teaching mathematics for teachers in Victorian schools. The Mathematics Curriculum and Standards Framework's (CSF's) list of desirable features in mathematics teaching includes things such as teaching from a base of concrete experience, recognising mathematics as concrete and general and using a variety of modes of classroom activity. Teachers are urge to incorporate approaches that ensure equity and equal opportunity for all students regardless of gender, ethnicity or socioeconomic factors. The CSF recognises that individual differences between children should be appreciated and catered for and urges teachers to allow children time to develop the capacity to link mathematical ideas together, to see connections and to use ideas (CSF, p 11-14). The importance of language, integration of mathematics with other subject areas and problem solving in mathematics is reinforced by the fact that almost an entire strand, tools and procedures, is devoted to them. The CSF reflects views that are reinforced in teacher training institutions, professional development courses and conferences, that changes in mathematics teaching are unavoidable with the advent of calculators and computer software. These have forced a major re-evaluation of school mathematics curricula in terms of content and strategies for teaching and learning mathematics.' (CSF, p.14).

Victorian teachers are currently coming to grips with the prescriptives of the mathematics CSF. While some aspects of the CSF are contentious, many mathematics educators would agree with the list of characteristics of a successful mathematics program. Examination of the CSF provides an indication of what is generally agreed to be important in teaching mathematics.

What do primary school teachers think is important in teaching mathematics? What do they find difficult? How would they design professional development when given the opportunity? This paper is a preliminary analysis of research that asks primary school teachers for their views of what is important in teaching primary mathematics, what are the difficulties in teaching mathematics to primary school children and what helps in professional development? The research also asks teachers how they feel about themselves and addresses ideas of change in teachers' views by asking them about significant agents of change in their teaching. This research is part of a larger study.

Method

One hundred and eight teachers working in primary schools were surveyed. The sample included 88 teachers working at 13 suburban schools in Melbourne and a group of 20 teachers who were enrolled in graduate diplomas in Education at Royal Melbourne Institute of Technology (RMIT). The teachers' experience ranged from first year out to thirty five years teaching with an average of fourteen years in the classroom. [The RMIT sample was included after the first 88 surveys were returned as they represented a much younger, less experienced group, while many of the teachers at suburban schools had been teaching for many years, and it was felt that the younger teachers were not well represented in the original sample.]

The survey included ten questions of a biographical nature, seven written response questions and thirty questions requiring responses on a five point Likert-type scale. The research reported on in this paper is an analysis of the written responses to the following five survey questions:

> What do you consider to be most important when teaching maths to primary school children?
>
> What is the most difficult aspect of teaching maths in primary school for you?
>
> How do you feel about yourself as a teacher of mathematics?
>
> Which things, people or events have been most influential in your teaching of mathematics?
>
> If you could design the perfect professional development for teaching mathematics in primary school, what would it be like?

The data were analysed by compiling all the responses to each question. These were then analysed to reveal patterns or themes in the data, using Nudist (1994). This method of analysis has been derived from the approach described by Marton (1993) in which the data is treated as a whole and patterns identified.

To provide an indication of the strengths of themes in the data percentages have often been used in the results. The percentages represent the proportion of the one hundred and eight teachers who mentioned a particular theme in their response. As any one response may contain references to a number of themes, the total percentage of respondents to any one question is often much greater than one hundred. For example, respondent one's responses to the question 'what do you consider to be most important when teaching maths to primary school children' were as follows: 'use of concrete materials, allowing children to solve problems in their own way, discussion and explanations in small groups after to clarify their thinking, real life experiences'. This response referred to four different ideas; concrete materials, solving problems, using language and real life experiences that would be sorted into four different categories.

Results

What Do You Consider To Be Most Important When Teaching Maths To Primary School Children?

The most important factors in teaching mathematics were developing knowledge and understanding of mathematical skills and concepts, generating enthusiasm for and confidence in mathematics by making maths fun and enjoyable, using concrete materials, and making mathematics meaningful for children. Other aspects that were important were solving problems, the use of language and providing adequate individual attention.

Knowledge And Understanding Of Skills And Concepts: Sixty percent of the teachers used the words 'knowledge' or 'understanding' when answering this question. They stated that it was important for children to have 'appropriate', 'sound', 'basic', 'clear', understanding and knowledge. Two thirds of the sample referred to the importance of skills and concepts, for example, 'a good understanding of the four operations', 'firm foundations in all areas of maths' and 'tables and automatic responses are extremely important'. Knowledge and understanding were also said to be important for teachers, who need an understanding and knowledge of the whole primary course, the strand they are teaching, of individual differences in children, of how children learn and of where children are at.

Concrete Materials: Concrete materials and hands on activity were said to be important, by 40% of participants, in practical work at all levels and for developing a good understanding of skills and concepts by providing a bridge from the real to abstract.

The teachers indicated that lots of hands on experience was required to introduce new concepts, for modelling mathematical ideas and for reinforcement.

Enjoyment and Enthusiasm: Affective factors, like keeping children interested, fostering enthusiasm and enjoyment of maths by offering challenges to keep children interested and using fun activities were included in responses to this question. Thirty seven percent of responses referred to the importance of things like developing a love of mathematics and confidence in children about their ability to do mathematics.

Real Life Maths: Making mathematics relevant to children was a part of the responses of thirty six percent of participants, who agreed that linking mathematics with real concepts and children's everyday lives was a way of making mathematics meaningful to children.

Other Factors: Only seven percent of teachers mentioned the importance of problem solving and most of those don't discuss what makes problem solving important although one teacher suggested that it extends children's abilities. Language was mentioned in *only* 6% of the responses as a way of clarifying ideas, explaining strategies and linking with other language activities. Catering for individual differences arose in 7% of the responses.

What Is The Most Difficult Aspect Of Teaching Maths In Primary School For You?

These teachers found it difficult to cater for the range of abilities of the children in their classes. This was made more complex by composite classes and multiage teaching. Difficulties presented by lack of materials and resources for teaching were mentioned by many teachers as a major problem for them. However the most significant difficulty identified by the teachers, because it impinges upon all of the other difficulties mentioned, was a lack of time.

Time: By far the most frequently mentioned difficulty for the teachers was finding the time to cover the content and scope of mathematics thoroughly, finding the time to organise practical activities, time to teach as effectively as they would like, time to assist individuals and time to enable children to work at their own pace. Thirty eight percent of the teachers referred to a lack of time as the most difficult aspect of teaching maths in primary school. They needed more time to plan, to teach and to try new ideas.

Catering For Individual Differences: Although only 7% of the sample nominated catering for individuals in their most important list, the difficulties associated with catering for individual children in the classroom were referred to by 30% of the sample. Concerns ranged from children at lower levels who hadn't covered the course to extending gifted children. Teachers were challenged by planning programs to cater for the wide range of abilities to be found in classrooms.

Resources And Concrete Materials: Twenty percent of the participants stated that resources and materials presented difficulties. The problems included availability of interesting materials, lack of and access to equipment, organising and storing equipment.

Other Difficulties: Other difficulties mentioned were problems associated with large classes (9%) and attitude factors (6%) such as negative attitudes, hostility towards the subject, and making maths enjoyable .

How Do You Feel About Yourself As A Teacher Of Mathematics?

The responses to this question were sorted into five groups. It can be seen from Table 1 that the majority of respondents (52%) are in the first two categories; 'good' and 'good but...'. All of these teachers used terms like good, confident, comfortable or enjoy to describe their feelings about themselves as teachers of mathematics. Forty one percent of the teachers seemed to be content with themselves as teachers of mathematics. The category 'good but...' arose because a subgroup of the teachers (11%) who considered themselves to be 'good' teachers of mathematics also added a qualification to their statements with the use of words like '...but (I) get frustrated sometimes', '...but I adopt a fairly traditional approach that suits me' and '...but there's always room for improvement through PD'. These qualifications were significant because they indicated that these people were not content just to be good. They were continually evaluating themselves and seeking improvements in their teaching.

Good or enjoy	41.6%
Good but...	11.1%
Feeling better - Improving	19.4%
Adequate	2.8%
Could be better	10.2%

Table 1: Responses to *How do you feel about yourself as a teacher of mathematics?*

Nineteen percent of the teachers described themselves as feeling better or *improving* as maths teachers. Many of them spoke of improving in confidence over time. Some mentioned the influence of new programs, books and inservice programs. Only 3% of people felt they just managed or were just adequate as maths teachers while 10% felt that they could be better teachers of mathematics. Their comments included:

> ...I would certainly be inadequate if I had to teach maths above grade 4 level...
>
> ...I feel it is not my strength but I enjoy integrating maths into my unit and making it meaningful. I am worried how my knowledge will be if I have to take a senior school class...
>
> ...I believe that I don't address the maths need of the each child in the group equally. There are too many invisible children who do things to an average or minimalist level. I need to motivate myself and the children to think more mathematically ..

Which Things, People Or Events Have Been Most Influential In Your Teaching Of Mathematics?

Planned professional development courses and conferences were most influential in improvement for these teachers. Colleagues - teachers working together - sharing and learning from each other, also rated as significant influences. Teachers' histories, arose also, in the descriptions of influences. Almost one third of the teachers mentioned their own school experiences were important, while a quarter of the sample referred to teacher training experiences.

Professional Development Courses And Conferences: Professional development programs, inservice courses and Maths Association of Victoria conferences were said to be influential in the teaching of mathematics for 43% of respondents. In this question 23% of the sample referred specifically to the EMIC programs (an ongoing professional development package that involved reading research papers, discussing current approaches, sharing of good ideas by teachers and the trialing of material in classrooms) that were conducted in Victorian schools in the early 1990's. Twenty two prominent presenters of professional development were named as being particularly influential.

School Experiences: Personal experiences as school students were mentioned by 32% of the teachers as being most influential. Of these 51% recounted positive experiences at school, while 31.3% recalled negative school experiences. The rest of the participants (17.4%) described experiences that were either both positive and negative (I was good at primary school but fair in secondary' or it was not possible to tell ('secondary maths teacher').

Those whose recollections of school experiences were positive spoke of their own enjoyment and love of learning maths in school and of teachers who allowed them to explore first, developed their understanding and who showed patience. Recollections of negative school experiences were of feelings of inadequacy, fear of being thought 'dumb' and low self esteem.

Colleagues: Working with colleagues in various ways was said to be influential for 29% of the participants. This category includes teaching colleagues who passed on hints, shared ideas and were involved in team planning. Some teachers mentioned colleagues who had provided them with effective modelling of good practice and others spoke of the benefits of working with experienced teachers and the positive effect of encouragement from other teachers.

Teacher Training Experiences: One quarter of respondents mentioned teacher training experiences as being influential in their teaching. Three people felt their teacher training had a negative influence, while the majority in this group stated that their teacher training had influenced their teaching in positive ways such as development of understanding of concepts that previously had been learned by rote. Many teachers referred to a particular lecturer who had made a difference.

Curriculum Materials: Text book series like Eureka, Young Australia Maths, Rigby, MacMillan and other books which provided information and activities involving current approaches, as well as curriculum support materials and the Curriculum and Standards Framework were regarded as influential by 19% of the teachers

Other Influences: Learning from children, about how they think and learn, was an important influence for 9% of the teachers. Seven percent named professional reading as most significant for them and four percent wrote of the importance of their fathers as positive influences on their maths learning. Others described the influence of experimenting with various teaching techniques and being involved with the calculator project.

If You Could Design The Perfect Professional Development For Teaching Mathematics In Primary School, What Would It Be Like?

Given the emphasis that these teachers have put on planned professional development (PD) courses and conferences as influences, their views on the perfect professional development will be relevant when planning for future change. These teachers, for whom time is of the essence wanted practical, relevant PD that is ongoing and provides them with activities and ideas that they can readily use. They wanted help with planning and programming, particularly concerning catering for children's individual needs and are interested in trying out ideas that have worked for other teachers.

Activities And Ideas: Thirty seven percent of the teachers mentioned that professional development for teaching maths in primary school needed to provide maths activities and ideas. Many (21%) used the words practical or relevant in answering this question. These words were usually coupled with fun, interesting, activities or ideas. Activities in the ideal professional development course should be hands on, previously trialed by teachers and readily implemented.

Programming And Planning: Programming, sequential planning that ensures 'proper' development of all aspects of mathematics and assessment, with support from other teachers working at the same level would be included in the perfect PD program designed by 25% of these teachers. The ideal PD would be ongoing, for 13% of the respondents, rather than one off sessions. An ongoing program, like EMIC, they felt, provides opportunities to trial ideas, share resources and keep teachers in touch with current issues.

Other Approaches: Twelve percent of the teachers wanted PD to provide them with information on what to teach and how to teach it based on current research on how children learn. Sharing 'what works for me' between teachers was mentioned in 9% of the responses as being a worthwhile aspect of professional development, while seven of the teachers (6%) thought that observing other teachers who were highly skilled would be of use in professional development.

Discussion

How do the teachers' views of teaching mathematics in primary schools compare with the view presented by curriculum developers, in the mathematics CSF? Thirty seven percent of the teachers involved in this research felt that affective factors like enjoyment, enthusiasm and confidence were important yet these are overlooked in the CSF. The issue of 'time' is addressed in the CSF in the context of allowing time for the growth of mathematical ideas in children. There is almost a plea emerging from the teachers' data for more time, for this, and so many other aspects of the teachers' work.

While teachers were concerned that mathematics should be made relevant to the children, issues of integration of mathematics with other subject areas were not a concern for them. There is quite a difference in the importance placed upon the roles of problem solving and language when comparing the teachers' view with the perspective of the CSF. Only 7% of the teachers felt problem solving was important, and only 6% of the teachers mentioned the importance of language. By comparison these things are stressed quite heavily in the Mathematics CSF. Gender equity, equal opportunity and the use of calculators and computer software do not arise as important for these teachers.

There are obviously differences in the issues that are important to the teachers involved in this research and those who determine curriculum for primary schools. Do these differences constitute the basis for further professional development for these teachers or should the areas of difficulty like catering for individual differences and organising and using concrete materials, which were identified by the teachers, determine their professional development? Who should decide what teachers need for professional development? While debate over what teachers need should continue, serious consideration must be given to professional development that is ongoing, practical and includes peer support programs. Solving the problems associated with the shortage of time could take a little longer!

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Student Understandings in Differential Calculus

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This paper considers the effectiveness of the traditional "first principles" approach to differential calculus and investigates some of the conceptual difficulties associated with it. Materials developed by Mary Barnes were adapted to produce a sequence of lessons characterised by investigative exercises and realistic calculus problems for a group of year 11 students. The lessons proved successful in improving students' conceptual understanding without impairing their ability to perform standard calculus techniques. They also reported positive feelings towards their study of calculus.

In recent years there has been a trend towards developing a more meaningful mathematics curriculum which is relevant to the needs and experiences of students. This has involved designing activities which are of interest to the students and which allow them to explore mathematical concepts for themselves as they play a more active role in their own learning. There has been a shift away from traditional teacher-centred methods in which the rote learning of rules and formula predominates, to one where the emphasis is on discovering patterns and understanding concepts. One area where such curriculum reform has been very clearly demonstrated is the study of calculus and a great deal of resources has been allocated to designing more effective means of teaching this topic.

There are important reasons why the calculus curriculum is in need of reform. The calculus component included in secondary schools has gradually been increased and made available to a wider cross-section of the student population (Grimison, 1990). As well, higher retention rates in secondary schools have meant that larger numbers of students are exposed to calculus instruction than has previously been the case (White, 1992) and for the majority of these students an abstract teaching of calculus concepts is inappropriate to their mathematical ability and educational needs.

Improvements in our understanding of the ways that students best learn new mathematical concepts have also called into question the effectiveness of the traditional approaches to calculus instruction. Many students still view mathematics as a body of rules having no relation to each other and they must be challenged to see the relevance of the mathematics they are studying as well as its connection to the mathematical concepts they are already familiar with (Mitchelmore & White, 1995).

The Traditional Approach

The traditional "first principles" approach to calculus instruction begins with a diagram like that shown in Figure 1 where the gradient of the tangent to the curve at the point P is the limiting value of the gradient of the secant drawn through P as Q approaches P. This is the method outlined in the New South Wales Department of Education syllabus for the 2/3 unit course (1982) and it is followed closely in the majority of high school texts which give a purely algebraic explanation using function notation and limits. There is generally no attempt to relate numerical and graphical explorations of the concepts and the main purpose seems to be to proceed immediately to the rules for differentiation (Wilkins, 1987).

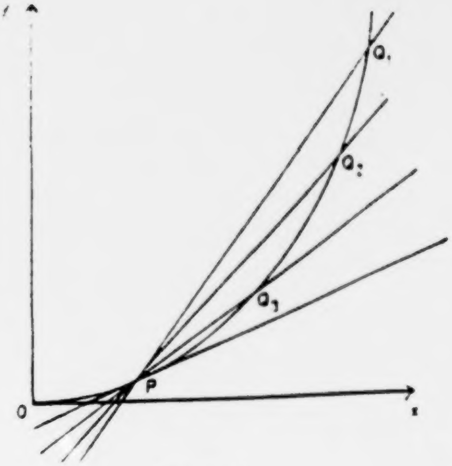


Figure 1.

Ryan (1991) reports that a large number of students who are instructed using a diagram like Figure 1 do not see the limiting process at all since they focus almost exclusively on the *chord* PQ and therefore do not see the *secant* approaching the tangent. In the minds of these students, the interval PQ is gradually getting shorter and shorter until it disappears altogether, and the central idea of the limiting process is lost.

Although the essential aim of the first principles approach is to develop a method for finding the gradient of the tangent to a curve, the usual manner of presenting this is almost completely abstract and given without any real justification as to why it might be useful or important. The applications treated are also generally mathematical in nature and no attempt is made to relate them to the experiences of the students or even the mathematical knowledge which they already have. Presenting new concepts in such a way, devoid of any real meaning for the students, can only increase the likelihood of misunderstandings and make the learning process more complicated.

The traditional approach is further characterised by abstract and complex algebraic manipulations which expose students to some highly demanding mathematics such as functions and limits from the very beginning. The difficulties associated with these concepts and the complex notation used to present them obscure the calculus itself making it unlikely that students will gain any appreciation for its significance. There is an increased chance that the students will too quickly come to rely on learning the rules for differentiation by rote without any real understanding of their meaning or significance.

One of the hardest concepts the students encounter in the first principles process is that of a limit. The difficulty begins with the many informal notions of the word "limit" itself which the students already possess (Barnes, 1991). Many of these capture some aspect of the formal mathematical concept but may also distort the finer nuances of the definition creating a conceptual barrier which makes learning about limits more difficult.

Tall and Vinner (1981) describe the kind of cognitive conflict which can develop due to discrepancies between the *concept image* (preconceived ideas in the mind of the student) and the *concept definition* (the more precise mathematical idea). They found that students who held strongly entrenched concept images which conflicted in varying degrees with the concept definition were unable to resolve the incompatibility.

Trying to water-down the limit concept and use a more informal version of the definition does not help because of the danger of becoming too colloquial and imprecise (Tall & Schwarzenberger, 1978). The idea of a limit loses its precision and the conceptual difficulty associated with limits is actually increased. There is also the problem of the notation used in writing limits which can be problematic for some students. Even those who can successfully evaluate limits often show a serious lack of understanding of the geometric significance of what they are doing (Ferrini-Mundy & Graham, 1991). Yet despite all of these difficulties, a proper understanding of the limit concept is crucial to the first principles approach.

The first principles definition also incorporates a quite detailed and involved use of functions, however many students commence their study of calculus with a poorly developed understanding of functions and function notation. Students tend to view functions in a rather static way, regarding the algebraic and graphical representations of functions independently (Dreyfus & Eisenberg, 1982). They often regard a function as a formula or equation into which a single value should be substituted and lack any real appreciation of the function as having meaning for a number of values.

Eisenberg (1991) found that while most students could draw the graph of simple functions, they were unable to see the connection with its algebraic form and regarded the graph as something quite apart from the function itself. He described the power of visual images in developing the function concept and proposed that the lack of a strong visual image of a function can restrict the students to purely symbolic representations which may impede a full appreciation of the more dynamic nature of functions. The requirement of the first principles approach that students be able to recognise the secant lines moving along the curve as they approach the tangent is thus unlikely to be met.

One of the key building blocks for work involving functions and limits is the concept of a variable. White (1992) studied students' understanding of variables in a calculus setting and reported that their lack of success in solving rate of change problems often arose from a poor understanding of variables. When problems were posed in symbolic form, few students had difficulty in executing the correct procedures. However, when similar problems were presented without the algebraic framework, the number of incorrect responses increased significantly. His research suggests that a more graphically oriented approach to introductory calculus based on rates of change provides a better alternative than the more traditional algebraic approach.

Clearly the traditional first principles method of introducing students to the concepts of differential calculus is problematic and calculus teaching which continues to rely heavily on this approach will not provide an adequate basis from which to develop strong conceptual understanding. The combination of increased numbers of students who are inadequately prepared for a study of calculus and a method of instruction which incorporates too many difficult mathematical ideas right from the very beginning exacerbates the situation. As a result, the first principles approach too easily lends itself to a teacher-centred form of instruction where students rote learn the rules for differentiation without understanding them.

Methods of calculus instruction which take account of the conceptual difficulties outlined above and present the calculus material in a more realistic context which reflects the interests and experiences of the students are clearly required.

A New Approach

The materials used to produce the teaching program for this research were largely derived from the work of Mary Barnes, particularly her unit *Investigating Change* (1991). The essential elements of her approach are to avoid the conceptual difficulties associated with the first principles method and to give the calculus a real-world context through the use of meaningful examples. Discovery learning and group investigations are included to give the students a sense of the power of the calculus they are studying to solve realistic problems.

The sequence of lessons created as part of this research is further distinguished by some important ideas developed by David Tall (1991). The first is the notion that sufficiently magnifying a small section of certain curves shows them to be locally "straight". The limiting process is implicitly contained in the magnification of the curve but no direct mention of limits is required. Instead, the students use the zoom function on a graphics calculator to obtain a visual image of this local straightness and by examining the curve in this way, they can recognise the gradient of this "straight" magnified portion as an approximation to the gradient of the curve.

This local straightness of curves then allows us to distinguish between the *practical* tangent which is a line drawn through two points very close to each other on the curve, and the *theoretical* tangent which is the straight line through the point having the same gradient as the curve at that point. Students can obtain a good approximation to the gradient of the curve by calculating the gradient of the practical tangent numerically using the standard result from co-ordinate geometry and this approximation can be used to measure the instantaneous rate of change.

Method

Subjects

The subjects were a group of 24 secondary school students in an average ability Year 11 3-unit mathematics class at an independent girls' high school in metropolitan Sydney. None of the students had any previous experience in calculus. The class was taught by the researcher and the teaching program was conducted for 13 fifty minute lessons over 3 weeks. The students were aware of the experimental nature of the work they were doing.

Instruments

Two tests were designed for this study. The pre-test was administered immediately prior to the commencement of the teaching program and was a written test consisting of 4 questions which the students had twenty minutes to complete. The purpose of this test was to assess the students' understanding of some fundamental pre-calculus concepts. The post-test was administered twice, once at the conclusion of the teaching program (August) and again after a further period of ten weeks (October). The post-test was exactly the same on both occasions and did not include any of the questions from the pre-test. It was a written test consisting of 8 questions which the students had to complete in thirty minutes. The purpose of the post-test was to assess the students' competence with and understanding of the calculus concepts they had studied. The questions from the post-test are found in Table 1.

Data Collection

Information about the understandings which the students were gaining of the calculus concepts they studied was also gathered from other sources. Four students whose ability reflected the range of the class were selected to take part in two interviews. The students were interviewed individually for 10 minutes at the end of the first week of the teaching experiment and then again at its conclusion.

Another separate group of four students volunteered to keep a diary of their impressions and feelings for the calculus they were studying. These students were also of mixed ability. As well, the researcher kept his own diary in which observations of the classroom activities and the ways the students responded to them were recorded.

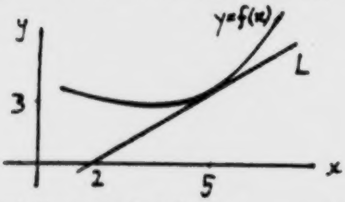
Table 1
Post-test questions

1. What is the gradient of the curve at the point (2,10)?

2. Find the equation of the tangent to the curve at the point where x=1.

3. Find the stationary points on and determine their nature. Sketch the curve.

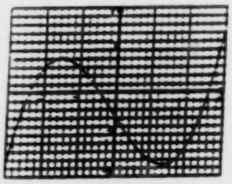
4. Suppose the line L is drawn as a tangent to the curve at the point (5,3) as shown in the diagram.
What is the value of
(a)
(b)



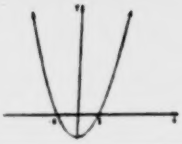
5. A fire has begun to spread through an area of bushland. Suppose that the area covered by the fire is given by where *t* is the number of days since the fire began. Explain how you would calculate the rate at which the area covered by the fire was increasing 10 days after it began.

6. Suppose that the function represents the number of people unemployed *t* months after the election of a new government. Translate each fact about the graph of into statements about the unemployment situation.
(a) The y-intercept of is 200 000
(b)
(c) The gradient of at is 1000

7. Here is a graph of a function
From the graph, estimate the value of the derivative at
Now, from looking at the graph of
sketch a graph of its derivative.



8. The diagram shows the graph of the derivative of a function . At what value(s) does
have a minimum turning point?
Give a reason for your answer.



Results

The results of the pre-test suggested that the students generally had a good understanding of distance-time graphs and that they were able to match the gradient of these graphs to written information about rates of change. One of the questions on the pre-test directly imitated the first principles approach as demonstrated in Figure 1 in order to test whether the students were indeed able to see the limiting process occurring for themselves. The most common response was that the chord became progressively smaller until it disappeared which supports the findings of Ryan (1991) quoted earlier.

The results of the post-test can be considered in terms of the questions which related to standard differential calculus procedures and those which examined conceptual understanding. The former type were found in the first three questions of the post-test which required the students to (1) find a numerical value for the gradient of a curve whose equation was given (100% of students correct in both administrations of the test), (2) find the equation of the tangent to a curve at a given point (92% correct in August and 83% in October), and (3) locate and classify the stationary points on a curve in order to sketch its graph (79% correct in August and 88% in October). The results of these questions suggest that the students were proficient in performing these calculus techniques and that they successfully retained this proficiency over time.

The remaining five questions of the post-test considered the levels of understanding of the calculus concepts which the students had attained. Question 4 tested whether the students could interpret a geometric representation of the derivative given in function notation. In August, 29% of the students were correct and in October, 54% were correct. This indicates that the students had difficulty with function notation.

Questions 5 and 6 were designed to test the students' understanding of the relationship between the derivative, the gradient of a curve at a point and the rate of change. In both questions, 80% of the students gave a correct response indicating that the students had a very good grasp of these ideas.

Question 7 asked the students to sketch the graph of the derivative for a function whose graph was given. Although this is a conceptually demanding task, the students had practised it during the teaching program and this was reflected in their high success rate (75% in August and 88% in October). The increased percentage of correct responses suggests that their success was not merely because they had seen it recently in class.

The final question on the post-test, question 8, gave the students the graph of a derivative and asked them to interpret it. This was a difficult question which was of a type that the students had not previously seen (46% of students were successful in August and 58% in October). The phenomenon of an increased number of correct responses between the two tests for this question occurred in other conceptually difficult questions as well suggesting that the concepts were being gradually developed over the period of time between the two administrations of the post-test. The students did not study any calculus topics in the intervening period so the improved results could not be due to any classwork done in this time.

Discussion

Conclusions and Limitations

The results obtained in questions 1 to 3 of the post-test demonstrate that the students were able to successfully perform the standard algorithms of differential calculus. This was despite the fact that significantly less time was given to these techniques in the teaching program than is normally the case and supports the view that time spent developing concepts is not wasted in terms of improving students' skills. If the students' conceptual understanding is enhanced in this way then they do not need to spend as much time learning how to solve standard problems. Furthermore, if they attempt fewer exercises, treated in greater depth so that the underlying concepts can be more fully explored, similar or better results can be achieved without completing numerous exercises by rote.

The results of the remaining questions of the post-test also generally support the view that the methods developed in the teaching program are more successful in building conceptual understanding than the traditional first principles approach. The avoidance of difficult algebraic manipulations, complicated notation and demanding concepts such as functions and limits in the initial stages of calculus instruction is clearly beneficial. Moreover, it is also advantageous to give the calculus material a realistic context which more appropriately reflects the interests and experiences of the students.

The approach based on magnifying curves and practical tangents not only avoids these difficulties but also has advantages of its own. It provides powerful visual images which give the calculus concepts a more dynamic character and allow the students to explore these concepts in a numerical and graphical setting as well. This encourages a slower pace which affords the students more time to assimilate the new concepts and also helps to reduce the reliance on algebra and notation in the early stages (Ryan, 1991).

The principles which governed the work of Mary Barnes (1991) of group work, discovery learning and using meaningful examples presented in an appropriate context also proved to be important aspects of the teaching program. Through the interviews and diary entries it became clear that many of the students who participated in the study gained more from the experience than improved mathematical understanding. The students achieved a greater appreciation of the historical development of calculus and the ways which it can be used to solve everyday problems. They also reported positive feelings toward their work and generally found it more interesting and enjoyable than other topics they had studied.

The teaching program included a number of aspects which departed from the traditional first principles approach but it is beyond the scope of this research to identify which of these had the greater influence in improving the level of conceptual understanding attained by the students. For instance, perhaps the notion of local straightness of curves was important or it may have simply been that the use of the graphics calculators to provide visual images was in itself the key factor. No firm conclusion can be drawn at this point.

It is also not possible to infer generalisations from this study as the research is based on the study of one class without any comparison group. While it can be seen that this particular class benefited from the teaching program, this may not apply in all cases. The fact that the teacher also operated in the role of researcher may have contributed unduly to the results obtained. The positive rapport which existed between the teacher and students may have influenced in some way the willingness of the students to ensure the success of the study. Such influences are virtually impossible to identify, but they should be acknowledged in such a small study.

Implications

There are some important implications for classroom practice which follow from the results of this study. Effective calculus instruction should be characterised by placing less emphasis on algebraic processes and functions, and introducing notation in a gradual way. Limits should not be treated until after the students are more confident of their understanding of the basic calculus concepts. Instead of relying on drill and rote learning procedures, students should be given the opportunity to investigate problems and discover many of the significant calculus concepts for themselves.

Graphical and numerical approaches should be used in conjunction with algebraic methods to investigate concepts. The use of graphs when students first consider rates of change is very important and numerical calculations can be used to help establish patterns from which the general algebraic results can be derived. Time should be spent carefully developing concepts in the initial stages of calculus teaching in order to reduce some of the misunderstandings which might arise later when students consider the applications of calculus. These applications should be presented using appropriate real-world examples which show the power of calculus to solve problems.

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Pre-Service Teachers' Pedagogical And Content Knowledge About Trigonometry And Geometry: An Initial Investigation

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In this paper, we report on our initial application of a theoretical model (Nason, Chinnappan & Lawson, 1996) which explicates the relationship between the organization and accessability of teachers' subject-matter knowledge, the nature of their teaching and the nature and quality of student learning within the domain of Analytical Geometry and Trigonometry in a Pilot Study. The major aim of this study was to begin the process of validating the hypothesized relationships and causal connections between the source knowledge elements of the model. This was done by examining the nature of a pre-service student teacher's (a) substantive mathematical knowledge, (b) pedagogical content knowledge, (c) knowledge about the learner, and (d) the relationships and causal connections between these three types of knowledge in the domain of geometry and trigonometry.

Although a significant number of recent research studies have attempted to examine teachers' subject-matter knowledge (see Ball & McDiarmid, 1990 for a review of this research), there has been a dearth of work that has examined the issue of how the teachers' subject-matter knowledge affects the quality of their teaching and the quality of their students' learning.

A corpus of research related to this particular issue has focussed on teacher instructional explanations (e.g., Leinhardt, 1987, 1988, 1989). This research has generated some significant information about the nature of teacher explanations. However, it has not offered a theory of the role of teacher subject-matter knowledge in classrooom instruction nor has it provided a specification of how teacher subject-matter knowledge influences instruction (Leinhardt, Putnam, Stein & Baxter, 1991). It also has provided little information about what kinds of subject-matter knowledge are particularly salient and how subject-matter knowledge is accessed and exploited during the course of teaching and how this in turn affects the quality of students' learning.

In 1995, Nason, Lawson and Chinnappan began the Ramanugan Project. The overall aim of this project is to begin to fill this void by the development of a theoretical model that explicates the relationship between the organization and accessability of teachers' subject-matter knowledge, the nature of their teaching and the nature and quality of student learning within the domain of Analytical Geometry and Trigonometry. In this paper, we present our initial conceptualization of the model and report on an initial investigation in which we investigated a pre-service secondary mathematics teacher's pedagogical and content knowledge about trigonometry and geometry.

Initial Conceptualization of the Model

The initial conceptualization of the model (presented in Figure 1 below) is based on our analysis and synthesis of the research literature (See Nason, Chinnappan & Lawson, 1996). This model (1) identifies key elements that underlie teacher expertise such as substantive mathematical knowledge, pedagogical content knowledge, teachers' perceptions and beliefs about mathematics etc., and (2) attempts to state in graphical form the relationships and causal connections between these elements. A more detailed presentation of this model can be found in Nason, Chinnappan and Lawson (1996).

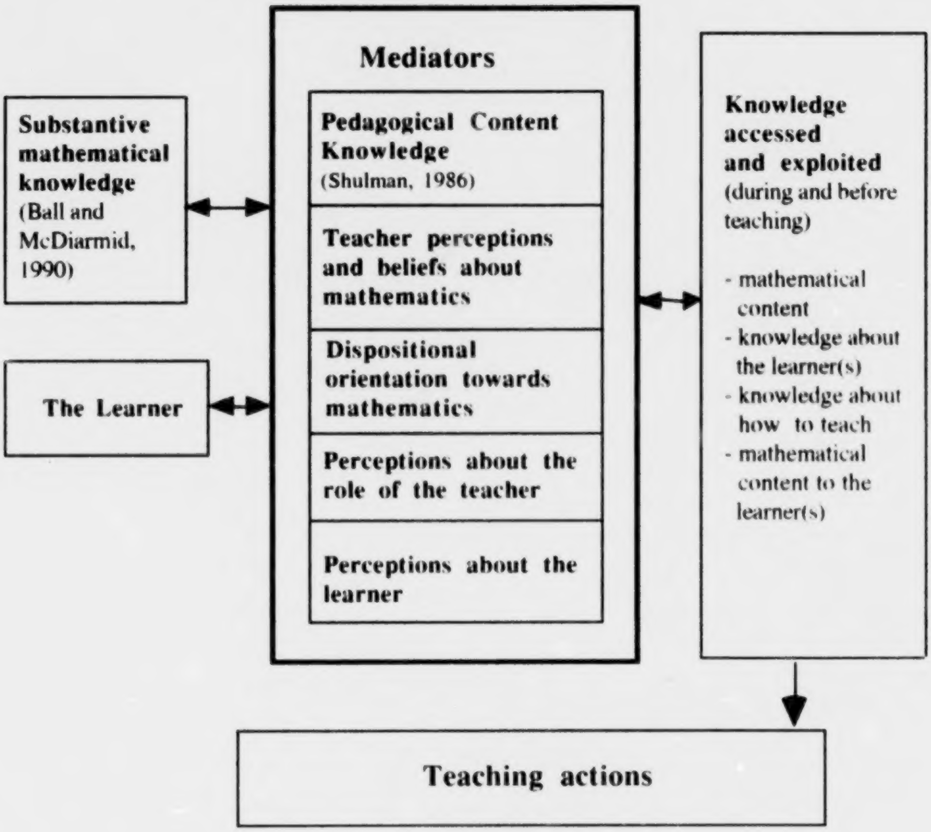
Mediators
Substantive mathematical knowledge (Ball and McDiarmid, 1990)
The Learner
Pedagogical Content Knowledge (Shulman, 1986)
Teacher perceptions and beliefs about mathematics
Dispositional orientation towards mathematics
Perceptions about the role of the teacher
Perceptions about the learner
Knowledge accessed and exploited (during and before teaching)
- mathematical content
- knowledge about the learner(s)
- knowledge about how to teach
- mathematical content to the learner(s)
Teaching actions

Figure 1

From our initial conceptualization of the model, we have generated the following hypotheses:

(1) that the quality and the quantity of the mathematical content accessed and exploited by teachers both before and during teaching depends to a large extent on the organizational quality of the teachers' repertoires of substantive mathematical knowledge and their reflective awareness of that knowledge;

(2) that the mathematical content accessed and exploited by teachers is related to the teacher's repertoire of pedagogical content knowledge and their dispositional orientation towards mathematics as indicated by the teachers' perceptions and beliefs about mathematics ;

(3) that the quantity and the quality of the knowledge about the learners is related to the teachers' pedagogical content knowledge, perceptions about the learners and perceptions about their roles as a teacher;

(4) that the quality and the quantity of the knowledge about how to teach the mathematical content accessed and exploited by teachers is related to the organizational quality of the teachers' repertoires of pedagogical content knowledge and their reflective awareness of that knowledge ; and

(5) that the knowledge about how to teach the mathematical content accessed and exploited by teachers before and during teaching is related to the teachers' perceptions and beliefs about the role of a teacher and the teachers' perceptions and beliefs about the learners.

In the next section of this paper, we report on our initial examination of the hypothesized relationships and causal connections between the source components shown in our model. This involved an attempt to identify a pre-service student teacher's (a) substantive mathematical knowledge, (b) pedagogical content knowledge, (c) knowledge about the learner, and (d) to map the relationshipss and causal connections between these three types of knowledge in the domain of geometry and trigonometry.

The Study - Method

A third year BEd preservice teacher who was majoring in secondary mathematics and computing(David) volunteered to participate in this study.

One week prior to the interview sessions, the investigator (Chinnappan) and David met for about thirty minutes. At this initial meeting, David was informed that he would be required to talk about concepts related to the areas of geometry, trigonometry and analytical geometry and that he also would be required to respond to question(s) relating the identification of these concepts, the importance of these concepts and how best to teach these concepts to his students. David was invited to raise questions relevant to these issues during this pre-interview meeting and during the course of the rest of the week before two interviews.

The first interview session began with David being asked to talk about all the concepts related to the areas of geometry, trigonometry and analytical geometry that he could recall. As he discussed each of these concepts, the interviewer asked follow-up questions in order to probe the width and depth of David's knowledge about each of the concepts and also to identify the degree of connectedness between these concepts. Then David was presented with an analytical geometry problem and asked to solve the problem out aloud. This problem task was administered in order to ascertain how David accessed and interconnected his repertoire of geometrical concepts in a problem-solving context. After he had completed the problem, he was asked to investigate other ways of solving the problem. The probing for alternative solutions focussed on the flexibility, the depth and the interconnectedness of his repertoire of concepts.

During the second interview, David was first reminded about mathematical concepts that had been identified and discussed during the first interview. Then he was asked to describe how he would teach those concepts. As this interview progressed, the investigator (Nason) frequently asked probing questions in order to get David to elaborate on and/or clarify points he had made about the mathematical content and his perceptions about learners and about how he would teach the mathematical concepts.

Both interviews were video- and audio-recorded and transcribed. The analysis of the data from Interview 1 focussed on the major concepts that David identified in relation to geometry and trigonometry, and the relations or links that he was able to construct. The analysis of data from Interview 2 focussed on David's pedagogical content knowledge and his knowledge and perceptions about learners. The initial analysis of the data was done by two of the investigators. The conclusions drawn from their analysis of the data were confirmed by having the third investigator at a later date go through the video-recordings and transcripts of the two interviews looking for negative evidence. As Sowden and Keeves (1990) point out, while the failure to find negative evidence after a deliberate search does not and cannot establish the "truth" of a conclusion, it however, does increase the probability that the original conclusion is sound. Some modifications were made to the set of conclusions following this process of verification.

Results

Substantive Mathematical knowledge

In this section, we present the results of our analysis of David's response to the investigator's questions about the identification of concepts in geometry, trigonometry and analytical geometry, and other concepts that could be related to these three areas.

Table 1 shows a list of the major concepts activated by David. In broad terms, on these topics of geometry, trigonometry and analytical geometry, one can isolate four major concepts: gradient, trigonometric ratios, similar triangles and Pythagoras theorem. It is interesting to note that David considered the notion of gradient as a starting point. His explanation of this concept included a procedure for working out the gradient of two points under two situations: when the coordinates of the points were not given, when the coordinates of the points were given. David also explained how a numerical value of gradient could be interpreted. For instance, he attempted to illustrate the meaning of a line with a gradient of 1/4 using the notions of vertical and horizontal distances. This idea was subsequently related to one of trigonometric ratios, namely the tangent ratio.

In this context , David quite appropriately mentioned how the idea of tangent of an angle could be used to identify similar triangles. He did mention the sine and cosine ratios but did not explain how the gradient and tangent ratio could be utilised in the derivation of sine and cosine ratios. That is, the link between tangent of the angle, gradient of the line and sine/cosine of the angle was not made explicit.

David then moved on to Pythagoras' theorem for which he gave the algebraic statement (a2+b2=c2). Although he did provide an alternative way to understand the

CONCEPTS IDENTIFIED	LINKS IDENTIFIED	LINKS NOT MENTIONED
Geometrical shapes		Relationship between two dimensional figures and their properties
Mensuration		Relationships between the concepts of area and perimenter; Areas and perimeters of geometrical figrues
Cartesian coordinate system	coordinates of a point, vertical and horizontal distance; gradient and tangent ration of angle	Coordinate system and analytical geometry; Coordiante system and graphing of polynomials
Gradients	Gradient, tangent ratioand similar triangles	Gradient and sine or cosine ratio
Trigonometric ratios	Tangent and gradient	Relationship between tangent, sine and cosine ratios
Pythagoras theorem	Pythagoras' theorem and areas of squares	Pythagoras' theorem and trigonometric identities; Pythagoras' theorem and coordinate system; Pythagoras' theorem and the right-angled triangle
Linear functions	functions and calculators; link between the x and y variables	Linear functions and gradient; linear functions and coordinates of points; graphing of linear functions
Equations	subtitution of numerical values into an equation	Equations and functions; equations and Pythagoras' theorem; solution of system of equations

Table 1

algebraic statement by way of considering areas of squares that could be constructed by using the three sides of the right-angled triangle, there is no evidence of David making any links between this representation and other related concepts such as the Pythagorean triplets, the converse of Pythagoras theorem. David could have also used this relationship in the context of a coordinate system to work out for instance the radius of a circle with its centre located at the origin, but he did not do so showing a lack of understanding of application of this idea in a range of situations.

The list in Table 1 shows that David had built up a range of mathematical concepts in relation to the topic of geometry, trigonometry and analytical geometry. However, he did not make or attempt to establish any meaningful mathematical connections between the various areas despite instructions to do so. This pattern suggests that his mathematical content knowledge base lacks a high degree of organisation and integration.

Pedagogical content knowledge

Interview transcripts were analysed for instances of activation of pedagogical content knowledge (Shulman, 1986). This analysis led to the identification three subgroups of knowledge within this category:

1) Pedagogical content knowledge showing the need for historical background;
(2) Pedagogical content knowledge showing the need for discussing application; and
(3) Pedagogical content knowledge showing the need for using particular method.

The need for historical background

When asked to explain how he would go about teaching some of the concepts mentioned in the first interview, David showed an interest in getting his students to become aware of the history of certain concepts. For instance, in relation to the teaching of the Pythagoras theorem, David made the following remark:

> *I would probably the night before the lesson do a little bit of research on Pythagoras and probably establish a bit of history so they know a little bit of (about) where it originated and why this formula ($c2 = a2+b2$) is used right throughout mathematics.*

The importance of application

David was also interested in sharing with this students the applications of mathematics. He mentioned that Pythagoras' theorem is used to solve different geometrical problems and that engineers also use it in their work. In response to a question about the importance of studying gradient, David's reply was:

> *Well I've had experience in surveying and to me that's totally important if you get a job with a surveyor*

Pedagogical content knowledge showing the need for using particular method

In the examination of teachers' pedagogical content knowledge, we were interested to find out about David's understanding of various methods he would employ in presenting ideas and initiating discussions about these ideas. In this context, David's remark that he would solve the problems on the chalkboard (*'I can do problems for them on the board'*) is interesting. How to intrepret this comment is not yet clear - it could reflect merely his observation that he had observed his own teachers or lecturers doing this, or it could represent a view that it is important for the teacher to model the problem solving process for students- this indicates an area where further probes are necessasy in the interview

David favoured the use of examples to teach mathematical concepts. For example, in one part of the interview, David was asked about teaching the concept of linear function. He selected one example of this class of function and decided to explain it. At the end of this explanation, David remarked that :

> *Alright let's put in in the form of that, so it's going to be, may be I'll pick another example, it might be easier. Something like, alright let's say something in the form of this*

The above reaction indicated that David did consider the use of examples to an effective and appropriate instructional strategy for teaching linear functions. This part of his pedagogical content knowledge also shows that care needs to be exercised in the selection of examples so that the more simpler one are used first.

While the above points highlight that David did attend to components of pedagogical content knowledge as theorised by our model, such instances were too few to have any significant bearing on his teaching. Analysis of the rest of the interview showed that David's knowledge of geometry and trigonometry, and how to teach these concepts was diffuse. At the end of his interviews, we were unable to clearly determine how his available mathematical knowledge would be deployed during teaching.

Knowledge about the Learner

As shown in Figure 1, a teachers' understanding of and assumptions about the learner exerts a major influence in the learning process. Towards this end, we were interested to examine whether and to what extent David showed consideration for the characteristics of his students. Our analysis showed that there was very little talk about the learner, or about the requirements of the learner, or of what learners could be doing during lessons while all the content is being presented.

What discussions there was of the learner referred only indirectly to knowledge of the learner, but not at all to the actions that a learner might engage in to develop an understanding of this content. For example, David mentioned that in the teaching of geometry, he will start with two dimensional shapes before investigating other concepts such as finding the areas and perimeters of such shapes. This seems to be a reasonable approach, but there is no evidence of how he would help the learner move from understanding properties of two-dimensional figures to working out their areas and perimeters.

During the interview, David made several assumptions about the learner. For example, when asked to explain how he would teach the notion of functions, David observed that it would better to start with an example that might be easier, perhaps indicating some appreciation of cognitive load.

> *'maybe I'll pick another example,it might be easier'.*

He was also aware that some students might find certain concepts harder to process but does not explain why this is so. Several times, he made the comment that

> *'if they understand they will feel comfortable'.*

This observation implies that helping students understand the concepts would engender a feeling of satisfaction and confidence in dealing with similar concepts.

What was not mentioned about the learner

In analysing the interview transcripts, we found that David did not talk about a number of characteristics about the learner. Firstly, there was no indication about the desire to understand the beliefs his students might hold about geometry or trigonometry. Secondly, students' dispositions towards this area of mathematics did not get any mention. David also did not consider a) the actions that the learners can engage in, b) the strategies that can be used by the learner, and c) the different ways of problem solving. In general, David's approach suggested that beyond the presentation core concepts, the teacher plays very little further role in the learning of geometry and trigonometry.

Discussions and Conclusions

A principal concern of the Ramanugan Project is to examine the nature of teachers' mathematical content knowledge, how this knowledge is structured for the purposes of teaching mathematics concepts, principles and procedures, and the assumptions teachers make about their students. A preliminary evaluation showed that the framework set out in Figure 1 is useful for classifying the contents of a teacher's talk.

The model has provided us with a useful tool for analysing some of the ways in which characteristics of a student teacher's knowledge might be different from those argued to be characteristic of expert teachers. One area of this difference involves the extent and coherence of mathematical content knowledge. As expected, this student teacher showed acceptable levels of this knowledge. However, the components of this knowledge were found to be relatively discrete. There was little evidence that the knowledge components were connected in a manner that would lead to the establishment of more extended networks of mathematical knowledge that are argued to be characteristic of expert teachers mathematical knowledge base (Shulman, 1986).

The type and sequence of mathematical concepts activated by our student teacher suggests that he had not reflected upon these in order to construct multiple representations for these concepts, an important requirement for the effective activation and of use the concepts in the solution of novel problems (Larkin, 1979; Shoenfeld, 1985).

The data analysed here indicated that, despite our instructions to highlight the various relationships among knowledge components, the student teacher merely retrieved isolated bits of mathematical information without any consideration to how they could anchored within the general mathematical curriculum, a point that is stressed in current documents about effective mathematics teaching (National Council of Teachers of Mathematics, 1989). This behaviour suggests that there is the possibility of the students of this student teacher being 'bombarded' with unrelated, or poorly related, mathematical information.

The knowledge base of the student teacher also showed a general lack of integration of mathematical knowledge with pedagogical principles. There was little evidence that the student teacher had considered the possibility of reconceptualising or restructuring the mathematical content knowledge in preparation for presentation to students. There was also no sense that this novice teacher was concerned to engage his students in active exploration of the subject-matter knowledge. The teacher's pedagogical content knowledge seemed to be poorly developed.

The third key feature of our student teachers' knowledge was that the role of the learner was under-conceptualised. Although there are instances where he had shown consideration to the learner in terms of how he or she would handle the concepts, there is a lack of understanding about how students activate and exploit mathematical knowledge acquired in the classroom.

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Expecting the Unexpected: Professional Growth through the Highlights and Challenges of a Problem Solving Classroom

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Two teachers, good friends and colleagues, and yet with quite different approaches to the teaching of mathematics, exploring what was for them "new territory" (pedagogically and mathematically) as they taught a problem based unit of work for the first time. Using classroom observations and post-observational interviews, this study described the challenges of the teaching of the unit through teacher identification of "critical incidents" that were encountered. This paper tells the story of professional growth through highlights and challenges, and explores issues of support provided by colleagues and the researcher.

The Call for Change

In recent years, around the world there have been calls for reform in the teaching of mathematics (e.g., Australian Education Council, 1990; National Council of Teachers of Mathematics, 1989). Many factors are influencing such pressures for change including new technology, the needs of a 21st century workforce, new insights from research into how children learn and changing perceptions of what it means to know and do mathematics. Current calls for reform present a contrasting picture of the mathematics classroom from that of traditional mathematics instruction. In the United States, the *Curriculum and Evaluation Standards* (NCTM, 1989, pp. 5-6) presented five general goals for all students:

- that they learn to value mathematics;
- that they become confident in their ability to do mathematics;
- that they become mathematical problem solvers;
- that they learn to communicate mathematically; and
- that they learn to reason mathematically.

The implementation of these goals requires a different classroom environment from that which has come to be called "traditional practice". *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990, pp. 18-20) reflects similar goals, with a strong emphasis on the student being actively involved ("students should experience the reward of arriving at a solution through their own initiative and persistence and not simply through imitation"), with the teacher as a facilitator or guide of learning ("helping students learn to seek imaginative solutions to problems in constructive ways") (AEC, 1990, p. 21).

Though teaching can never be viewed as a completely predictable act, the role of the teacher in the traditional mathematics classroom is to structure the lesson in such a way as to minimise unexpected events. However, when a teacher is responding to students' thinking, and challenging students to explore and conjecture, it becomes a case of "expecting the unexpected."

Background to the Study

The study described in this paper was undertaken in the United States. It was a case study of two experienced seventh-grade mathematics teachers as they taught an innovative unit of work, the mathematical content of which was angles, similar triangles, and introduction to the tangent ratio embedded in context which included hang gliding, shadows and "blind areas" in shipping. The two teachers (Ms Brennan and Ms Whitten—fictitious names) were colleagues who worked closely together but whose approach to teaching provided a considerable contrast.

The questions that guided the study were:

1. What kinds of critical incidents for the teachers arise during the teaching of a unit of mathematics built around realistic situations?
2. In what contexts do such critical incidents arise?
3. How do teachers respond to these critical incidents and what influences their responses?

The unit of work that provided the mathematical content for this study was one taken from a National Science Foundation curriculum development initiative. *Mathematics in Context. A Connected Curriculum* (Romberg et al., 1991) was a mathematics project for grades 5-8. The stated philosophy underlying the project was the belief that mathematics is fallible, changing, and like any other body of knowledge, the product of human inventiveness (Ernest, 1991).

Instructional units which were built around non-routine problems were initially developed by the Freudenthal Institute at the University of Utrecht in The Netherlands, and "Americanised" as required. These units provided opportunities for the active involvement of the student, both physically and mentally. The mathematics content was challenging and yet Dutch experience showed that the realistic contexts seemed to make the problems accessible to the students (de Lange, 1987; Treffers, 1987). Alternative solutions to problems were encouraged. Although this was a curriculum development project, its purpose was not to produce "teacher-proof" materials, but rather to encourage teachers to experiment with a new style of teaching by providing appropriate curriculum materials and ongoing inservice and within-class support.

Methodology

This study was based on the belief that teaching is a complex activity (Koeler & Grouws, 1992). The decisions teachers make are largely situation-specific and as such need to be looked at in the context in which they occur. The perspective taken by the researcher in this study was one of participant-observer which Erickson (1986, p. 121) described as *interpretive*.

Erickson argued that this perspective is best at answering questions like "What is happening, specifically, in social action that takes place in this particular setting? [and] What do these actions mean to the actors involved in them, at the moment the actions took place?" (p. 121). These are clearly the type of questions being asked in this study.

The research involved a case study of two teachers as they taught a particular *Mathematics in Context* unit to a particular class of students and met together after most lessons for planning and reflection with the researcher. Initial observations and interviews prior to the teaching of the unit provided background information as well as familiarisation with student and teacher routines. Once the teaching of the unit began, information was gathered in the form of field notes, post-observational interviews with the teachers, and audiotaping of lessons.

In total, 58 lessons were observed and 23 observational and two summary interviews were conducted. Interviews provided the opportunity to document critical incidents as identified by the teachers, and also enabled the teachers to discuss the reasons for the teaching and planning decisions made in response to these.

"Critical incidents" were those classroom situations identified by the teachers as memorable and significant—both challenges and highlights. As the task of identification rested solely with the teachers, this process was clearly influenced by their past experience, their teaching style, and their mathematical confidence and competence.

Audiotapes of interviews and some classroom interactions were transcribed. As these critical incidents were noted from the transcripts, the corresponding points in both the field notes and journal were combined with the teachers' comments from the interviews. Together, these formed a "story" of each incident. As the transcripts were being read and re-read for critical incidents, relevant background information, and general themes were noted. All data, interviews, field notes and journals were electronically "cut and paste" into different areas of interest. These included the critical incidents, information on background and classroom practices for each of the teachers, general school issues, mathematical challenges, and assessment issues, and these formed a starting point for the writing process.

Critical Incidents

This study attempted to identify and describe those situations that occur in the teaching of a "realistic" unit that were challenging and difficult for the teacher. Although other studies (see, e.g., Ball, 1988) and pilot testing of *Mathematics in Context* units provided anecdotal examples, a more in-depth study of the unexpected situations that arise on a day-to-day basis had the potential to give a clearer picture of what the challenges are in the teaching of this type of unit. These were not isolated incidents but occurred within the context of a particular classroom environment.

The incidents were grouped into broad categories, where the particular focus was on what constituted the "trigger" for the incident. This should not be considered a definitive categorization, the decision of categorization being based on my interpretation, drawing on all the available data relevant to each critical incident, including field notes from classroom observations, classroom audio tapes, interviews transcripts, and journal entries.

The broad categories were:

- Student insightful response.
- Student incorrect response.
- Student affective response.
- Challenge to the teacher's knowledge.
- Challenge to the teacher's beliefs.

There were incidents which involved more than one component of this categorization. For example, an incorrect response from a student may challenge a teacher's knowledge, clearly cutting across at least two categories. Of equal importance with the categorization were the detailed descriptions of the incidents themselves as they represented important and significant classroom incidents for these teachers as they taught an innovative curriculum unit for the first time (see Clarke, 1995).

For both teachers, the insightful responses of students were the most common source of critical incidents. These included incidents where students were conjecturing and posing questions. They were student-initiated solutions of considerable variety and sophistication, which were often highlights, but sometimes challenging in their resolution.

The teachers identified only a small number of situations resulting from students' incorrect solutions, probably because many of the student errors were of the kind commonly experienced in all mathematics classrooms. For example, students misreading the scale on a protractor in Ms Brennan's class was not reported in post-lesson interviews. Both teachers identified situations involving student mistakes where the source of the error was difficult to determine at the time.

Incidents that were triggered by students' affective responses included situations where the students were so engaged in the thinking and context of the problem that lively debate turned into heated discussion about a particular mathematical issue.

Other critical incidents arose because of challenges to the teachers' mathematical understanding. Their prior mathematical experience (both as teacher and student) had been largely the presentation of an algorithm followed by twenty or thirty exercises for which the algorithm was a perfect match. When faced with problems requiring flexible, conceptually-based understanding, they found that their mathematical knowledge was inadequate. The interviews provided a non-threatening environment in which to discuss any difficulties the teachers were having, particularly such mathematical ones.

Ms Brennan's Response to Incidents

The teaching of this unit yielded many unexpected situations for Ms Brennan that were different from the types of incidents associated with traditional mathematics teaching.

> Those are good surprises that come up because you haven't said, "this is how to do it". You never do that, so having them come up with those generalisations, that's good. The discovery part of it. This is real open-ended. So you really don't know what you might end up with. (Interview, Ms Brennan)

This unit was also challenging for the students. "It's harder for them because you're forcing them to have to think" (Interview, Ms Brennan). Ms Brennan viewed this as a positive aspect of the use of this type of unit. "I like it a lot better. I think it makes them think and makes them use what they know".

Ms Brennan showed a willingness throughout the unit to be a risk taker, both pedagogically and mathematically. When discussing in general terms the situations that arose, she was asked if she saw herself as a risk taker:

> I'm probably more of a risk taker than most, I guess. I wasn't before. I would, you know, only do certain things, but this [material] makes it easier to take the risks and do it with the kids.

(Interview, Ms Brennan)

Although the unit covered content with which she was not familiar, she was not discouraged. On the contrary it was a positive experience for her:

> You have to love to learn. I think that's one of the things you have to do. If you don't like learning then this isn't what you need to be in. You have to like learning and you learn each time you do it with them. (Interview, Ms Brennan)

The incidents identified by Ms Brennan were not only those that were challenging, but also those that were memorable and affirming. There was a sense of anticipation, almost excitement as she taught this unit, and critical incidents were often viewed very positively, being seen as indicators that there was worthwhile mathematical activity going on.

Ms Whitten's Response to Incidents

Ms Whitten was not quite as comfortable with this type of unit as Ms Brennan. During one interview, she discussed her comfort level with this unit and those like it. She had had a discussion with another teacher at the school who was currently teaching her first unit. They had compared the teaching of this with the teaching of a Social Studies program called "Quest."

> Well it's very much like this in terms of preparation. The materials, the activity level, the grouping and you are always go go. She said it's exhausting. I said that that's really true. . . It's hard on the teacher doing it all the time. (Interview, Ms Whitten)

Ms Whitten found that the teaching of these units was more demanding, both physically and emotionally:

> It is dealing with questions, dealing with more activity. In that way it's demanding of your energy, physically and emotionally. There's the preparation before, the thinking it through, the trying, setting up the materials. It's always clean it up, next class, clean it up, . . .
>
> Making it understandable for that at least 50% of the kids that are going to struggle somewhat. . . . I found they were catching on pretty well but I kept feeling I'm hand-feeding them more than I

> should. I would have liked to [provide] less help and yet then it would go so slowly. . . You know, there's pluses and minuses to helping them along. . . (Interview, Ms Whitten)

The role of the traditional mathematics teacher is to structure each lesson with particular outcomes in mind. Ms Whitten was more comfortable with the pace of a traditional classroom. The thinking and on-the-spot decision making of this new approach were demanding for her:

> The teachable moment, that kind of occurs more. Having a variety of answers, sometimes surprising answers coming back at you. Definitely more thinking situations. (Interview, Ms Whitten)

In summary, Ms Whitten, a thoroughly-prepared, dedicated teacher was a willing participant in the project, but interviews and observations revealed a commitment to traditional practice. The importance placed on students' success which tended to tie her to traditional practice was also an influence in her growing commitment to the need for a change: "seeing them do well and understanding gives the program more validity and I'm willing to put the time into it" (Interview, Ms Whitten).

During the teaching of this unit and our discussions, Ms Whitten came to question aspects of her teaching. Though always reflective of her practice and willing to listen to new ideas, this particular experience had been pivotal. She was more prepared to allow the students to explore the mathematics themselves.

Toward the end of the study, Ms Whitten expressed the view that she may look to take a year off teaching to rethink what she was doing. She did this and at the time of writing I am unsure as to whether she has returned to teaching. The factors in such a decision are complex, but her realisation that different expectations were now required of her was expressed to me as a major concern. During the time of the study, Ms Whitten gained confidence and commitment to this approach. Though it presented a number of challenges, the regular discussions and support of both Ms Brennan and myself seemed to make these manageable. Ms Whitten was an excellent classroom manager who was genuinely concerned for her students and it would be of concern if such expertise was lost to the teaching profession.

The Professional Growth of the Two Teachers

The teachers' responses to the incidents and the subsequent discussion clearly showed a change in their practice and a growth in their understanding of the role of the teacher. It is reasonable to assume that challenges are likely to be present in any classroom where a teacher is "wrestling" with creating this kind of "constructivist" environment. In fact, the data in this study supported the hypothesis that the more the classroom resembles such an environment, the greater the incidence of critical incidents of the kind discussed in this study. This is supported by the work of Shroyer (1981), who suggested that there was a greater frequency of critical incidents in a problem solving classroom than in a traditional one.

Structuring Classroom Discussions and Presentations

The amount of structure varied between teachers, and both too little and too much structure seemed to limit the students' thinking. For example, with the initial presentation of problems there was a need to clarify, but at the same time not to present complete explanations. The challenge was to present a problem in such a way that the students could make a start, while not limiting their thinking by over explanation; that is, clarification without explanation. This may sound an easy distinction, but in practice it was not, as it depended on the problem context, the problem itself, the students' prior experiences, and the teacher's knowledge of the students and the mathematics.

There is no set formula for the amount of structure that is desirable in a given situation. Romagnano (1991) considered this to be an unsolvable dilemma. Nevertheless, it is likely that teachers will become more confident in determining the level of structure, as they acquire more experience with the use of innovative materials.

Student Comfort Level

Another challenge the teaching of this unit presented for one teacher in particular was that of student comfort versus struggle. The emotional well being of the students was very important to this teacher, and she worked very hard at making the classroom environment comfortable and successful. She would tend to react quietly to an incorrect student response so as not to embarrass the student. Students were not generally encouraged to struggle and it is this very struggle, the grappling with mathematical ideas, that reform documents encourage. This represented a contradiction between the practices which this teacher believed allowed students to be comfortable and successful and the intended purpose of the unit.

Issues of student affect were very important to both teachers, though in quite different ways. For one teacher (Ms Whitten), the emphasis was on removing anxiety and providing regular success. For the other teacher (Ms Brennan), it was important that students were motivated and she was more willing to embrace the teaching approaches advocated by the unit's authors because the students were more enthusiastic, and it reinforced what she valued.

Limited Mathematical Understanding

The teachers found the mathematical content of the unit challenging. They were required to understand an innovative unit of work and a consequent range of insightful and incorrect student solutions, for which their previous mathematical background had not necessarily equipped them. Creating an environment of conjecturing and inventing is particularly difficult where teachers feel they are at the limits of their own knowledge of content. As Brophy (1991) argued:

> Where (teachers') knowledge is more explicit, better connected, and more integrated, they will tend to teach the subject more dynamically, represent it in more varied ways, and encourage and respond fully to student comments and questions. Where their knowledge is limited, they will tend to depend on the text for content, de-emphasize interactive discourse in favor of seatwork assignments, and in general, portray the subject as a collection of static, factual knowledge. (p. 352)

Teachers' lack of mathematical knowledge led to critical incidents but it also affected the responses to incidents where the teachers were unsure how to proceed. Both teachers relied heavily on the regular discussions with the researcher and each other to clarify their own mathematical understandings. They were able to learn through the process and were encouraged by this, but the mathematical demands of the unit of work remained a constant challenge for them.

It is interesting to reflect however that in Ms Brennan's case, her own struggles with mathematics sometimes provided greater opportunities for the students. There were several occasions when Ms Brennan's doubt about mathematical content led to her opening up the discussion to students' ideas, in the hope of resolution of the difficulty. Despite this, while connected and integrated mathematical knowledge on the part of the teacher is not a sufficient condition for a classroom environment that is of the kind recommended in reform documents, the data from this study lend support to Brophy's argument that it is desirable.

Physical, Mental, and Emotional Demands on the Teacher

The requirements of this type of teaching made great physical, mental, and emotional demands on both teachers, more intense than those of traditional mathematics teaching. Excitement, frustration, uncertainty, and enthusiasm were common emotions felt by the teachers, sometimes all in the one lesson. Ms Whitten, though increasingly realising the value of both the classroom materials and the recommended approach to teaching, found the demands of managing a fast-paced and more boisterous classroom where students were, for example, making models of the Grand Canyon, measuring boats, creating and flying paper aeroplanes, and offering a variety of unexpected solutions to problems, a considerable contrast from her usual, orderly classroom. Ms Whitten took the opportunity occasionally to "catch her breath" by providing review lessons.

Ms Brennan, much more comfortable with the unexpected nature of the interactions and willing to "go with the flow", nevertheless found the work quite intensive. It is very difficult to be interacting, listening in an attentive way, trying to understand students' solutions, and thinking of an appropriate response, many times during a lesson. In a more conventional classroom, the teacher knows the "right" way to do a problem because they were the ones who explained it to the students in the first place.

Some Implications

This study provides illustrations of the kinds of critical incidents that arise in a problem solving classroom as well as considerations that a teacher must balance as she makes spur of the moment decisions. Although each situation is a unique combination of multiple influences, it is hoped that the story of these teachers' struggles will provide insight into this type of classroom, and will therefore be useful for the professional development of mathematics teachers. Particular incidents have the potential to be used to engage teachers in discussion and debate about what it means to know and to do mathematics and how student learning can be enhanced in such a classroom.

Shulman (1991) discusses the use of "cases" as a promising tool to "infuse reality and concreteness into theory-laden courses of teacher preparation and staff development" (p. 28). Such cases represent readable accounts written by teachers of teaching episodes or series of events that represent recurring challenges for teachers. These include the context in which the situations occur as well as the thoughts and feelings of the teachers involved. They provide an opportunity for teachers to reflect in a manner that is difficult in the classroom setting, as "too much is happening too swiftly in the messy world of practice for teachers to think deeply about what they are doing" (p. 29). This study generated similar rich descriptions of classroom situations which, although not teacher narratives, provide insight into the struggles for teachers in a new classroom environment.

The combination of curriculum materials, previous staff development, and support by project staff encouraged the teachers to pursue a different approach to their teaching. However, it is a major demand on teachers to make significant changes to their practices when they may have to teach five classes every day, as well as their other administrative responsibilities. There is little time for the kind of reflection that is needed if they are seriously going to look at the implications of these critical incidents and the units on their practice. One of the positive aspects for the teachers in this study was the opportunity to share with each other and the researcher on a regular basis. Both teachers commented on the value of having a support person to encourage them in what was a difficult task. In this study, the role of the researcher could not easily be separated from that of project support person, as the teachers struggled with both mathematical and pedagogical issues.

Conclusion

When students are encouraged to invent, conjecture, and solve realistic mathematical problems, there will be situations that are unexpected and challenging for the teacher. Avoiding these may lead to a limiting of students' thinking, but confronting them leads to a very demanding role for the teacher. This study affirms the view that teaching is a complex enterprise where decisions must be made for which no predetermined formula exists.

One of the most powerful affirmations of this unit for the teachers was the displays of insightful student thinking. Although the situations were challenging, the richness of student work and the frequent insights they demonstrated encouraged the teachers to "expect the unexpected" and to persist with the aim of making mathematics meaningful and relevant for their students.

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The Concerns and Issues in Primary and Secondary Mathematics Teaching as Reflected in Questions to Mathematics Educators

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Using email, academics, consultants and other teachers of mathematics were surveyed as to the most difficult question they had ever been asked by a mathematics teacher. The 110 questions that were received in response to this request were then coded using "Hyperqual". A range of themes emerged, with interesting similarity across the four contributing countries; assessment issues being the most common theme. Implications are discussed.

Background

Many readers of this article will frequently be in the situation of making presentations or leading workshops concerning their research or trends in mathematics education with teachers or preservice students. In the course of such workshops or presentations, we are often faced with questions which challenge our beliefs about mathematics teaching and learning, and make clear the concerns of practitioners.

Such questions can leave us stammering, interrupt the flow of a session, and have us looking for a hole in the floor into which to jump. However, on the positive side, such questions enable us to "tap into" the issues that are of immediate concern to teachers, enabling us to make our seminars, workshops and teaching responsive to the needs of the group.

In seeking to gain an overview of the more difficult questions that colleagues in mathematics education have been asked and therefore a new perspective on issues of concern to teachers, I encouraged colleagues with whom I had an "email association" to share their most difficult question. This paper discusses those responses and the implications of these for the work of researchers and others involved in the professional development of mathematics teachers.

Methodology

The Sample

Being somewhat naive about the ways in which email networks work, I sent an email request to all 105 people in my email "Nickname" file and to all of those people on the MERGA and MESH (Mathematics Educators Sharing Network—Victoria) email list, there being some overlap in these three "directories". I asked them the following question:

> Could you please give me the most difficult (or interesting!) question you have ever been asked by a teacher (let's keep it to professional questions, preferably in the mathematics education domain)?

The next day, I received 85 replies; the following day another 59, with responses continuing for the next two weeks. Although more than half the replies were "undeliverable mail", I received many replies from people to whom I hadn't even sent directly the original request. I had someone who I had never heard of send all of my colleagues an advertisement for an algebra package, and a request from someone to take her off my mailing list immediately, when I didn't even realise that she was on it. This is a dangerous business. Despite these drawbacks, I was delighted with the range and interest of the kinds of challenging and interesting questions that I received.

Responses were received from colleagues in Australia, Germany, New Zealand, Sweden, the United States of America, and the United Kingdom. Of these, 75 included one or more questions.

Data Analysis

Using Hyperqual (Padilla, 1991), I coded each question or part of question according to the themes that were evident within it. Not surprisingly, some questions or parts of questions were coded in two or more different ways. For example, the following question from Tony Herrington (Edith Cowan University, Perth), was coded "Problem Solving" and "Assessment and reporting": "Problem solving is great—but how do I know the kids are learning anything?"

There is no claim that this was other than "convenience sampling", and given this and the fact that I had not asked for details on the setting in which the question was asked and by whom, this information was not able to be included in the coding process.

I sent a copy of all of the questions to anyone who had provided a question to me. A further stage in this small study was to consider the reactions of colleagues to the set of questions that emerged. Some of these will be discussed later in the paper.

Results

The following codes (expanded from the actual code used in *Hyperqual*, which are required to be no more than one word) were used to categorise the themes that emerged from the data. With each code, the total number of questions or parts of questions that were coded this way is given in parentheses, as is one example of such a question:

- ***Assessment and reporting (23)***

"How can I teach maths so children can converse, question, explore, discover, etc., and still prepare them to pass national standardised, multiple-choice tests?"

Joseph Zilliox, University of Hawaii, Honolulu.

- ***Difficult/unmotivated/special needs students (11)***

"I have these two (insert acronym or dysfunctional label here) boys/girls. They just won't (choose one of the following: sit still/listen/cooperate/learn). If I worry about them, I have to spend all of my time just keeping them on-task, so the rest of the class suffers. How can I teach with these types of kids integrated in my class?"

James Middleton, Arizona State University.

- ***The content of the mathematics curriculum (10)***

"Assume for a moment, that a student will only be able to master five critical concepts in mathematics. What concepts would you recommend?"

Robert Ronau, University of Louisville.

- ***Catering for mixed abilities (9)***

"I worry that in this climate of math reform, we are only really serving the best and brightest. Is this approach the right one for 'at-risk' kids? These kids really need the skills if they are going to get ahead."

Ron Ritchart, Harvard University.

- ***Teachers' pedagogical content knowledge (9)***

"How could I illustrate division of a fraction by a fraction when I mean 'shared between' and not 'how many'?"

Judy Mousley, Deakin University, Geelong, Victoria.

- ***Teachers' own mathematical content knowledge (8)***

"This [rectangle, made up of a series of smaller pieces] is not a real square metre—you cannot say that one square metre is one square metre if is not a square, can you?"

Ulla Oberg, Lund University, Sweden.

- ***Student understanding (8)***

"Isn't it easier just to tell (or show) them how to do it?"

Clarence Dockweiler, Texas A&M University.

- ***Teacher time (8)***

"I'm really confused and frustrated. I work very hard as a teacher. It seems that all I do is work day and night. You've given us so many wonderful ideas today, but what it all boils down to is I'll have to work even harder. Well, I can't work any harder, I simply can't do everything you're asking of me. How do I decide what is really important?"

Frank Lester, Indiana University, Bloomington.

- ***Professional growth of teachers/the change process (7)***

"My kids are doing well; why do I have to change?"

Jack Price, President, National Council of Teachers of Mathematics, USA.

- ***Organisation/management issues (5)***

[In situations in which new programs, guidelines or policies need to be addressed and it is true that they do involve some extra effort]:

"I can't get a turn to use the computer lab"/"I teach in a different room every lesson"/"Do you expect me to carry these materials with me every lesson?"

Howard Reeves, Tasmanian Department of Education.

- ***Parents (4)***

"What can you do about parents who show no interest in their children's mathematics learning?"

Rosanne Fulton, Boulder Public Schools, Colorado.

- ***General pedagogy (3)***

"How can a teacher possibly teach in such an active way with so much stimuli coming in from so many students?"

Frederick Silverman, University of Northern Colorado.

- ***Research (2)***

"What are some of the latest studies in learning?"

Gaea Leinhardt, University of Pittsburgh.

- ***External support/pressure (2)***

"Who is pushing this?" [by "this", teachers meant the unit standards of the New Zealand Qualifications Authority].

Derek Holton, University of Otago, New Zealand.

- ***Resources (2)***

"How am I going to have my students do 1996 math stuff when they are using 1986 textbooks?"

James Bierden, Rhode Island College.

- ***"The basics" (2)***

"How do I get students to remember "the basics"? They all use calculators to perform simple additions these days".

Vince Geiger, Hillbrook Anglican School, Enoggera, Queensland.

- ***Time for mathematics (1)***

"You don't really expect me to teach this stuff (or use this stuff) and teach all the math I have to cover, do you?"

Shari Coston, Arkansas Educational Renewal Consortium, Arkansas.

- **Gender (1)**

"What do I do to eliminate gender differences in math in my classroom?"
Elizabeth Fennema, University of Wisconsin.

- ***Problem solving (1)***

"Problem solving is great—but how do I know the kids are learning anything?"
Tony Herrington, Edith Cowan University, Perth.

- ***Integration with other disciplines (1)***

"How do you integrate mathematics with the other school disciplines?"
Frederick Silverman, University of Northern Colorado.

- ***Relevance (1)***

"Under what circumstances does the erstwhile engaging context of a mathematizable situation become distracting to my students rather than empowering?"
Jon Manon, University of Delaware.

I did not use any code that related to teachers' beliefs about mathematics learning or a philosophy of teaching and learning, as, in a sense, all questions reveal something about where the questioner is coming from.

A careful examination of these questions reinforces the statement that a given question can relate to a variety of themes. In the following section, I will discuss several of these themes and implications for researchers, mathematics educators, and others involved in the professional development of mathematics teachers.

Discussion

As indicated earlier, these questions were drawn from a convenience sample, and therefore there is not an even spread of questions from the five countries involved. I acknowledge the helpful comment of one reviewer that the data are not necessarily questions asked by a large number of students or questions frequently asked, but, as appropriate to my original request, questions that respondents found interesting or difficult to answer. It is quite conceivable that other questions are asked more frequently, but are considered by respondents to be neither difficult nor interesting. Nevertheless, several important points can be drawn from the data.

Commonality of issues

Although there are several aspects of particular questions that tie them to local contexts (state and national), a frequent comment from those who have considered the list subsequently is that the issues raised appear to be "universal".

Assessment was the most common issue to be raised, and this probably reflects a growing worldwide realisation that given new content and pedagogy in the mathematics classroom, pencil and paper tests are limited in providing information on growth of mathematical understanding (Ellerton & Clements, 1995). Questions such as "how do we know when the pupils know?" (Laurinda Brown, University of Bristol) were common. Interestingly, this realisation was balanced by questions which referred to the pressures created by external authorities in, for example, the use of standardised tests across a state or country in measuring performance.

If the number of questions related to difficult/unmotivated/special needs students were combined with those relating to catering for mixed abilities, the total would exceed the number of assessment questions. In my experience, this concern with catering for the wide range of students that arrive in our classrooms is longstanding, and any answers can only be partial ones. One apparent difference across countries however (one which is also reinforced by personal experience), is that such concerns in the United States tend to reside more with the lack of motivation and interest of students than with competence.

Do these questions have answers?

Laurinda Brown of the University of Bristol made the following important point relating to context: "You seem to think that it is possible to 'answer' such questions but surely any (inevitably?) partial solution is going to be embedded in a context and is therefore not easily transferable?" (email communication, February 16, 1996). Local factors and issues are clearly important, but hopefully a response to such questions by someone who is unaware of the local situation can nevertheless provide some food for thought, and we must give teachers credit for adapting a response to their own situation.

The point should also be made that we are unwise to claim complete answers in any situation, and one of the points that has been made to me is that many of the questioners appear to presume that there *is* an answer to each of these questions, and that it is only a matter of finding the right person to ask.

Bill Juraschek from the University of Colorado—Denver in commenting that the questions seemed similar to those that he was asked 25 years ago, described his reaction to the following question offered in his master's class: "Assessing by probing for conceptual understanding seems a good idea, but how do you make it easy?" Bill describes his reaction:

> After I calmed down and allowed the poor fellow to return to the room . . . Actually this query, variants of which appear throughout your list, belies the major paradox: humans always want to "make things easy" in the sense that they want to reduce problems to predictable routines—which is a major goal of mathematical research—yet teaching, historically, is intractable in this regard. As long as the system assumes the factory-production model the questions will never go away.
>
> (email communication, February 27, 1996)

The notion that *someone* has the easy answer is also evident in this question provided by Marj Horne (Australian Catholic University, Victoria): "Which is the best textbook? And if there isn't one, why not?" Many teachers continue to seek "teacher-proof" materials and methods, particularly in a climate of increased class sizes and increased teaching allotments, but a discussion of the limitations of such a concept is important.

On the theme of teacher-proof materials, I recall leading a two part workshop with teachers in Providence, Rhode Island, with the sessions spaced two months apart. On the first day, a teacher asked a question similar to that raised by Marj Horne: "Why doesn't someone just put all this good stuff together, give it to me, and I'll teach it?"

Two months later, the same teacher had by then tried a range of ideas that he had gained from myself and other teachers. However, in seeing the power of problem solving approaches in his classroom, he had then, in his words, "thrown away the textbook", and in the recall session shared photographs and samples of student work from five or six wonderful mathematical investigations that *he* had developed. This teacher, who had sought to be shown the easy way, had learned that with the provision of some innovative models of what the mathematics classroom might be like, he was then in a position to take these models and create his own activities and tasks, as he responded to the needs and interests of his students.

Teachers' content knowledge

A question offered by Joseph Zilliox from the University of Hawaii ("How can I teach topics in maths that I myself do not know very well—content about which I am not very confident?") reflects an important issue at a time when the numbers of qualified mathematics teachers appears inadequate, and relatively few mathematics teachers are in training in Australia.

I believe that one of the reasons why content knowledge becomes even more important in an era of curriculum reform is because it is less common for a typical lesson to consist of worked examples on the blackboard followed by students practising a single algorithm over and over. Such teaching requires minimum preparation and mathematical understanding on the part of the teacher. On the other hand, teaching in a problem solving environment demands a *connected* view of mathematics. As Brophy (1991) claimed:

> Where [teachers'] knowledge is more explicit, better connected, and more integrated, they will tend to teach the subject more dynamically, represent it in more varied ways, and encourage and respond fully to student comments and questions. Where their knowledge is limited, they will tend to depend on the text for content, de-emphasize interactive discourse in favor of seatwork assignments, and in general, portray the subject as a collection of static, factual knowledge. (p. 352)

Having said that, I believe that the difficulties that teachers experience with content are not due to a lack of study of traditional university mathematics units (see Fennema & Franke 1992), but rather the lack of opportunity for teachers to study mathematics in the kind of problem solving environment which they are being encouraged to create for their own students.

In a study of teachers struggling with the content demands of a problem solving environment (Clarke, in press), I asked an experienced middle school teacher to suggest the best way in which she could be supported in her own mathematical growth. She commented that she had learned more mathematics over the previous two years during which she had been teaching problem solving units than during any previous time, and she attributed that to the opportunity to team-teach with someone "strong mathematically." Of equivalent importance to this teacher was the chance to work collaboratively with three colleagues, when the teachers planned their teaching of particular activities or units of work. She felt that the provision of such opportunities for other teachers would be the most helpful strategy in helping teachers to understand important mathematical ideas (Interview, Ms Bartlett, September 10, 1992).

A lack of focus on pedagogical content knowledge

In recent years, I have been concerned that there is relatively little discussion at conferences and inservice programs of mathematics teachers (as reflected in session titles) on matters relating to *pedagogical content knowledge*. This term has been coined by Shulman to refer to

> The most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others. . . . [including] an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning. (Shulman, 1987, p. 9)

Only eight questions out of the 110 which I received were of the kind that sought advice or help on student understanding, misconceptions and possible growth paths, or the best explanations, examples and illustrations for particular content. I have also noticed that few students wishing to undertake research studies through our institution identify such areas as those upon which they wish to focus, preferring rather to concentrate on student affect, classroom organisational structures, and assessment alternatives, for example.

If a major part of the task of mathematics teaching is to find the best ways to provide support to the mathematical growth of students in particular content areas, then it is reasonable to assume that many of the questions asked of mathematics educators and others working with teachers might relate to the challenges of providing such appropriate support, and building pedagogical content knowledge.

Implications for Further Research and the Work of Mathematics Educators

Although this study was small and no claim is made for representativeness other than the extent to which the questions "ring bells" for colleagues with whom I have shared the list, it nevertheless provides a starting point for a study of teachers' concerns as reflected in the questions. Other researchers may wish to conduct similar, larger, and more systematic studies, which seek to extend the range of participants to other countries, and to define more precisely the setting in which the question was asked. Another study might seek responses to such questions from eminent mathematics researchers and educators.

As well as giving an idea of the universality of issues in mathematics education, these questions have the potential to be used by academics and consultants working with networks of teachers, while teaching courses or leading inservice programs. Readers of this paper could usefully share the questions with teachers in their own districts or work settings, encouraging them to identify those of personal interest and to share their own "answers" (if answers are possible), and, perhaps more importantly, their views on the issues that underpin these questions. So many of the questions reveal much about the teachers' beliefs and philosophies about mathematics teaching and learning.

A final quote from a teacher that was coded under the heading of "Teachers' Content Knowledge" is my favourite question so far, and one which illustrates the benefits of this small study, showing as it does the way in which a short question can encapsulate a whole range of issues in mathematics education:

> Could you show me how to add fractions? I know how to teach it, I'm just not sure how to do it".
> (Alistair McIntosh, Edith Cowan University, Perth).

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Standardising the Mathematics Curriculum: Some Historical and Comparative Perspectives

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This paper provides historical and comparative perspectives on current pressures to standardise mathematics curricula. Two main questions are considered: (a) What are the forces behind the move towards core mathematics curricula? and (b) Are core curricula likely to generate more equitable school mathematics programs? The first question is discussed in terms of accountability, colonialism, and politics. For the second question, evidence on equity considerations is provided from developments in the United Kingdom and in the United States.

Over the past decade Australian mathematicians, mathematics educators, and mathematics teachers—the traditional guardians of Australian mathematics curricula—have been confronted by influential education bureaucrats who believe that the authority and responsibility for making policy decisions on curriculum issues, including decisions on the content, structure and assessment of school mathematics, belongs with them and *not* with the mathematics community (Thomas & Clements, 1994).

This paper seeks to provide historical and comparative perspectives on current moves to standardise mathematics curriculum in the States and Territories of Australia. The following two questions, not entirely independent of each other, will be examined:

1. What are the forces influencing the move towards core mathematics curricula?
2. Is there any evidence to support the view that a legislated core curriculum is a prerequisite for the establishment of more equitable school mathematics programs?

Forces Influencing the Move Towards Core Mathematics Curricula

Four main categories of forces will be identified: (a) the drive towards accountability in Australian education systems (b) colonialist tendencies, (c) political dimensions, and (d) equity and social justice considerations. The first three will be dealt with in this section, and the last in the section of the paper dealing with equity.

The Drive Towards Accountability in Australian Education Systems

Throughout the 1980s, many State and Territory education bureaucrats, especially curriculum directors, believed that standards in literacy and numeracy were falling across Australia (see, for example, Priest, 1981). They were concerned that they had no systematic way of proving that this was the case (Bessant, 1989). Arguments had been put forward in the United Kingdom (Graham, 1993) and in the United States of America (National Commission on Excellence in Education, 1983) that the absence of central control over curricula in these nations had resulted in lower standards, and since Australia also did not have a national curriculum, there were fears that its school programs were not generating the highest possible standards of literacy and numeracy (Marsh, 1994).

The need to specify clear indicators for desired learning outcomes is largely accepted by Australian education bureaucrats (Cuttance, 1995). During the 1990s calls for the development of value-added accountability methods of ranking Australian schools, whereby schools are compared according to relative gains on tests made several years apart, have been regularly heard in all States and Territories. If the performances of schools and school systems were to be compared then a core curriculum was needed from which pencil-and-paper items could be constructed. In the domain of mathematics education, the national *Mathematics Profile* (Curriculum Corporation, 1994) appears to have provided an adequate framework for this purpose (McGaw, 1995).

International comparisons of achievement. Concerns for standards and for the need to provide accountability of schools and school systems, were fuelled by studies which suggested that the mathematics curricula in schools in the United States of America were inferior to the national curricula in Japan, Korea, and Taiwan (Berliner, 1995; McDonnell, 1995; McKnight et al., 1987; Stevenson & Stigler, 1992).

Australian extrapolations. In the late 1980s Australian politicians and education bureaucrats seemed to accept the view that curricula in the United Kingdom, the United States, and therefore probably Australia too, had been found wanting (Department of Employment, Education and Training, 1988), and that something needed to be done if Australian education systems were to be able to demonstrate appropriate levels of accountability (Bartlett, Knight, Lingard & Porter, 1994). This view was strengthened by the assumption—which was probably without foundation (Berliner, 1995; Ellerton & Clements, 1991; McGaw, 1994, Priest, 1981)—that over the past 20 years there had been significant declines in mathematical performance in schools in the UK, in the USA, and in Australia. Therefore, the Australian educators reasoned, it would be wise to follow the British move towards a more standardised education agenda (Piper, 1989).

Such a move was not entirely logical. Anyone wishing to argue that a national curriculum is likely to be associated with better teaching and learning of mathematics is faced with the problem of explaining why so many nations in the Asia-Pacific region with legislated national mathematics curricula (for example, Malaysia, Papua New Guinea, The Philippines, and Thailand) have never obtained above-average results in international surveys of mathematics achievement.

Colonialist Tendencies

Truran (1994) has disputed Clements, Grimison and Ellerton's (1989) claim that important developments in the history of Australian mathematics education have often occurred because of the strength of "colonial echo" forces. Truran maintained that such colonial echo explanations are "too simplistic and too general" (p. 632). On the basis of his historical analysis of the introduction of probability into school mathematics programs in South Australia, he argued that "muddling through" would provide a better description of curriculum development than either "colonial echo" or "mature development" (p. 638).

The national curriculum framework developed in Australia in the late 1980s and early 1990s was similar to the national curriculum which had been introduced in the United Kingdom in the second half of the 1980s (Ellerton & Clements, 1994; Piper, 1989, 1992). There can also be little doubt that standards-based approaches, deriving largely from the United States of America, are increasingly influential in Australia (McDonnell, 1995; Pascoe, 1995). Given these similarities, it is worth considering whether allegations of colonialism in the Australian education context can be substantiated.

Without resorting to a "colonial echo" interpretation, how can one explain why the structure of the national curriculum framework developed in Australia was so remarkably similar to the outcomes-based education (OBE) structure of the national curriculum in the United Kingdom which had been developed five years earlier (Ellerton & Clements, 1994; Hughes, 1990)? The same five major content Strands (Number, Measurement, Space, Chance and Data, and Algebra) as those incorporated in the UK national mathematics curriculum were to be found in the nationally developed mathematics curricula for Australia. Each curriculum assumed four "bands" of schooling (lower primary, upper primary, lower secondary, and upper secondary), and the notion of a profile based on "levels" (10 in the United Kingdom, and 8 in Australia) was common.

Expressions of colonialism. Colonialist tendencies in education were often expressed through the power of money and association. In 1989, for example, Phillip Hughes, Professor of Education at the University of Tasmania and President of the Australian College of Education, participated in a three-month study, sponsored by the British Council, into the national curriculum in England and Wales. Hughes (1990) subsequently wrote an influential booklet in which he stated that Australia could learn much from the national curriculum in the United Kingdom which permitted "individual schools and clusters of schools to undertake very effective curriculum planning" (p. 17).

During the period 1988–1994 a succession of senior educators from the United Kingdom visited Australia, and delivered invited plenary addresses on national curriculum issues. In September 1990, for example, Professor David Hargreaves, of the University of Cambridge, addressed a "National Conference of Australian Educators," on how the national testing component of the UK national curriculum provided straightforward answers to three questions asked by parents: (a) What were the achievements of their children relative to other children of the same age? (b) How well were their children progressing in key learning areas? and (c) Has there been a rise or decline in general student achievement in their children's school, and in the nation's schools in general? (Hughes, 1990). When, in January 1994, the recently established Australian Teaching Council conducted a Summer School aimed at helping teachers understand teaching and assessment implications of the nationally developed curriculum statements, profiles and key competencies, two of the four keynote speakers were Professors of Education from the United Kingdom (Cooper, 1994).

There were seven plenary speakers at a conference on "Efficiency and Equity in Education Policy" held in Canberra in September 1995, comprising two senior educators from England, two professors from the United States, and three senior academics from Economics Departments in Australian universities. One of the educators from England, a Dr Geraint Johnes, of the Centre for Research in the Economics of Education at the University of Lancaster, told those at the Conference, that in the United Kingdom "as a means of facilitating parental choice, publication of a series of performance indicators had been introduced." Johnes (1995) continued:

> The examination performance and absenteeism records of pupils at each school are now published annually. These are well publicised; indeed detailed league tables appear in the daily newspapers. Upon their introduction, such performance indicators were controversial, not least because they make no allowance for inter-school differences in the quality of pupil intake. National tests have been introduced for secondary school entrants which will, in a few years time, make possible the development of value-added performance indicators. (p. 129)

Johnes drew attention to how schools could, by selective entry and exclusion, manipulate their position in performance tables. According to Johnes, schools which performed well in the league tables did so partly because they are able to deny access to weaker pupils. He concluded his address by stating that Britain had been to the fore in the development of performance indicators in education, and that "the rest of the world can benefit from both positive and negative aspects of Britain's experience" (p. 139).

One of the American professors who spoke at the Conference, Lorraine McDonnell (Professor of Political Science and Education at the University of California), made many references to policies affecting mathematics education in the United States of America. McDonnell quoted extensively from the National Council of Teachers of Mathematics' (1989) *Standards* document, and argued that research indicated unambiguously that mandated national or statewide testing has a powerful effect on classroom teaching, in that "teachers teach to the test" (McDonnell, 1995, p. 246).

In responding to the paper by McDonnell, Bill Scales (1995), Chair of the Industry Commission in Australia, spoke of the parallels that could be drawn between Australian and United States' experiences with national curricula. He also spoke of the need for Australia to define carefully student learning outcomes and indicators in order that "an overall assessment of school system performance" could be made. Statewide assessments of student learning were needed, Scales said, because the Review of the Commonwealth-State Service Provision, established by the Premiers, Chief Ministers, and Prime Minister, had stipulated that such data would enable comparisons to be made between school systems. Scales (1995) went on to say that although there was no system of national assessment of student learning in Australia, plans were being made to solve the problem:

> The Review is currently examining whether bridges built between the States' test instruments to establish some comparability over time (for example, by embedding similar questions, administering existing tests to samples of students in other education systems) ... Comparisons between systems will ultimately enhance the transparency that will arise from reporting state-specific learning outcomes. (p. 280)

There was not one mathematics educator present at this high-level conference, and the thinking of those who would formulate policies which were likely to influence school mathematics in Australian was being shaped by economists, politicians, and visiting professors from the United Kingdom and the United States of America.

The evidence suggests that although the process of "muddling through" in education policy occurs within Australia, those controlling the direction of the "muddling" still look to the United Kingdom, the United States of America and, increasingly, to the discipline of economics for inspiration. In particular, policies affecting practices in school mathematics have been, and continue to be, based on decisions made by governments in the United Kingdom and the United States of America.

Political Dimensions

Although the national curriculum framework was not approved by the AEC in 1993 (the vote in July 1993 being lost by 5 votes to 4, with voting being along party lines), the Federal Labor Party used tied grants to ensure that the national curriculum statements and profiles became a basis for curriculum development in the States and Territories. As the Deputy Prime Minister, Kim Beazley, was quoted as saying after the July 1993 decision: "The two billion dollars Canberra provides to the States each year provided plenty of ice for the Commonwealth to skate on if it has to," and "schools would get the national curriculum whether through a sensible co-operative arrangement or ... the old way" (quoted in Ellerton & Clements, 1994, pp. 269–270). Subsequently, State and Territory governments were denied adequate funding for the professional development of their teachers unless they participated in a Federally-funded National Professional Development Program (NPDP). The Federal Government also funded the creation of the Australian Teaching Council (ATC) which, among other things, conducted residential workshops at which teachers were familiarised with the national curriculum documents (Ellerton & Clements, 1994). Over the past ten years, political considerations have never been far away from all matters to do with the development of the national curriculum framework in Australia (Ellerton & Clements, 1994; Eltis, 1995; Marsh, 1994).

Less than the most expert advice. It will be useful, here, to provide an example of the kinds of action which resulted from the politicisation of the national curriculum. In August 1995 Faculties of Education in Australian universities received a copy of a report written by Susan Pascoe, entitled "The standards movement in America: A United States/Australia Comparison." An enclosure signed by Alan Ruby, Deputy Secretary of DEET, stated that the writing of the report had been funded by DEET, and Pascoe's (1995) report indicated that the research project had been undertaken by the Victorian Catholic Education Office for DEET. The purpose of Pascoe's project had been to investigate, during a visit to the United States, four important educational issues: (a) the strategies used by federal agencies to achieve the involvement of states and localities; (b) the kinds of assessment procedures being used and their relationship to benchmarks and standards; (c) examples of best practice in standards monitoring; and (d) the impact of international imperatives on moves towards national literacy assessments.

In her comprehensive and balanced report, Pascoe (1995) referred extensively to NCTM's (1989) *Standards* document, and therefore clearly entered the domain of mathematics education. Yet, in August 1993 the Mathematics Education Research Group of Australasia (MERGA) had received a letter from Alan Ruby indicating that DEET was not prepared to accept an offer by MERGA to advise DEET on how mathematics education research findings might inform government policy in respect of contentious mathematics education issues (Ellerton & Clements, 1994). The funding of Pascoe's investigation, and of the preparation and dissemination of her report, indicated, however, that DEET was prepared to place some credence in an investigation which would be carried out by someone unknown in the Australian mathematics education community, into issues which were associated with, among other things, important recent mathematics education developments in the United States.

This episode is consistent with the unwillingness, evident between 1987 and 1995, of most Canberra education bureaucrats to seek guidance from the nation's top mathematicians and mathematics education researchers on policies affecting mathematics education (Ellerton & Clements, 1994). Much of Federal government policy on the national curriculum and the quality movement in schools and higher education was imported from the UK and the USA. One explanation of why this happened is that, from the mid-1980s, the Departments of Employment, Education and Training in Canberra and in the States and Territories were, increasingly, controlled by economists or Total Quality Management (TQM) experts who had little direct experience in education settings but had large knowledge of overseas trends linking economics and education (Pusey, 1991).

Politics and sincerity. For all the questionable tactics it used in its struggle to achieve a national curriculum, the Australian Labor Party clearly believed that a national curriculum was needed to achieve important education, social, and economic goals. University-based mathematicians and mathematics educators, in particular, were consistently regarded as representing privilege. As one Commonwealth Minister stated in 1993, "with school retention rates at 80%, the idea that schools are nothing more than a source of recruits for high level university study in Maths and Physics is no longer sensible" (quoted in Ellerton & Clements, 1994, p. 270).

The Labor Party gained the cooperation of trade unions, employers, and special interest groups, as well as State and Territory education bureaucrats, in its quest to gain acceptance not only for the national curriculum concept, but also for the development of work-related competencies. This was Labor Party policy, and Marsh's (1994) book makes it clear that the politicians and bureaucrats saw themselves as leading the nation's thinking on educational matters. That is why they adopted what Marsh has called an "Authority Model" in their efforts to achieve a national curriculum. They saw it as appropriate for them to use their political clout to develop and implement curriculum policy. After all, they had been elected to provide strong and decisive leadership.

Core Curriculum and Equity in School Mathematics

Equity and Social Justice in Mathematics Education

In the 1970s and 1980s those who advocated school-based curriculum development believed that the closer those developing curricula were to the "action" in schools the more likely it was that the curricula would meet the needs of individual students (Hannan, 1985). The kind of data reported by Bourke and Parkin (1977), in which the mean performance of Aboriginal children on tests of basic literacy and numeracy skills was shown to be well below the mean performance of other Australian children, was taken as proof that equity in education was unlikely to be achieved by a core curriculum approach.

However, towards the end of the 1980s and into the 1990s some influential educators who had previously argued against core curricula began to shift ground. Hannan (1993), in an obituary for Garth Boomer, stated that Boomer had been accurate when he recognised that the progressive ideas and methods of the 1970s and early 1980s—which had been needed to free a cramped system in Australian education—"became soft" (p. 8). Too often, Hannan said, school-based curricula had resulted in working-class children and other potentially disadvantaged children taking the "soft options," which effectively made it impossible for them to gain entry to prestigious tertiary courses.

The notion of a "core" or "entitlements" mathematics curriculum has become a *cause celebre* of the NCTM in the United States of America. The NCTM's (1995a) position statement on "mathematics for language minority students" is explicit:

> Every student is entitled to a mathematics education that provides preparation for a full range of careers having mathematical prerequisites and to be an informed citizen. Cultural background and language must not be a barrier to full participation in mathematics programs. Therefore ... all students, regardless of their language background and language must study a core curriculum in mathematics based on the NCTM standards. (p. 20)

This viewpoint is consistent with "Goals 2000," a Clinton administration initiative passed by Congress in 1994, which called for states to develop voluntary content and performance standards to be applied to *all* students (McDonnell, 1995).

The December 1995 NCTM News Bulletin. The political dimensions associated with the three *Standards* documents (NCTM, 1989, 1991, 1995a) in the United States need to be understood in Australia. NCTM, its affiliated organisations, and US academic mathematics educators have been in a privileged position in that they have been well supported by governments and education authorities in providing quality professional development programs aimed at assisting the implementation of the *Standards.*

The politics of the *Standards* could be recognised in a recent (December 1995) NCTM *News Bulletin,* in which Richard Long, the NCTM's "government relations specialist," warned members that there were proposals to take all of the funds away from "Goals 2000," or to turn it into a block grant. According to Long, such a change would "have a significant impact on the mathematics education community," in that Federal support for the professional development of teachers of mathematics would become more difficult to obtain. NCTM members were advised to lobby their local members of Congress, and if they wanted advice on what to say they could contact NCTM headquarters.

The same NCTM *News Bulletin* reported that, from an equity perspective, positive changes were taking place in American science and mathematics education. Nevertheless, the *News Bulletin* said, cuts in Washington threatened to slow down the changes (NCTM, 1995b). American students, as a whole, were doing better at elementary mathematics, yet "26% of black students scored at or above the basic level in mathematics on the NAEP in 1992, whereas 73% of white students scored at or above this level." Enrolments in Algebra 2 classes were up, and this was especially true for Hispanic and American Indian students. By contrast, the national percentage of high-school students taking general, remedial, or consumer mathematics had declined by one-third since 1990.

In another section of the *News Bulletin*, a "roving reporter" quoted answers given by five mathematics educators to the question: "Has there been a downside to expecting students to take higher-level mathematics courses? If so, what is it?" The five respondents found very few negative effects, although one made the ominous comment that many teachers were feeling challenged, because they had "to work with a more diverse student population and that causes more work for them."

None of the "advances" mentioned in the *News Bulletin* could be directly attributed to the *Standards* documents, and no convincing data were provided on the issue of whether most teachers were coping with the diversity. Interestingly, it was precisely this "coping-with-diversity" issue on which the national curriculum in the United Kingdom, and the nationally developed curriculum in Australia has foundered (Cooper, 1994; Ellerton & Clements, 1994; Groundwater-Smith, 1993; Kelly, 1990; Lanyon, 1995; Truran, 1993).

"The Burden." Some evidence is available related to the "coping-with-diversity" issue with respect to the implementation of the *Standards* in the United States. An evaluation study, directed by Robert Stake, into the effectiveness of a Mathematics and Science teacher professional development program in Chicago in the early 1990s, provides commentary on the implementation of NCTM's (1989, 1991) *Standards* documents (see Stake et al. 1994). Questions such as "Were the schools and teachers most needing help the ones participating?" and "How has teacher knowledge, pedagogy, and belief been affected by participation in professional development activities?" were investigated.

The evaluation team's report, a 120-page document titled *The Burden,* provided a powerful commentary on the negative effects of the extra work imposed on teachers by the volume of mandatory external projects including standardised testing programs, and detailed prescription of curriculum outcomes (Stake et al., 1994). The report indicated that the load schools could bear had not adequately been assessed, and that unrealistically, the State, the reform movement, and those organisations providing assistance contributed to the burden. The following quotation illustrates the tenor of the evaluation:

> They are not helping school people decide, given the resources available, what responsibilities to diminish. Part of the burden is those long lists of state, national, community, and professional goals and standards, idealistic views of what ought to be. Voltaire's words were "The best is the enemy of the good." Some schools like Westlake have been getting better gradually but far, far too slowly. Long term failure is almost certain, partly because the outsiders, as well as the Westlake LSC [Local School Council], have too grand a view of what they should be. The vision of democracy-in-action to effect school reform is not possible until committees are setting the goals as well as the programs. (pp. 84-85)

Although the report is about schools and a professional development program in a district in Chicago, the reader is almost compelled to reflect on the health of mathematics education in the United States, even under a *Standards* régime.

Lessons for Australia? It is too early to know whether the NCTM's belief that a core mathematics curriculum will result in more students being empowered to use and enjoy the mathematics they study at school. In Australia, there is certainly a belief that the *Standards* have had "a powerful effect on classroom teaching in the United States," and that a similar approach in Australia, based on a core curriculum with "a complete set of rigorous indicators" and associated statewide tests, would have a strong and positive impact on teachers in schools (Scales, 1995). Those in high places in Australia also seem to believe that standards setting can play a role in raising the expectations that teachers hold for the achievement of students in potentially disadvantaged target groups, thereby "removing the risk of institutionalised underperformance" (Scales, 1995, p. 281).

A National Statement on Mathematics for Australian Schools (Australian Education Council, 1991) stated that in Australia we need to "be vigilant that mathematical interest and talent are able to flourish regardless of geographic location, gender, social class or ethnic origins" (p. 10). Although there is no evidence that the OBE approach adopted in the *Mathematics Profile* (Curriculum Corporation, 1994) offers a suitable framework for the variety of different curriculum provisions and teaching strategies "needed in order to accommodate a diversity of individual needs" (Australian Education Council, 1991, p. 10), the *Profile* has constituted an excellent base for statewide testing and the implementation of value-added accountability schemes (McGaw, 1995; Scales 1995).

The balance of evidence from the United States, the United Kingdom, and from Australia, does not support the view that core mathematics curriculum approaches will assist the introduction of fairer, more equitable, school mathematics programs.

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Using Data Maps to Analyse Students' Strategy Use in Problem Solving: A Visual Technique

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Qualitative data analysis has become increasingly important in mathematics education research. However the conceptualisation of the data and the fulfillment of the interactive assumption of qualitative analysis are a concern. An alternative to a linguistic approach to analysis is a visual approach. The visual analysis of data utilises wholistic data displays, referred to as data maps, and employs visual reasoning. The data maps, which are produced using drawing software, can enhance conceptualisation and facilitate the interactive process of analysis.

In recent years the use of qualitative data analysis in mathematics education research has expanded and provided useful insights into the teaching and learning of mathematics. However the use of text as the "unit" in analysis is recognised as labour intensive (Bogdan & Biklen, 1982), time-consuming and difficult (Pitman & Maxwell, 1992). An alternative to the use of a linguistic representations and sequential reasoning in data analysis is the use of visual representations and visual reasoning. Such a perspective does not seek to degrade the *status quo* but rather to recognise the limitations of tools and techniques and attempt to overcome them: "New perspectives provide new order to the things before us" (Webb & Glesne, 1992, p. 795). Therefore, the purpose of this paper is to explore whether there is any benefit in representing and analysing qualitative data visually. In order to investigate this question a pilot analysis of data is reported from a study of students' use of diagrams in problem solving. This paper specifically addresses 1) the rationale for the visual analysis of data, 2) the development of a visual tool, and 3) the technique of visual analysis.

The Rationale for the Visual Analysis of Data

Both qualitative and quantitative data analysis consist of three phases: data reduction, data display, and drawing inferences. However it is too simplistic to consider the differences between the outcomes of the analyses as a difference between the use of words and the use of numbers. The fundamental difference between these forms of analyses is in the *process* of data analysis and the *state of development* of the data analysis techniques (Miles & Huberman, 1994). Quantitative data analysis is essentially a sequential process which utilises statistical techniques that are reported as, for example, means, correlation tables and levels of significance. However the validity of the data analysis is dependent on an understanding of the assumptions of the techniques and their appropriate use. In contrast to the well-developed process of quantitative analysis, qualitative analysis is still in a "pioneering" state. Although guidance is provided for qualitative analysis, there appears to be no specific guidance on the visual analysis of data (e.g., LeCompte, Millroy & Preissle, 1992; Miles & Huberman, 1994). Quantitative analysis however utilises both well known and lesser known visual display techniques (Tufte, 1983; 1990), for example graphs and Chernoff faces respectively (Chernoff, 1973). Unlike quantitative analysis there are no shared conventions for the analysis or reporting of qualitative data, but there is agreement that the analysis should be an interactive and recursive process (Miles & Huberman, 1994). The lack of convention however is not to be interpreted as a need for convention. Qualitative studies are by their very nature diverse and accordingly require an analysis that suits the study. Thus, qualitative analysis is based on a criterion reference system rather than a norm reference system. Implicit in qualitative analysis is the obligation to provide criteria as a backdrop for an audit trail, which provides sufficient detail about the data and the analysis to enable the reader to evaluate the trustworthiness of the conclusions (Miles & Huberman, 1994).

In qualitative analysis the use of software, such as NUD.IST (1994), facilitates the exploration of patterns and relationships by positing "what if" scenarios. The utility of such software supports Pea's (1985) notion that technology is a powerful tool that extends thinking "beyond amplification" by reorganising mental functioning and thus engaging cognitive processes in qualitatively different ways. Special purpose software packages for qualitative data analyses differ, but text is normally used in the display, for example in a concept map (Richards & Richards, 1994). There appears to be no software in common use in data display that employs visual representations as a means of exploring "what if" scenarios (Miles & Weitzman, 1994), although inexpensive and easily managed general purpose drawing packages do exist (e.g., SmartDraw Pro, 1995). As cognition is affected by the use of visual representations (Barwise & Etchemendy, 1991) and visual reasoning is implicated in problem solving (National Council of Teachers of Mathematics, 1989) the visual analysis of data appears to have potential.

Visualisation and the use of diagrams are recommended in mathematics education (National Council of Teachers of Mathematics, 1989) and are credited as the source of insight for many eminent people, such as Einstein (Boden, 1990). A diagram is an external manifestation of visualisation (Presmeg, 1986) and is defined as an abstract visual representation that exploits spatial layout in a meaningful way, enabling complex processes and structures to be represented wholistically (Winn, 1987). Diagram use in problem solving has particular advantages related to working memory (van Essen & Hamaker, 1990), the conceptualisation of the problem (Yancey, 1981), and the informational content of the representation (Larkin & Simon, 1987). A further advantage of diagrams is that they provide a visual alternative to words (Mayer & Gallini, 1990). The utility of visual-spatial methods is recognised both by expert mathematicians (Eisenberg & Dreyfus, 1986) and in cross-cultural studies (Shigematsu & Sowder, 1994). Changing the mode of a representation, for example from a linguistic to a visual form, may also be advantageous as a means of knowledge generation (Karmiloff-Smith, 1990). However whereas linguistic representations utilise sequential reasoning, visual representations utilise visual reasoning (Barwise & Etchemendy, 1991). The key difference between sequential and visual reasoning is the respective fragmentary and wholistic use of the representations.

The Development of a Visual Tool: The Data Map

Communication systems are not developed arbitrarily but are dictated by the tools available and the mode of communication. A diagram utilises graphic elements, including points, lines, shapes, shading, direction and location. Symbols that are visually associated with actions can also be incorporated to facilitate communication, for example, a hexagon can be used to signal a "stop" in a procedure. In a diagram, all elements of the display are available simultaneously and thus interpretation requires visual reasoning. In decoding diagrams, attention can oscillate between the whole diagram and sections of the diagram. Diagrams that serve to organise and display the data, and orient the decoder are henceforth referred to as data maps. These maps are not to be considered as absolute depictions of data but are relative to the question they seek to explore. Hence a variety of maps could be produced from the same data.

A Pilot Analysis of Qualitative Data Using the Data Map: A Visual Technique

Background to the study

The subjects of the study were 12 Year 5 students who were participants in a teaching experiment within the context of a case study. All subjects were interviewed individually twice and in each interview were presented with five novel problems. The problems in the two interviews were isomorphic. The first interview was conducted before the commencement of an instructional program of 12 half hour lessons spread over a four week period which was designed to facilitate the use of the strategy *draw a diagram* in novel problem solving. The second interview was conducted at the conclusion of the instructional program. The interview sessions were video-taped and subsequently transcribed. The interviews were of approximately thirty minutes duration and consisted of two phases. In Phase A of the interview the subjects attempted each of the tasks independently. During Phase B the subjects were asked to explain their solution strategies. The interviewer also probed for meaning and if warranted, provided minimal support in order to determine whether scaffolding (Vygotsky, 1978) facilitated the solution of tasks which were unable to be solved independently. The interviewer was known to the subjects through prior classroom involvement.

The evaluation of a teaching experiment requires the detection of changes that have presumably occurred as a result of instruction. Therefore, the change in the selection and success of problem solving strategies used by the subjects is of particular interest in the analysis. Although teaching experiments may employ pre- and post-tests as end product measures, a comparison of the strategies used in the interviews was considered as a more appropriate measure of change when evaluating diagram use (Diezmann, 1995).

The data maps

Two data maps are presented to exemplify the visual analysis of data from the study. The graphic code, which was developed to represent the components of the subjects' behaviour during the interviews, is shown in Figure 1. The first map displays the strategies used by a single subject across tasks and time, see Figure 2. The tasks and interviews are identified numerically at the top of the map. For example, 1/2 means Task 1 in the second interview, whereas 2/1 means Task 2 in the first interview. Although time sequence is represented by the vertical chain of strategy use, the duration of time has not been included as a component because the subjects' solution times were affected by the interviewer probing responses. The line lengths between shapes, which represent behaviours, were determined by the display and accordingly are not correlated with time taken. This particular subject was selected because of the gross differences in her strategy use between the first and second interviews. The second map displays the performance of four students, from the group of twelve subjects, who used a restricted number of strategies on a pair of isomorphic tasks, see Figure 3. These two maps are intended to be illustrative rather than exhaustive of the possible data maps which can support various mathematical tactics for generating meaning, including the discernment of patterns and relationships, counting and noting relationships between variables (Miles & Huberman, 1994). The interpretation of the data maps necessitates the dual consideration of visual and linguistic cues. The analysis that follows is by necessity brief, but provides an example of the potential use of data maps in visual analysis.

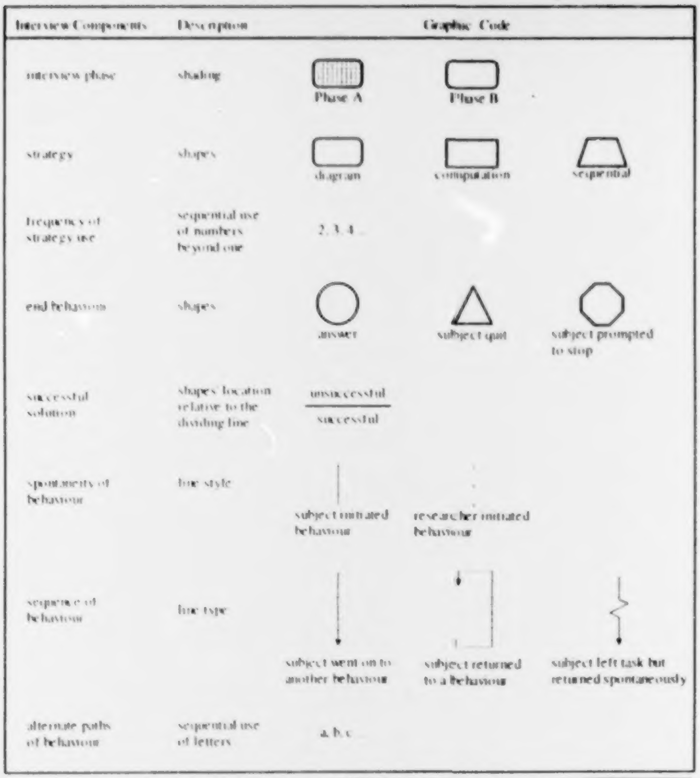
Interview Components
Description
Graphic Code
interview phase
shading
Phase A
Phase B
strategy
shapes
diagram
computation
sequential
frequency of strategy use
sequential use of numbers beyond one
2, 3, 4 ...
end behaviour
shapes
answer
subject quit
subject prompted to stop
successful solution
shapes' location relative to the dividing line
unsuccessful
successful
spontaneity of behaviour
line style
subject initiated behaviour
researcher initiated behaviour
sequence of behaviour
line type
subject went on to another behaviour
subject returned to a behaviour
subject left task but returned spontaneously
alternate paths of behaviour
sequential use of letters
a, b, c ...

Figure 1. The graphic code used in the data maps

An example of the data analysis

Figure 2 was designed to display the strategies used by a single subject during the two interviews on the five pairs of isomorphic tasks. Performance can be compared between the phases of the interviews and between the two interviews. The analysis focuses on three areas. Firstly, what patterns and relationships can be identified from the data map? Secondly, what questions are suggested by the analysis of the data map? Thirdly, what were the critical events during the interviews?

At least four themes are evident from Figure 2. Firstly, Karen's *spontaneous and appropriate choice of the strategy, draw a diagram,* in 1/1, 3/1 and 4/2, does not mean that she can implement the strategy successfully because in each task her solutions were unsuccessful. Secondly, her *repeated use of a strategy* in 1/1, 2/1, 3/2 and 5/1 does not culminate in success, either with or without interviewer intervention, suggesting that either her choice of or implementation of the strategies was inappropriate. Thirdly, the *instructional program* seems to have had a positive effect on Karen's problem solving. After the instruction she was able to solve two problems, 1/2 and 2/2, which were isomorphs of 1/1 and 2/1 respectively, problems on which she was previously unsuccessful. However her lack of success on 4/2 needs explanation as she was successful on an isomorphic task, 4/1. A comparison of her strategy selection in 4/1 and 4/2 reveals that her initial strategy selections were different in these tasks. Thus, her success may be strategy dependent. Fourthly, in four of the ten tasks Karen "*quit*": 2/1, 2/2, 4/2 and 5/1. Although she spontaneously restarted in 2/1, 2/2 and 4/2, and was in fact successful in 2/2. Hence, "quitting" for Karen may be a period of incubation rather than necessarily the end point of an unsuccessful solution process.

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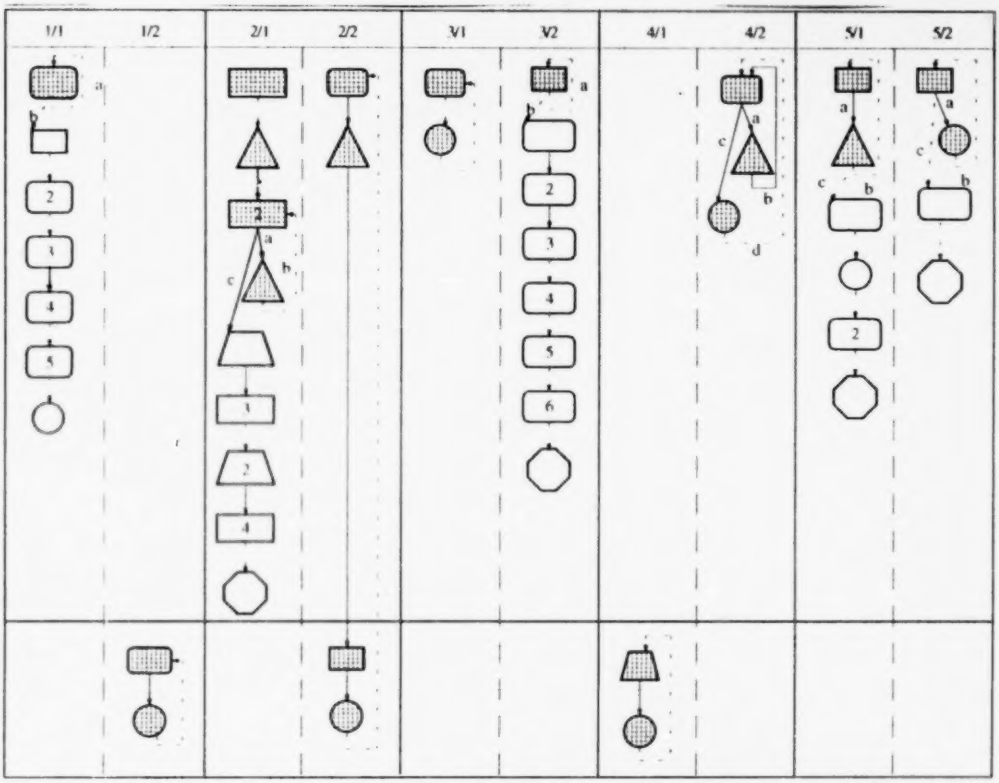

Figure 2. A comparison of a Karen's performance on five pairs of isomorphic tasks

The data map suggests further questions to explore. For example, was her unsuccessful use of diagrams after instruction in Task 4 an isolated event? No, the map reveals that she was also unsuccessful after instruction in Tasks 3 and 5. Did she ever use the strategy successfully after the instruction? Yes, the map shows in Task 1. Hence, Karen's erratic performance using diagrams needs explanation and requires a return to a more detailed data source, such as the interview video-tape or associated transcript. However there is an alternative that can be explored visually, that is Karen's performance relative to her peers on the same task, as perhaps the task was simply a difficult task for students of this age. Errors in diagram use on this task include both generating and using the diagrams (Diezmann, 1995).

The specific details of Karen's difficulties with some of these tasks is not evident from the map but what is evident is that during this interview there were some critical events which warrant further exploration. What made Karen quit? What made Karen restart after quitting? Why did the interviewer stop Karen from proceeding? The interview video-tape or transcript should provide some understanding of these events.

In order to follow up Karen's performance on a task, a second data map was prepared to show her performance relative to her peers on the first tasks in each of the interviews (See Figure 3). Figure 3 is interpreted similarly to Figure 2 and displays the performance of four students, including Karen, on the first task in both interviews.

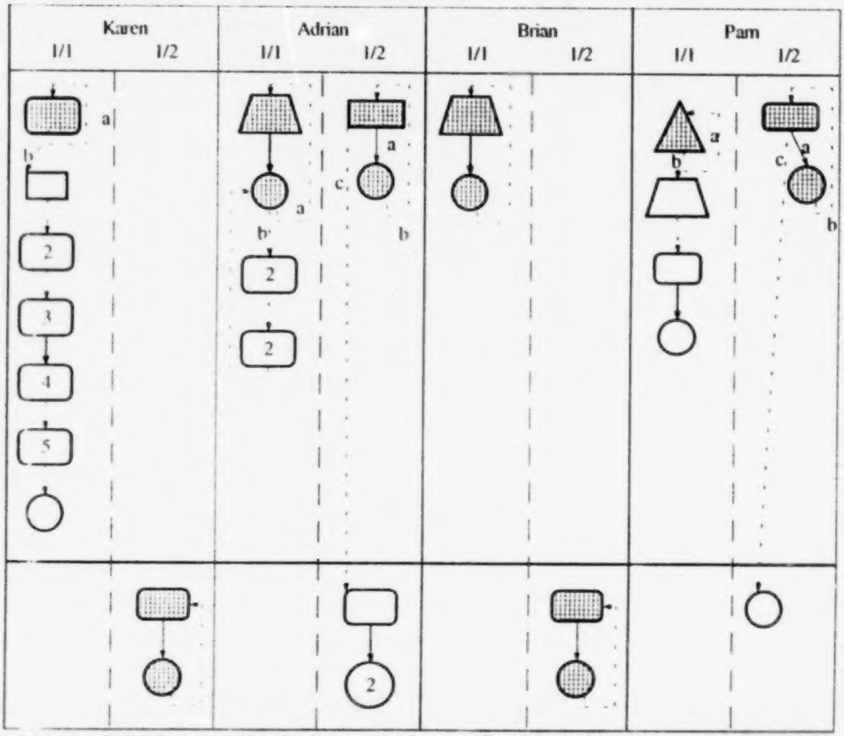

Figure 3. Using vertical cross sections of subjects' maps to compare the whole groups' performance on a pair of isomorphic tasks

A comparison of Karen's performance with three other students indicates that prior to instruction no students were able to complete the task successfully. Karen however was the only student to spontaneously use the strategy *draw a diagram* prior to instruction. All students however were able to use the strategy *draw a diagram* successfully on this task after instruction, two independently and two with interviewer intervention.

In summary, although this visual analysis was only illustrative of what can be done some understanding of the subjects' strategy use in novel problem solving was revealed. The analysis was a two way process in that specific questions were explored through the maps and the maps themselves revealed patterns and relationships. The process was interactive as the interpretation of the maps suggested follow up questions to explore. For example, the comparison between subjects' performances on Figure 3 resulted from a query in Figure 2 regarding the task difficulty. Additional questions could be investigated rapidly by generating further maps using the cut and paste facility of Smartdraw Pro (1995). For example, which strategies resulted in a successful solution? Thus, the software provides a tool to generate the data map and the data map in turn becomes the tool for analysis.

The facility of data maps to provide a wholistic account of the data is particularly useful. When a data map is considered together with the linguistic data, for example a video interview or a transcript, the map can act as an overlay "pinpointing" critical periods within the interview that can be followed up.

Conclusion

The question under discussion has been-is there any advantage in analysing data visually? In the spirit of qualitative research this question is rhetorical and can only be answered personally, although the process can be evaluated by the reader, given the criteria of the study and an adequate audit trail. This study has however served its purpose with regard to the development of a visual tool for data display and a visual analysis technique. The investigation has, to my mind at least, advanced the issue of whether visual analysis of qualitative data is plausible to-when is visual analysis useful or even preferable?

Pragmatically, the visual analysis of data is dependent on well organised and information rich data maps. Although the data maps can be created by hand and for convenience may be if coding directly onto a transcript, the software enables the rapid generation of further data maps to investigate specific questions thereby fulfilling the interactive assumption of qualitative analysis. Data maps and the technology appear closely entwined as demands on the technology or development within the technology may impact upon the data maps which in turn may affect the visual analysis.

The creation of data maps and the associated technique of visual analysis were borne out of need and are still in their infancy. However with further development and more understanding of their utility, the visual display and analysis of data appears to have enormous potential in qualitative studies.

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When There Isn't Enough Time For An Interview: How To Analyse Open Assessment Tasks

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This paper describes open assessment tasks and their place in interview-based research. Open assessment tasks are designed to elicit children's understandings and are interview-like in their intent. Responses to open assessment tasks are extremely diverse, and our analysis provides an order that allows inferences about children's understanding to be made. Our experiences to date with this approach indicate that open assessment tasks offer a highly reliable, valid approach to situations where it is not possible to implement interviews.

Introduction

Let it be understood at the outset that we believe the clinical interview has no peer for assessing a child's capabilities in mathematics, particularly their cognitive capabilities. However it must also be said that *interviewing* has severe limitations in terms of time and the skills required of the researcher. Our own research needs have led us to devising a methodology that provides an efficient alternative. Our methodology is based on the use of open assessment tasks.

The open assessment task puts questions to children, in written or oral form, and requires a written, drawn, or constructed response that can be analysed later by the researcher. Unlike a clinical interview, there is no 'next question' to probe further, but to compensate for this, there is no need for the researcher to make an instant decision about what is the next 'best question'. The strength of the open assessment task lies in the analysis of children's responses to the first, and only, question posed. This analysis preserves the richness and diversity of the data, rather than distilling it to simply a numerical form as in traditional assessment analyses.

Two major implementations of open assessment tasks have taken place in Australia; one study investigated aspects of science, a second looked at aspects of social education. In mathematics there have been several small-scale implementations; these were in number, measurement, space, and data. It is from the mathematics research that our examples are drawn. In this paper we shall look at the structure of open assessment tasks, the types of responses they elicit, and most importantly, the analysis of these responses.

Structure of open assessment tasks

An open assessment task requires the child to write, draw, or construct a response to a question; this is the strength of the open assessment task. With a permanent response, the interviewer–researcher is able to interpret responses and draw inferences uninterrupted by the need to find the 'next' question. The clinical interview requires a one–to–one arrangement; this is a strength of the clinical interview. On the other hand, the open assessment task is best suited to groups of children; this is a strength of the open assessment task.

An open assessment task requires that each child has a copy of the question or questions, and the means to record or construct their response; it also requires that the researcher is able to collect the children's responses (in some cases we have used photographs to record constructed responses).

The analysis of the children's responses is the critical step in open assessment task methodology. The analysis combines art and science; the art is in discerning a child's underlying conceptual meaning in their response, the science in the application of mathematical knowledge in classifying these meanings. The essence of the analysis is to focus upon the conceptual meaning that a response appears to represent. We say 'appears to represent' because it is our interpretation of the response that we are dealing with, not necessarily the child's 'reality'. This analysis, although not complex, is best explained through example; two examples are detailed below.

The children who provided the data

The children who provided the data came from a socio-economic cross-section of inner-city and suburban government and private schools. The schools used a wide variety of curriculum materials and resources and included many different teaching situations and styles, and as such were representative of schools across the nation. In each case approximately 200 8-year-old children attempted the task.

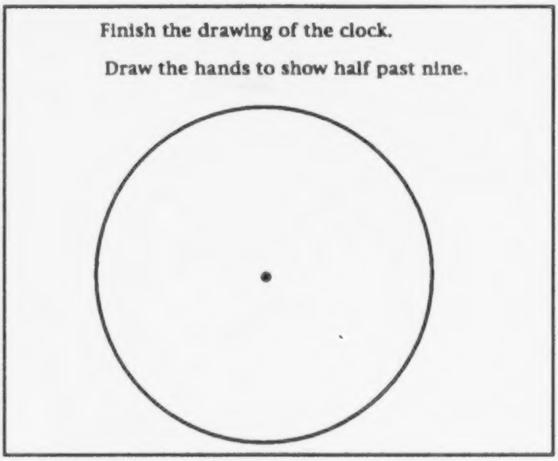
Example One: The Complete a Clock task

The first example described uses an unfinished clock–face. This task was based on an activity, described by Pengelly (1992), that was originally intended as a teaching activity. The activity was for use by junior primary children and was called 'Make a Clock'. The activity was adapted as a open assessment task for middle primary children and renamed 'Complete a Clock'. An advantage in using this task was that it was already known that it elicited a range of responses.

Curriculum basis for Example One

The national curriculum *Mathematics – a curriculum profile for Australian schools* (Curriculum Corporation, 1994) acknowledges that understanding the conventions of analogue clocks develops over time, implying that continuing assessment of these developing skills is important.

The task was developed to assess the curriculum outcome in the Measurement strand on Time for Level 2: "recognise key times on an analogue clock and tell the time of day on digital clocks in hours and minutes" (Curriculum Corporation 1994: p. 45). This outcome is one which is expected to be achieved by children who have completed their third year of schooling; that is they are usually about 7 years of age. By the end of Level 3 (the fifth year of schooling, when children are usually about 9 years of age) the curriculum outcome suggests that children should be able to "tell the time on digital and analogue clocks" (Curriculum Corporation 1994: p. 61); hence it is reasonable to assume that most 8 year-old children could show 9:30 on a drawing of an analogue clock. The task was presented as is shown in Figure 1.



Finish the drawing of the clock.

Draw the hands to show half past nine.

Figure 1: The Complete a Clock task.

Analysing responses to open assessment tasks

The examples described in this paper were designed to elicit a range of responses, with the intention of providing, as with any interview, opportunities to "see" children's understandings.

However the diversity of sights presented is at first bewildering! Some order needs to be imposed to enable sense to be made. The methodology used for responses to these examples is that first used for analysing conceptual understanding in science (Adams, Doig, and Rosier, 1992), and subsequently in mathematics (Doig, 1993a, 1993b) and in social education (Doig, Piper, Mellor, and Masters, 1994). The procedure described is extremely simple to understand. It should be noted, though, that for any set of data there is not, necessarily, a "correct" analysis.

The essence of the methodology is to read each child's response (whether text, diagram, or construction) with a view to understanding wholistically the ideas behind each response; a knowledge of the results of related research is important to this aspect of the methodology.

Next the responses are placed into categories wherein all the responses appear to have the same underlying idea; every time a response which suggests a substantially different idea from any before it is found, a new category of response is formed. Categories are thus mutually exclusive. Once all responses have been allocated to categories (that is, all necessary categories have been formed) short descriptions are written to identify categories for subsequent discussion, possible further analysis, and the drawing of inferences. Responses are then re-examined to make sure that the categorisation has been done consistently; if possible an independent judge is asked to use the category descriptions to sort the responses. Differences of opinion about categorisation provide the opportunity to clarify categories. The critical idea is that the categorisation should both preserve and clarify the differences in children's thinking revealed by the responses.

Analysis of responses for Example One

Table 1 sets out the categories constructed to organise the data from Example One. The categories are represented by the rows of the table. The columns are further sub–divisions of the rows designed to help us understand the diversity that exists even within the categories; for example, within the group of responses giving a correct clock–face, there are three possibilities. The same is true for other categories set out by row. Alternatively, if the focus desired is on 'correctness' then a column by column reading may be preferable; for example this approach shows that within the category of readable (that is, interpretable as 9:30) responses, there are at least four variations. The best option is to consider the table from both aspects; this gives the most complete understanding of the data. Examples of each category of response are given in Table 1, together with the percentage occurrence of each category of response.

Inferences from the analysis of Example One

The most obvious category to construct was that with correct responses. Nearly two–thirds of children could represent, acceptably, 9:30 on an analogue clock–face. However, a small number of responses (6) showed minute divisions on the clock-face. All of these displayed the correct time, but a number of these (4) were flawed because they had five marks between the numerals, thereby creating six intervals. These responses are examples of a misunderstanding of differences between continuous and discrete quantities; that is, a misunderstanding of the difference between the rôle of the space between marks, and the rôle of the marks themselves.

A further category of response was that of problems with the hands of the clock. The problems were of three types: there were those where insufficient hands were drawn; those where the hands were not differentiated; and those where the hands were reversed. The conventions associated with the hands of a clock are quite subtle. Some children appear to be unable to deal with this subtlety. Responses in this category would seem to indicate that these children are unaware that hands of different lengths refer to different scales. In some cases children do not appear to understand which hand refers to which scale.

Other responses were categorised as having problems with the numerals. When the numerals were incomplete or so incorrectly spaced around the circular region as to be visually at odds with conventional clocks, then the responses were categorised as having a problem with numerals. The fraction knowledge required to place twelve numerals equally around a circle is very sophisticated. Many children made good attempts at this aspect of the task, but it was clearly beyond their spatial and, or, fraction skills. Some responses had problems with both numerals and hands. These were combinations of the categories listed above. Despite the misunderstandings or lack of skill exhibited by these responses, some could be interpreted. They do reveal aspects of the child's understandings that need to be developed.

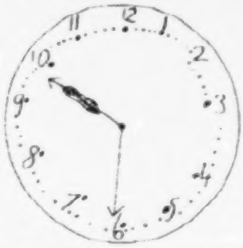
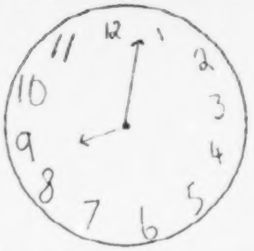
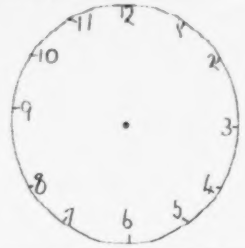
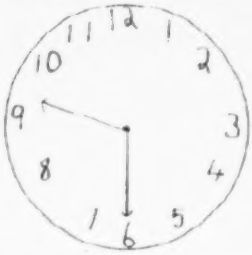
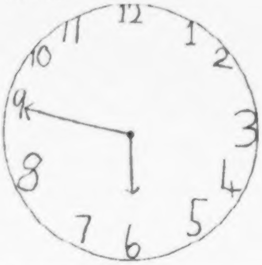
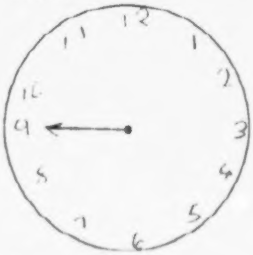
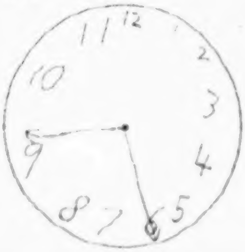
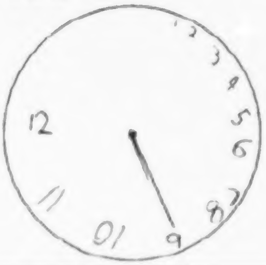
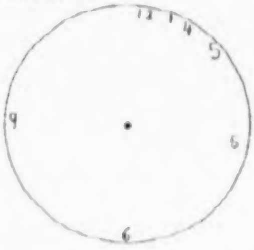
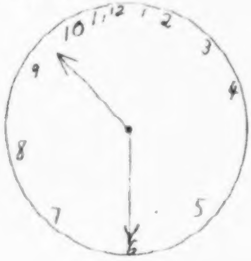
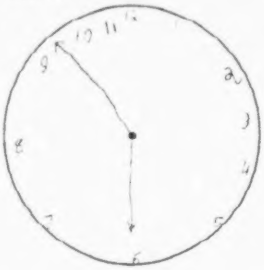
Category	**Representations of time**		
	Readable as 9:30	**Unreadable as 9:30**	**Incomplete**
Correct clock-face	60.8% 	3.2% 	0.5% 
Problems with clock hands	17% 	2.1% 	1.0% 
Problems with numerals	8% 	1.6% 	0.5% 
Hand and numeral problems	4.7% 	0.5% 	0%
Totals	90.5%	7.4%	2%

Table 1: Analysis of responses for Example One (n=189).

Example Two: The Make a Model House task

This task was based on the well-known problem that many people have in interpreting two-dimensional representations of three-dimensional objects and constructing the object. Mitchelmore (1976, cited in Eliot, 1987) in examining cross-cultural studies of spatial abilities found that cues for representing depth and perspective were poorly understood and difficult to understand. Children are being confronted with an increasing amount of two-dimensional material (printed and electronic), which they are expected to interpret, so this task was seen as adding extremely valuable information to teachers' portraits of children. The task used a 'plan' of a house that had a cube as its base and a triangular prism for its roof. The task required children to 'build the house' using match–sticks joined by plasticine (a type of modelling clay). The 'house' was the home of the Three Bears in the folk–tale 'Goldilocks and the Three Bears'.

Use the matchsticks and plasticine to make a model of the three bears' house.

Make your model just like the one in the drawing.

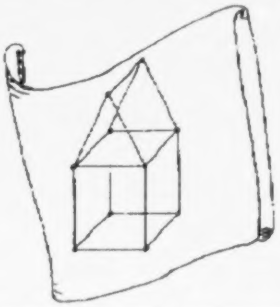


Figure 2: The Make a Model House task.

Curriculum basis for Example Two

This task was developed to assess the national curriculum *Mathematics – a curriculum profile for Australian schools* (Curriculum Corporation, 1994) sequence of outcomes in the Space strand. These outcomes were about using spatial ideas to interpret, draw and make. The profile suggests, that at Level 2 a child: "fulfils simple spatial criteria when making things from verbal and visual descriptions" (Curriculum Corporation, 1994: p. 38). This outcome is expected to be achieved by children who have completed their third year of schooling; that is they are usually about 7 years of age. By the end of Level 3 (the fifth year of schooling, when children are usually about 9 years of age) the curriculum profile suggests that children should be able to "pay[s] attention to the shape and placement of parts when matching, making and copying things, including matching nets with 3D shapes … Make polyhedra in solid (with clay), hollow (with provided nets) and skeleton (with straws) forms." (Curriculum Corporation, 1994: p. 54). The task was presented as shown in Figure 2.

Analysis of responses for Example Two

Sometimes ordering the categories of response makes the methodology even more effective. That is, once the categories are 'settled' they are arranged in order, with the category containing responses showing the most sophisticated understanding first, then categories with responses considered to show less and less sophisticated understanding after. It is crucial for this ordering that the purpose of the task is clear and well-defined.

This ordering of categories is not necessary for all tasks; while it makes understanding responses easier in most cases, it is only strictly necessary if responses are being used for creating developmental continua (see Adams *et al.* (1992) for a description of developmental continua and their construction).

The ordering of response categories builds a picture of the diversity of responses, yet allows a sense of the increasing proficiency displayed by children to emerge. Table 2 sets out the categories constructed to organise the data from Example Two.

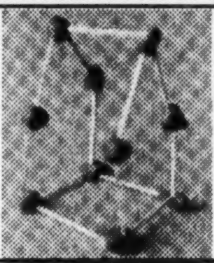
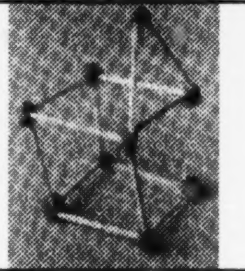
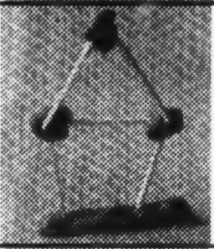
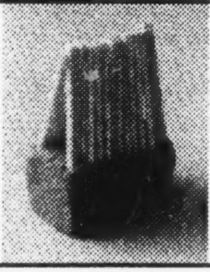
Category	Description of responses in this category	Percentage
Correct model	Correct 3-dimensional model of the house.	60%
3-dimensional model with subtlety problems		10%
Combination of dimensions		22%
2-dimensional interpretation		3%
Other responses		5%

Table 2: Analysis of responses for Example Two (n=227).

Inferences from the analysis of Example Two

Nearly two-thirds of children were able to construct the model from the plan correctly. This was very encouraging given that many children in the sample claimed never to have done this type of activity before (at least at school).

About ten per cent of the responses were substantially correct, but had minor deviations from the 'plan'. The responses indicate that these children can interpret this type of information and understand that it represents a three-dimensional object;

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however some details are either ignored or over–looked. This may indicate either an immature appreciation of the conventions, or that the context (of a house) was so well–known that children were building more from memory than the plan.

Approximately a quarter of responses show that children interpret the 'plan' as a mixture of two– and three–dimensional elements. It may be that these children are viewing the 'plan' as representing a series of individual planes, rather than representing a solid. These responses could also indicate that at this age (8 years-old) children, with respect to understanding drawing conventions, are in transition from two– to three–dimensional representations.

A very small group of responses depicted the 'plan' as a two–dimensional object; there was no indication that the children saw the 'plan' as representing a three–dimensional object. A small number of these two–dimensional models were in reality 'drawings' of the plan, using matchsticks to represent the lines, and plasticine for the 'dots'. These were considered to be strictly two–dimensional.

Some responses did have the 'flat' house (two–dimensional) standing upright; these tended to consist of representations of a single face or wall. Whether these are truly two–dimensional is open to question. The children who made upright walls are certainly not adept with the conventions of representing solid objects in two–dimensions.

We would argue that their responses show that they are at least aware that three–dimensional objects can be represented by drawings. On the other hand those children who simply 'drew' the plan in matchsticks, knowing that it was a house–plan, knowing that they were to build a house, appear not to be aware of the most rudimentary aspects of this form of representation. In such a small sample (n=227) and with only 3 per cent of children's responses falling into this category, it would be rash to suggest more than this.

Whilst responses were categorised by the similarity of their spatial attributes, the final categories (if ordered by the sophistication of their understandings) bear remarkable resemblance to Mitchelmore's stages of development in three–dimensional representation (looking at three–dimensional objects and drawing them). In the case of this task, the process is the reverse (reading two–dimensional drawings and making three–dimensional objects), but the two schema appear to be isomorphic (Mitchelmore, 1980).

Discussion

The two examples discussed are meant to illustrate how open assessment tasks can reveal conceptual understanding and mathematical strategies on a par with clinical interviews. The form of analysis described preserves the richness of the data and makes coherent its apparent chaos. Further, during the process of analysis, the analyst is building theories about a child's (or groups of children's) mathematical understanding.

Commonalities revealed in children's responses, however, may or may not be developmental; it is not inherent in the data that more complex understandings naturally arise from more naïve understandings; what is revealed is simply the diversity of thought existing within the group being investigated. In some cases it may be legitimate to infer a developmental ordering of ideas; the researcher needs to be wary of succumbing to a 'stages' mentality. It is salutary to remember that the interpretation one categorisation provides is not the only one possible; another researcher may well provide a different interpretation. This being said, it is encouraging to note the consistency between the results depicted here and those of other researchers (see in particular Mitchelmore, 1980 and the 'Make a Model House' example).

As we have argued elsewhere (Cheeseman and Doig, 1995) educational tasks that reveal more of children's thinking than the common alternatives, are worth the extra time and effort needed to construct and analyse them. The richness of the data provided by such tasks, the wealth of detail revealed about children's thinking, is only one aspect of using open assessment tasks. A further practical application is gathering *a priori* information for planning effective learning programmes.

Both those engaged in research and those working within the classroom may find that open assessment tasks offer advantages over more traditional methods of exploring children's thinking. Where time, or the number of children involved prevents the use of clinical interviews and similar probes, open assessment tasks can uncover children's mathematical understandings.

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Searching for classroom RATs (Rich Assessment Tasks)

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To implement the national profile for reporting student mathematical achievement requires alternative assessment strategies. A team of classroom teachers was brought together to prepare a package of rich assessment tasks (RATs), evaluated for their potential in assessing mathematical performance and aligning the profile. The search process highlighted issues as the need to match assessment to the classroom context, and the changing role of teacher and students through varying assessment procedures, resulting in production of tenets for developing and evaluating RATs.

Background

With the advent of the *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990), there has been increased emphasis on broadening approaches to teaching and learning mathematics in order to achieve the goals of school mathematics. The stated goals of school mathematics are that students will develop confidence and competence for dealing with situations involving mathematics in their daily lives and positive attitudes towards mathematics; be able to use mathematics to solve problems, communicate mathematically, and utilise modern mathematical tools and techniques; and experience the investigative process through which mathematical ideas have developed (Australian Education Council, 1990). Similarly, of the goals for school mathematics the National Council for Teachers of Mathematics (NCTM)(1989, p. 4-5) have stated that " today's society expects schools to insure that all students have an opportunity to become mathematically literate, are capable of extending their learning, have an equal opportunity to learn, and become informed citizens capable of understanding in a technological society." In order to achieve such goals, a variety of teaching approaches are advocated, including directed teaching, activity-based learning, discussion, application and problem solving, and open investigations (Australian Education Council, 1990; Cockcroft, 1982; Department of Education, Qld, 1987).

The organisational structure of various mathematics curriculum documents has seen the specification of components other than content/concept topics (e.g. number, measurement, space) including processes (e.g. justifying, inferring, validating) and affects (persistence, confidence, interest and enjoyment)(Department of Education, Qld 1987); mathematical reasoning, communication, problem solving, connections (NCTM, 1989); attitudes and appreciations, mathematical inquiry, choosing and using mathematics (Australian Education Council, 1990); working mathematically (Australian Education Council, 1994). With the inclusion of such strands within the mathematics curriculum, it would appear that teachers need to construct teaching and learning situations alternative to directed teaching approaches in order to provide students with opportunities to be engaged in and develop such mathematical processes.

With calls for variety in instruction for school mathematics there has been corresponding calls to make mathematics assessment match the teaching and learning process (e.g. Australian Education Council, 1990; Clarke, 1992; Department of Education, Qld. 1987; NCTM, 1989;Stenmark, 1989). Assessment has a significant influence on students' learning of mathematics, as Boud (1995, p. 3) has stated "every act of assessment gives a message to students about what they should be learning and how they should go about it." Assessment communicates to students what the teacher values most in the curriculum, thus, what the teacher emphasises and regularly assesses will be what students consider important aspects of instruction (Lester and Kroll, 1991). Therefore, if such processes as problem solving, communicating, reasoning, are valued mathematics goals and an integral part of the mathematics curriculum, then assessment must provide students with feedback on how well they are moving toward achieving such goals, while simultaneously communicating to students their value in the study of mathematics.

According to NCTM (1989), assessment in mathematics serves two functions; it is as a means of providing students with information on their achievements in mathematics, and it informs the teacher, providing direction for further teaching. As they have stated, "assessment should provide a biography of students' learning, a basis for improving the quality of instruction" (p. 203). Therefore, students and teachers are key stakeholders in the assessment process. It is the teacher's role to ensure that assessment is meaningful for students, and not a confidence deflating exercise. As Clarke (1995) has stated, "the purpose of assessment is to make explicit children's achievements, celebrate their achievements with them, then help them to move forward to the next goal." The notion of assessment as a celebration is a novel perspective, but aligns descriptions of change in emphasis in assessment in mathematics. As suggested by NCTM (1995), assessment should be concerned with what students know and how they think, rather than informing them of what they don't know; of providing potential for students to apply a number of mathematical ideas rather than testing isolated mathematical skills via exercises and work problems; of enabling students to respond mathematically in a variety of ways, including written, oral, symbolic forms; of providing opportunities for students to demonstrate their mathematical knowledge continually rather than restricting assessment to specific times on specific days.

In order to build a holistic picture of students' mathematical abilities, rather than focussing on areas of weakness, gathering of data from a variety of sources is required. Enabling students to engage in cooperative problem solving and open investigations for assessment purposes can provide teachers opportunities to collect data on which to draw together other fragmented assessment data into a cohesive picture (Stenmark, 1989). As Lovitt (1994, p. 5) has stated, problem solving and investigations, "by their nature can allow us to see how a pupil is developing in ways that fixed closed sets of exercise never can." The difficulty in using such problems for assessment purposes, Clarke and Reed (1992, p. 23) have suggested, is due to individual teacher's conceptions of the discipline of mathematics. As they have stated, there is a "conflict of paradigms between a conception of mathematics as a catalogue of distinct and disjoint skills assessed by targeted test items, and a conception of mathematics as a network of behaviours which are not acquired hierarchically, are not practiced independently, and cannot meaningfully be assessed in isolation from one another." Problem solving tasks as assessment items, Clarke and Reed have contended, provide truer indications of students' mathematical knowledge and understanding, because it is in such situations that students choose the mathematics they wish to use, not procedures they have studied prior to a formal test/exam. Observing students selecting mathematics thus provides a deep insight into the effectiveness of the teaching and learning process.

Of the assessment situation itself, Clarke (1992) has starkly distinguished the difference between formal and informal assessment, stating that *formal assessment* "involves the cessation of instruction, typically for the whole class, while the assessment 'event' is held"; contrasting *informal assessment* as the "collection of information about students' learning coincident with instruction, without disrupting the learning process." (p. 157). Clarke has further added that informal assessment should not necessarily be interpreted as unstructured, unfocused, unplanned, ad hoc, or non-legitimate. It appears that formal assessment in one means of gathering data on students' mathematical performance, but informal assessment is an ongoing process which melds fluently with the teaching and learning situation. Thus, gathering information about students' mathematical knowledge and ability in an informal, yet structured way, can occur whilst students are engaged in investigations and problem solving activities. Informal assessment then, enables a variety of assessment procedures to be utilised, such as observation schedules, checklists, student self-evaluation and reflective writing, mathematical reports, group investigations (Stenmark, 1989). In this way assessment aligns the teaching situation.

As previously stated, current mathematics curriculum documents, in describing the goals of school mathematics, provide a framework for curriculum development, stressing the importance of using a variety of teaching and assessment practices in order to achieve stated goals. Companion to the national statement on mathematics for curriculum development (Australian Education Council, 1990) has been the advent of the national profile (Mathematics - A curriculum profile for Australian School, Australian Education Council, 1994) as a framework for recording and reporting student achievement in mathematics. Developed at a national level, it has been adopted in varying degrees throughout states of Australia. In Queensland schools, the national profile was adopted completely in 1994, and retitled "Student Performance Standards in mathematics for Queensland schools" (Curriculum Corporation, 1994) and acronymed SPS. The national profile (SPS in Queensland), while aligning with the organisation of the national statement, is fundamentally different to the organisation of the Queensland Years 1-10 mathematics syllabus (Department of Education, Qld, 1987). As a reporting framework the national profile requires that student performance be assigned a level (on a scale of 1-8, with 1 being the lowest) for each of the six strands of *working mathematically*, *space*, *number*, *measurement*, *chance and data*, *algebra*. In 1995, Queensland state school teachers of Years 1-8 were directed to report on student achievement in the three strands of *space*, *number*, and *measurement*. Thus, by its very structure, the national profile required changes to the reporting practices employed by teachers in Queensland schools. It may be expected that a mandated change to reporting practices through the implementation of SPS would meet with a lot of teacher resistance, and this has been found to be the case (Bleicher, Cooper, Dole, Nisbet, & Warren, 1996). However, as Lovitt (1994) has suggested, the national profile does provide a type of curriculum audit framework, because the profile strand of *working mathematically* must become an integral part of teaching and learning. If teachers cannot determine the level to which students in their classrooms, for example, can apply problem solving strategies, test conjectures, investigate and explore problems, verify conclusions, then students have not been given the opportunity to do so. Of the national profile, he has stated "I like them. They seem [to me] like one of the last pieces of a big jigsaw, legitimising and confirming many of the fragments of educational growth over the past many years and importantly bringing those fragments together in a way that allow us to get an overview of the whole classroom process"(p. 4). Teachers found that, in order to report on student mathematical achievement using SPS, assessment data from, for example, traditional pen and paper tests, was difficult match to SPS outcomes.

To assist teachers with implementing SPS, systemic inservice was provided to teachers in Queensland state schools. In an effort to further support teachers to implement the national profile, the Queensland Association of Mathematics Teachers/Professional Development Project (QAMT/PDP) sought to produce a package of rich assessment tasks (RATs) which had been trialled in classrooms and found to provide rich data on students' mathematical performance, and thus align with SPS. This paper reports on this project in which practicing classroom teachers were brought together for the purpose of writing rich assessment tasks; tasks which had been developed and trialled at a 'grass-roots' level by teachers for students in their own classroom. This report addresses two main dimensions of the project: (i) particular episodes which influenced the process of developing and evaluating alternative assessment tasks; and (ii) teachers' perceptions of their changing role as a result of implementing alternative forms of assessment. A summary of the writing process is presented in the form of tenets for developing and evaluating RATs.

The study

The study was qualitative, using participant observation (Spradley, 1980). The researcher was co-ordinator of the writing team, participating in the writing process, and observing the team as it interacted. as well as providing guidance and direction for all team meetings. The writing team consisted of twelve teachers (6 female, 6 male) representing 12 primary schools in a regional cluster. The teachers had been nominated by their school principal to participate in the project. The years of teaching experience of the team ranged 3 to 35 years; 3 teachers were currently teaching in the lower school (Years 1-3), 4 teachers were currently teaching in the middle school (Years 4-5), and 5 teachers were currently teaching in the upper school (Years 6-8). The team meetings were held at a venue removed from the teachers' school situation. Each full-day team meeting was organised into 3 sessions. The first session was a whole group discussion/feedback session where participants described assessment initiatives trialled, and discussed issues relating to alternative assessment in mathematics. The search for rich assessment tasks (RATs) began in the second session. In groups, participants perused mathematical resources and brainstormed ideas for assessment tasks which could be used in the classroom. In the third session, having taken ideas from session 2, the participants worked either individually or in pairs to plan for implementation of the tasks developed in their classrooms. The trialing of tasks was by participants in their own classrooms, and results reported back to the group at the next meeting. There were four team meetings in all for the duration of the project, which spanned 4 months of the school year. Data for this study was collected by observation, fieldnotes and by gathering of teachers' work in progress.

Results and discussion

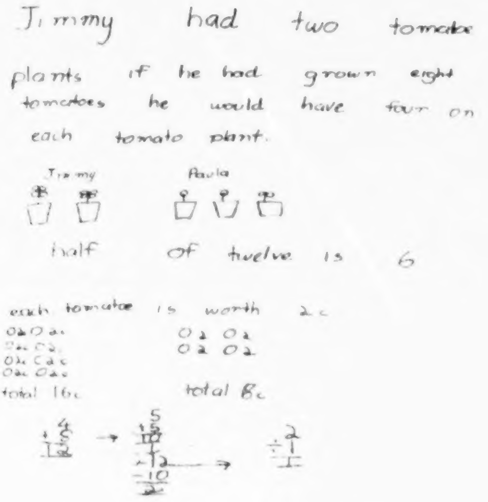
The process of trialing assessment tasks

The process of developing and trialing alternative assessment tasks highlighted particular issues as (i) the need to give students experience with novel tasks before using them for assessment purposes; (ii) the need to trial tasks with pupils to assess the value of the tasks for assessment purposes; (iii) the individual teacher's need to feel comfortable with alternative tasks, and how familiar assessment tasks can be built on to provide stepping-stones for implementing more open investigation; (iv) how alternative activities enabled students to demonstrate mathematics connections; and (v) how tasks developed by others, unrelated to any teacher's classroom context were trialled as 'activities' rather than assessment tasks. Each of these issues will be discussed under the following headings: (i) *familiarising students with mathematical investigations*; (ii) *trialing tasks*; (iii) *starting small, taking "risks"*; (iv) *valuable assessment data from "chaos"*; and (v) *teacher-made tasks; other tasks.*

Familiarising students with mathematical investigations: Upon implementation of certain investigative tasks developed, it was found that little mathematical data could be collected. One team member developed a Number activity for her Year 3 class involving manipulation of numbers shown on 3 dice. For this group activity, the students were required to roll 3 dice, and operate on the 3 numbers shown (using addition, subtraction, multiplication, and/or division) in an effort to reach a designated total. A record sheet was provided for the children to write their mathematical sentences and keep a running total of their score. As the activity was novel to the children, the teacher found that she was required to assist children in completing their record sheet, help children check their calculations, adjudicate on the legitimacy of the dice throwing, and so on. The teacher had anticipated that she could gather important information on children's mathematical performance as the students engaged in the activity, particularly by noting which children could competently operate on the numbers, which children used problem solving strategies, which children showed evidence of using mental computation. The teacher evaluated this activity as a fun and worthwhile mathematical activity, which gave all students an opportunity to work in groups. As an assessment task, the teacher found that it yielded little useable information as she had no time for observation of students' performance. Upon reflection of the classroom trial of this task, the teacher made modifications to the worksheet and the rules of the game. Through retrialing, the teacher found than the task provided the type of data she had anticipated to gather in the first instance.

Trialing tasks: Some tasks trialled were found to be unchallenging to students, or students did not provide depth of response anticipated by the teacher. One teacher developed a measurement activity related to a golf course. The task required students to measure distances to putting greens on a scale drawing of a golf course, to plot their course, and to throw a dice to get to the flag. The teacher produced a high quality game board plan of the golf course, detailing bunkers and water hazards. Upon implementing this activity, the teacher found that the students totally disregarded any measuring, and completed their round of golf by engaging in little mathematical calculation. In evaluating this activity, the teacher reported that, for his Year 6 class, it yielded little information about his students' measurement and number ability. The need to trial a task in order to determine its richness appeared to be an important element in the process. As the team member noted, "It is sometimes difficult to initiate tasks. What seems good here [writing team meetings] doesn't always translate into the classroom."

Starting small, taking "risks": The project encouraged participants to 'take risks' by trialing alternative teaching and assessment ideas in their classrooms. For one teacher, allowing students to engage in totally open investigations for assessment purposes was a quite daunting prospect, at odds with her teaching style. This teacher found that by modifying slightly an existing assessment task she used in her classroom, much information on children's mathematical performance could be obtained. The teacher presented her Year 3 class with a story situation about Jimmy Joker and Paula Pepperpot who had a tomato growing competition. In this story, Jimmy got two plants to grow, and Paula got three plants to grow. Jimmy had eight tomatoes on each plant, and Paula had four tomatoes on each plant. The students were required to answer a number of mathematical questions derived from the story. The teacher provided space for the students to provide reasons for their answers. One student responded that "Jimmy Joker" grew the most tomatoes; that Jimmy and Paula would each receive "6" tomatoes if they shared them equally; that if they sold their tomatoes for 2c each, Jimmy would make "6c" and Paula would make "16c"; and that if Jimmy and Paula ate 5 tomatoes, there would be "1" tomato each to give their own 'mum'. One student's reasons are presented in figure 1. Upon analysis of this particular student's response the teacher gained insight into the child's knowledge of decimal representation of money; of counting in two's; of addition, subtraction, division facts; the division concept, and written computation; knowledge which would not have been evident if only numerical answers had been written. By taking this first step in experimenting with alternative assessment forms, this teacher found that she could find out much more about her students' mathematical knowledge and understanding, and this assisted her begin to utilise more open investigations. As she stated: "I am introducing more varied and challenging tasks, not all for assessment purposes."



Jimmy had two tomatoe plants if he had grown eight tomatoes he would have four on each tomato plant.

half of twelve is 6

each tomatoe is worth 2c

total 16c

total 8c

Jimmy
Paula
02c 02c
02c 02c
02c 02c
02c 02c
02 02
02 02
4
+8
12
5
+5
10
+12
-10
2
2
÷1
1

Figure 1. Student response to a mathematical assessment task

Valuable assessment data from "chaos": As a result of trialing alternative tasks in the classroom 'surprising' evidence of students' mathematical knowledge appeared. One teacher developed a "Smarties" task for Year 8 students, trialled while students were studying a unit on Percent. In groups, the students were provided with a box of Smarties and asked to explore the contents, presenting information in any way they wanted. Out of the chaos the classroom was plunged into as a results of providing such a stimulus, one student spontaneously revealed understanding of ratio knowledge, writing the ratio of one colour Smartie to that of each of the others, and actually simplifying the ratios. The teacher observed that, of the two students working, only one was writing ratios while her partner watched. The teacher watched as the 'ratio writer' explained to her partner what she was doing. Adding this as an anecdotal note to the student's written response, the teacher found that evidence of this student's ratio understanding was obtained.

Teacher-made tasks; other tasks: During the project, team members were provided with ideas for tasks, which had been developed from various stimuli, including fairy stories, household goods, everyday children's experiences. The tasks were deliberately open to be suitable for a range of year levels. Writing team members were asked to adapt and trial any of the tasks in their classrooms. It was found that few team members actually trialled these tasks, or if they did, the activities were merely presented to students at times additional to classroom mathematics lessons, and students were given little direction on how to go about the activity. For example, one task, entitled "Does your shoe fit?" was designed for students to investigate the area of their foot and their shoe. Many students, in responding to the title of the task, answered "Yes", or "No". It was apparent that tasks 'out of context' in a teacher's mathematics program were not readily explored by teachers for assessment purposes. Thus, teacher-generated tasks that link directly to the mathematics program appear to be more easily trialled and evaluated by teachers in terms of richness for assessment purposes.

The process of change in teachers involved in writing and trialing assessment tasks

Throughout the course of the project meetings, teachers found that both their role and that of their students altered as they trialled alternative assessment tasks. Teachers' reported that they felt their role had become one of an observer of children, as suggested by the following comments:

> " [I make] keener observations of children as they work on assessment tasks."
>
> " [I am] more observant and understanding of the capabilities of the students. Less teacher direction."
>
> "[I am] looking at activities and seeing the children's responses to them."

Teachers perceived that alternative forms of assessment required students to write about their mathematical experiences. This change to the student role in mathematics necessitated that teachers provide students with opportunities to acquire such skills. The following comments show awareness of this dimension to the student role in assessment:

> " I now have children write more of what they verbalise during maths lessons because I see this skill as being a vital part of documentation and it must be taught so that children gain experience in it."
>
> " [I have an] awareness of children's difficulties when writing responses - this is a whole new ball game for them."
>
> " Children at present are having difficulty writing responses or explanations to activities, but I'm sure it will be practice makes perfect."

Reporting on how their involvement with the project had influenced teaching and assessment practices, it appears that the project enabled a change to more variety in assessment to take place. The following comments by team members provide this evidence:

> " By using the tasks designed at the meetings, I have used more group collaboration to find solutions, I now give more time to working through

problem areas with my children."
" [I now] use more ways in which students are given opportunities to verbalise and explain the outcomes of mathematics situations."
"I now try to include open ended tasks which require written assessable answers."
" [I am] directing my assessment away from the end of unit test to a program incorporated into my teaching program."

The product

At the conclusion of the project, approximately 50 assessment tasks had been trialled in classrooms, with approximately 25 suiting the purposes of the publication. The tasks were identified as RATs: due to their yielding a lot of data on student performance across a range of strands and substrands; they allowed all students in the classroom to respond across a range of levels; the activities were rated as worthwhile learning activities in their own right; and the students enjoyed working on the activities. The number of RATs produced for exemplary purposes was less than anticipated by the project initiators, and also by one team member, who stated that the "output in actual task writing has not been as great as expected."

As a result of working with the RAT writing team, the project co-ordinators developed eleven tenets (and their corollaries) for RAT (rich assessment task) development and evaluation (Dole & Tansley, 1996), which provide a summary of key issues in the process of searching for classroom RATs. The tenets are based on conclusions drawn from the RAT writing process. The first four tenets relate to the student in the assessment process; RAT tenet 5 relates to teaching, learning, and assessment and the mathematics curriculum, RAT tenets 6-8 relate to the teacher in the assessment process, and RAT tenet 9 cautions against collecting a RAT and labelling it as 'THE assessment task'. RAT tenets 10 and 11 are separated from the first 9 RAT tenets, as they specifically address assessment and reporting schemes. The tenets (and corollaries) are as follows:

1. A RAT allows students to reason mathematically, using logic and mathematical verification. *(Corollary: assessment tasks requiring students to find a single, correct answer are not RATS).*
2. A RAT enables students to draw together their mathematical knowledge. *(Corollary: assessment tasks requiring rehearsed performance are not RATs).*
3. A RAT is sufficiently novel to allow students to enjoy working mathematically. *(Corollary: assessment tasks which students find boring are not RATs).*
4. A RAT should be a worthwhile activity in its own right. *(Corollary: assessment tasks which do not promote student learning are not RATs).*
5. A RAT should reflect what really counts in learning and doing mathematics. *(Corollary: assessment tasks which test isolated mathematical skills are not RATs).*
6. A RAT integrates seamlessly into the learning environment. *(Corollary: assessment tasks which are unnatural to a teacher's teaching style are not RATs)*
7. A RAT is efficient of teachers' time. *(Corollary: assessment tasks which yield little data are not RATs).*
8. A RAT enhances teaching. *(Corollary: assessment tasks which are used for purely summative purposes are not RATs).*
9. A RAT has a limited life expectancy. *(Corollary: assessment tasks which have passed their shelf life are not RATs).*
10. A RAT allows performance across various strands. (Corollary: assessment tasks aimed at particular strands are not RATs).
11. A RAT allows performance at various levels. *(Corollary: assessment tasks aimed at a particular level are not RATs).*

Conclusion

Involvement of teachers in a RAT writing team appears to have assisted teachers explore issues in implementing alternative assessment tasks in their mathematics classrooms, and assisted in helping them adopt a more *informal* (Clarke & Reed, 1992) approach to mathematics assessment. Developing and evaluating RATs for the classroom is not a simple task. As a result of this project, it can be seen that for a task to be deemed a RAT, it must be trialled in the classroom. To be adequately trialled in a classroom, it must link with a teacher's mathematics program and the teacher's style of teaching. To explore the richness of the RAT requires change in both the teacher's and students' role in the assessment situation. Searching and implementing RATs in the classroom is an evolving process.

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Collaborative Team Writing of Assessment Tasks as Professional Development

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Teachers were brought together to produce a package of exemplary assessment tasks, rich in their potential to provide data on students' mathematical understanding and knowledge, and thus link with the national profile as a reporting framework. Through involvement in the writing team, it was found that teachers actually guided their own professional development. The writing team situation provided teachers with (i) support, (ii) feedback, (iii) opportunities for reflection; as well as assisting them gain (iv) confidence in using, and (v) knowledge and understanding of, the national profile.

The national mathematics profile (Australian Education Council, 1994) is a framework for recording and reporting student performance in mathematics. It follows from the national mathematics statement (Australian Education Council, 1990), and organises the mathematics curriculum into 6 strands (space, number, measurement, chance and data, algebra, and working mathematically) with 8 performance levels spanning the school years 1 - 10. The national profile document was adopted completely by the Queensland Education Department, and retitled Student Performance Standards (SPS) (Department of Education, Queensland, 1994). SPS as a reporting framework was partially utilised by Queensland state school teachers in 1995, with Years 3 - 8 teachers reporting on students' level of mathematical performance for the three strands of number, space and measurement.

The organisational structure of the SPS framework, while aligning the structure of the national statement, is fundamentally different to the Queensland mathematics syllabus for Years 1-10. Also, teachers found that mapping student assessment data to SPS levels was not a simple translation task. Alternative forms of assessment were required (Bleicher, Cooper, Dole, Nisbet & Warren, 1996). Thus, by its nature, SPS as a reporting framework required teachers to view the mathematics syllabus from a different perspective, and to change assessment practices.

This paper reports on the efficacy of collaborative writing of assessment tasks as a professional development activity. In particular, it evaluates the effect of the collaborative writing program on teachers' knowledge, beliefs and confidence with respect to assessment.

Professional development and teacher change

Implementation of SPS, in requiring change on two major fronts, met with a lot of teacher resistance (Bleicher, et al, 1996). Change in teaching practice comes about when teachers' beliefs and attitudes about aspects of their practice change (Clarke & Hollingsworth, 1992). Similarly, resistance to change is also due to attitudes, beliefs and experience (Grimison, 1993; Mousley, 1991). Berliner (1986) has suggested that teacher beliefs and attitudes change when student outcomes to teaching are observed to improve. It could be argued that teachers' beliefs and attitudes towards their current practices in mathematics would not lead to greater student understanding as a result of implementing SPS; hence resistance to SPS.

Overcoming teacher resistance to SPS required teacher change. A common method of effecting teacher change is to encourage teachers to participate in professional development programs involving various forms of inservice activities. As Guskey (1985), Guskey and Sparks (1991) and Smylie (1988) have shown, the effectiveness of inservice is enhanced if teachers have opportunities to share their best practices, trial new ideas in their own classrooms , assess student understanding and bring feedback to the next inservice session to share with colleagues. This is supported by deLange (1992) who argued that an integrated approach, which provides teachers with time to experiment, gain experience and build confidence and focus on assessment was crucial to the success of inservice projects. Research into effective professional development that promotes change in teaching practice (e.g. Clandenin & Connelly, 1991; McLaughlin, 1990) has identified the following as essential:

(1) teachers' prior beliefs and attitudes are important elements in the change process;
(2) worthwhile and enduring change is a slow process that requires commitment and risk from teachers;
(3) collegial support in the form of regular meetings and discussions is an important factor to the change process;
(4) experience and reflection are necessary for effective change and input is needed from sources outside the schools to facilitate this reflection and to clarify and introduce different ways of considering situations;
(5) teachers' perceptions of successful and improved student learning is crucial to the success of the change process;
(6) teachers need experience with new strategies before they will change their attitudes and beliefs to them (awareness and knowledge of new strategies is not sufficient for their adoption to the classroom); and
(7) senior school staff support and commitment is a crucial component of successful school change.

Clarke and Hollingsworth (1994) have suggested that teacher change can be viewed from six different perspectives: (i) change as training, where professional development programs provides teachers with appropriate teaching skills and/or where inappropriate teaching practices are identified and corrected in an 'evangelistical' manner (p. 154); (ii) change as adaptation, where teachers change as a result of a change to the working environment (e.g., increased class size, new school policy); (iii) change as personal development, where teachers themselves identify their own needs and seek to develop additional skills and strategies to improve their classroom performance; (iv) change as local reform, where teachers work together to change their working environment; (v) change as systemic restructuring, where teachers must respond to and implement change imposed by external bodies; and (vi) change as growth or learning where teachers come together as a group to discuss, work through, issues of significance to initiate and sustain change.

From Clarke and Hollingsworth's six change perspective categories, it can be seen that the introduction of the national profile for report student achievement in mathematics aligns with the change as systemic restructuring perspective, where the location of change is external, the teacher's role is that of implementer, the object of change is the curriculum and the inservice character is systemic (p. 160). However, as Clarke and Hollingsworth have suggested, the categories are not mutually exclusive, and it could be argued, that as a result of change through systemic restructuring (as with the adoption of the national profiles) may actually cause teachers to change in accordance with other change categories. For example, a teacher may be provided with systemic in-service on the national profiles and this may cause the teacher to reflect upon his/her own further professional development needs (change as personal development) and seek such professional development. Certainly, it appears that the implementation of the national profiles would cause many teachers to go through an adaptation process (change as adaptation).

Rich assessment tasks and collaborative writing

To adopt SPS, teachers required professional development at 3 main levels: (i) familiarisation with the structure of the SPS document; (ii) experience in using various and varied methods of mathematics assessment; and (iii) practice in recording student assessment data using the SPS framework. Systemic inservice was provided for state school teacher, in the form of systemic restructuring category of professional development (Clarke and Hollingsworth, 1994). In an effort to further support teachers to implement the national profile, the Queensland Association of Mathematics Teachers/Professional Development Project (QAMT/PDP) sought to produce a package of assessment tasks which had been trialled in classrooms and found to provide rich data on students' mathematical performance, and thus align with SPS. The purpose of such a package was to provide teachers with ideas for assessment and to exemplify how mathematical activities, alternative to pen and paper timed tests, could be used in the classroom for assessment purposes. The trialing of the activities would enable the richness of the task to be evaluated. For this initiative, practicing classroom teachers were brought together for the purpose of writing rich assessment tasks; tasks which had been developed and trialled at a 'grass-roots' level by teachers for students in their own classroom. A sub-purpose of this project was to provide participants with an opportunity to become more familiar with SPS and thus guide their own professional development This professional development aligned with the Clarke and Hollingsworth (1994) category of change as growth or learning.

There have been many professional development programs implemented in the Unites States that involve teachers in collaborative writing (e.g., Blau, 1988). The central features of these programs are that teachers work together in groups with a mentor, supporting each other as they write. When these programs focus on the development of curriculum ideas, they generally involve the teachers in trialing their ideas in classrooms and, therefore, in receiving feedback from both the students and the group. The sharing and discussion involved in the collaborative act ensures that the participating teachers reflect on their writing both before and after the trials.

Research has shown that teachers are able to establish communities of collaborative writers and that these communities have produced effective curriculum materials and increased general confidence in teaching (e.g., Santa Barbara Classroom Discourse Group, 1995). The question is can the collaborative writing process enable effective professional development of mathematics teachers and promote change in teaching practice, particularly with respect to assessment.

Method

The methodology used in the study was participant observation (Spradley, 1980). One member of the research team established this role with a group of teachers. She organised and facilitated the group's meetings. The level of involvement was what Spradley called "active participation" - in this case, the researcher interacted fully with the participants in her official role of group facilitator. Participants in this study were 12 primary school teachers, 6 female and 6 male, and two group facilitators, one of which was the researcher. Data were collected through fieldnotes based on researcher observations combined with ad hoc group interviews, plus an end of inservice survey to ascertain effect of the program on participants.

The writing team met together four times over a four month period during the school year. Meetings were for the whole day and were organised in a three session format:

(1) an opening session where participants discussed assessment issues in the first meeting and shared their experiences in trialing the assessment tasks with other teachers in the second, third and fourth meetings;

(2) a second session where the teachers brainstormed new assessment ideas and techniques in small groups in the first, second and third meetings and refined tasks in the last meeting; and

(3) a final session in which individuals or dyads planned the trials of the assessment ideas from the second session in the first, second and third meetings and organised final writing and publishing in the final session.

This format afforded the opportunity for teachers to share their classroom experiences of trialing new ideas and techniques of assessment with critical friends/colleagues. At the end of the final session, participants completed a feedback survey which covered positive and negative aspects of the program, changes in teaching and assessment practices, attitudes to SPS, support networks, suggestions for improvement, and confidence in running similar programs.

Results

Data were collated and discussed by the research sub-committee (composed of the authors of this paper). A constant comparative method of analysis (Guba & Lincoln, 1989) was employed with these data. Through the discussion and re-examination of the entire corpus of data, there emerged themes that helped explained the relationship between the inservice activity and the kinds of classroom experiences and attitudes reported by participants.

This analysis is reported in six parts. First, to contextualise the analysis, the teachers' responses to the collaborative writing process in the *meetings* are briefly described. Second to sixth, to structure the reporting of the results, the teachers' reactions to the collaborative writing as a professional development are presented under the five major components of the process. These components are inherent to the structure of the collaborative writing process, that it provides teachers with *support*, offers *feedback*, allows opportunities for *reflection*, particularly with respect to classroom practice, increased *confidence* in their own ability to write their own assessment tasks, and increased *knowledge and understanding of SPS.*

The meetings: Overall, participants reported that their knowledge and confidence grew in richness and depth as meetings progressed over time. Participants felt that the meetings were worthwhile and were important as learning experiences. They also appreciated the opportunities to trial ideas in their classrooms. They valued both the meetings and the trials, and stated that the good experiences in previous meetings provided a good measure of the motivation to attend subsequent meetings. They were able to prepare assessment tasks that they were happy to use in their own classrooms (Dole, 1996). In the third and fourth meetings, teachers expressed feelings of belonging to a valued group of colleagues. In the final meeting, the unanimous feeling of the group was the wish that such meetings could continue throughout the whole school year. For example, one participant commenting on how she felt about working with colleagues, said: "to recognise the wealth of collective knowledge is exhilarating".

Support: All participants indicated that having the opportunity of sharing both their successes and difficulties from their own classroom experiences was a highly appreciated aspect of the activity. For most, this was a new experience - they very rarely had been afforded the opportunity of sharing their teaching experiences with colleagues at any length in the past. This support for such sharing became a highly valued aspect of the model. All participants agreed that the genuine interest shown by their colleagues in listening to each other's classroom experiences was highly motivating and supportive.

The following comments from participants illustrate this theme and indicated how participants perceived the collegial support provided at meetings:

- meeting and working with teachers from other schools was a highlight
- the relaxed atmosphere made it easy to express all concerns about mathematics teaching and learning
- within a professional environment being able to share resources ideas and concerns was very supportive

It was evident that teachers developed increased self-efficacy and confidence in their teaching and assessment skills. They also stated that they now believed they could successfully add new teaching and assessment techniques to their repertoire.

Feedback: Participants consistently expressed their appreciation of the advice and constructive criticism they received from colleagues at the meetings. They felt that receiving immediate feedback through the meetings not only helped future planning, but further added to the belief that colleagues were interested in one another's work. They stated that they would have been disappointed if the sharing of experiences had stopped.

Participants clearly expressed that the bottom line for them was whether their practice made a difference to student learning in their classrooms. The time element turned out to be important - it was optimal for teacher's to be able to perceive an improvement in student outcomes during classroom activities, be they especially highlighted for assessment or otherwise. Instances where student interaction during an assessment activity was immediately perceived as positive were particularly motivating for the teachers. The teachers perceived interactions as positive if they felt there was an improvement on the kind of student interaction formerly elicited in such situations.

The following comments from participants illustrate the importance they gave to feedback:

- provided an opportunity to obtain feedback advice on assessment tasks
- re-affirmed my beliefs about teaching learning and assessing
- my ideal was always that teaching and learning tasks could be used for assessment, but that "nervousness" of "that it may not be enough" has been allayed more
- I'm more selective - willing to discard activities which are limiting in results
- feedback made me think more carefully about what I am assessing and adjust tasks to suit
- has given me more ideas on how to assess and set tasks that are relevant

Reflection: Participants were clearly changed by the collaborative writing process. The aspect of the collaborative writing program that seemed to produce these changes was the reflections that were an essential component of the group activities. The structure of the program was such that participants were given opportunities to share with their colleagues the tasks they had written and their students' responses to these tasks. This sharing encouraged reflection, both on the assessment practices and on the collaborative writing process itself. This had two outcomes. The reflection on assessment provided the participants with a depth of understanding that they would not have otherwise gained and improved the quality of the written products. The reflection on the process highlighted the positive role of the other teachers and the success of the program. This in turn improved the morale of participants and promoted confidence and self esteem. The following comments from participants lend support for this:

- I am aware of not limiting the students to what they can demonstrate they are capable of doing (i.e. certain forms of writing assessment as those which appear in the sourcebook)
- while trying to ensure the basic facts and concepts are thoroughly covered, I am introducing more varied and challenging task - not all for assessment
- I have gained more of an insight into assessment and how it affects me and my teaching style

- I'm more aware of varying abilities
- I now have the children write more of what they verbalise during maths lessons because I see this skill as being vital part of documentation and it must be taught so that children gain experience in it
- I am aware of children's difficulties when writing response - this is a whole new ballgame for them
- I use more ways in which students are given opportunities to verbalise and explain the outcomes of math situations.

Confidence: The project also appeared to provide participants with a sense of reassurance that their own personal feelings of insecurity in implementing SPS were common amongst fellow teachers, as the following comments show:

- Realisation that you are not alone (mathematics wise)
- the ultimate ego boost of seeing that what you're doing is OK
- appreciating that the same difficulties are faced by others
- to discover that people have same concerns as I have
- to know that other teachers were experiencing the same difficulties as oneself is encouraging and supportive
- exchanging fears and frustrations [the meetings] have increased my awareness of fellow teachers problems in applying SPS to their teaching

Knowledge and understanding of SPS: Participants commented that they found this professional development exercise had had a positive influence on their attitude to SPS, as supported by the following comments:

- definintely clarified the link between the syllabus and SPS outcomes
- demonstrated a more effective method of data collection
- I have oscillated from negative to positive to negative to positive to reserved positive as I have had more involvement with SPS
- probably my attitude is more positive than previously
- because I am much more familiar with SPS, I no longer find it threatening
- reaffirmed (rather than changed) I'm on the right track. [Meetings have] certainly made [SPS] less ominous, not to be dreaded; boosted confidence, even raised enthusiasm.
- I understand the positive aspect of SPS, ie children's control over assessment, children's responsibilties for assessment; assessment of what children can do.
- a broader understanding of SPS levels and strands, and therefore confidence in assigning levels to students work.

Discussion and Conclusions

The findings above support the conclusion that participation in the collaborative writing activity was an effective means of changing teacher practice and increasing confidence. In particular, participation in collaborative writing professional development appears to be a viable means for effecting change in practice while, at the same time, nurturing teacher confidence. The reasons for this are that all seven factors from the research of Clandenin and Connelly (1991 and McLaughlin (1990) that promote effective professional development are satisfied by collaborative writing. Especially prominent were the following: collegial support in the form of regular meetings and discussions is an important factor to the change process; experience and reflection are necessary for effective change and input is needed from sources outside the schools to facilitate this reflection and to clarify and introduce different ways of considering situations; teachers' perceptions of successful and improved student learning is crucial to the success of the change process; and teachers need experience with new strategies before they will change their attitudes and beliefs to them (awareness and knowledge of new strategies is not sufficient for their adoption to the classroom).

Participants were clearly influenced by whether their practice made a difference to student learning in their classrooms. The time element turned out to be important - it was optimal for teacher's to be able to perceive an improvement in student outcomes during classroom activities, be they especially highlighted for assessment or otherwise. Especially motivating were instances of students interacting directly with the teacher in the context of activity based assessment tasks in ways perceived by the teacher to be an improvement on the kind of interaction formerly elicited in such situations.

Therefore, the results support the following three conclusions:

(1) the collegial support of the collaborative writing process resulted in increased confidence for the participants with respect to teaching and assessment;

(2) the collegial feedback from the classroom trials and discussion with other teachers improved the teaching and assessment ideas of the participants; and

(3) the reflection encouraged by the process on classroom practice improved the teaching and assessment practices of the participants and improved students' learning outcomes.

Factors that were strongly affected throughout the program were the attitude and beliefs of the participants, particularly confidence and self-esteem. Therefore, the program supported the findings of Clarke and Hollingsworth (1994) that changes in teaching practice is preceded by changes in attitudes and beliefs. As well, the program supported the findings of Berliner (1986) that beliefs and attitudes change when student outcomes are seen to improve, and the findings of Guskey and Sparks (1991) that trialing and sharing are crucial for effective inservice.

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Teaching Statistics: Teacher Knowledge and Confidence

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This paper reports on research in progress that explores primary school teachers' ideas about statistics and the teaching of statistics. Data on teachers' attitudes and beliefs was collected through interviews and the use of belief and attitude scales. Issues related to teachers' lack of statistical training, confidence in teaching statistics and their views about essential knowledge for teaching statistics are explored.

Introduction

Recent changes to school curriculum documents in New Zealand place considerable emphasis on statistics. The mathematics curriculum, in particular, includes statistics as one of six major strands of study (Ministry of Education, 1992). Statistics is now no longer peripheral to the primary school curriculum but is part of 'mainstream' mathematics from the beginning of school onwards.

While changing a curriculum document may come at the 'stroke of a pen' successful implementation of curricula reforms is often problematic. A recognised key player in the curriculum reform process is the teacher (Shaughnessy, 1992). Recent work in the area of teacher thinking and mathematics education sees changes to teachers 'world view' of mathematics as important in bringing about change in mathematics education (Steffe, 1990). A teacher's 'world view' of mathematics is thought to be related to deeper beliefs and knowledge. Ernest (1989) for example, states that:

> It depends fundamentally on the teacher's system of beliefs, and in particular on the teacher's conception of the nature of mathematics and mental models of teaching and learning mathematics. Teaching reforms cannot take place unless teachers' deeply held beliefs about mathematics and its teaching and learning change (p 249).

This paper sees teachers as possessing a rich store of knowledge and experience in regard to statistics. Gaining a greater understanding of what teachers think and understand is seen as important as their 'ideas' form part of the dialectic in the process of learning. If we expect teachers to help students, we also need to deal with teachers' own ideas and understandings.

Teachers' Ideas about Statistics

At the wider societal level we know that statistical notions are pervasive (eg. in the media) and that people are easily influenced by common probabilistic misconceptions (eg. purchasing lotto tickets). As teachers and even mathematics education researchers are, apparently, not immune from these 'misconceptions' (Shaughnessy, 1992) it is important to explore the statistical notions that teachers take into their classrooms. A perusal of the literature reveals that teachers' ideas about teaching statistics have received little attention. In New Zealand research in this area is almost non-existent. Among the aspects highlighted by overseas researchers are such issues as; the lack of preparedness of teachers for teaching statistics (Greer & Ritson, 1993); teachers lacking in essential knowledge and understanding of many statistical concepts (Russell & Mokros, 1990 and Bright, Berenson & Friel, 1993); teachers 'avoiding' teaching statistics due to a lack of confidence (Gal, 1992); a correlation between 'personal confidence' and teaching confidence and the suggestion that teacher confidence may be 'misplaced' due to teachers understanding of concepts (Callingham, Watson, Collis and Moritz,1995); and lastly, work by Rubin and Rosebery (1988) in the USA concluded that teacher's prior knowledge of statistics is not necessarily structured to facilitate the statistical reasoning required by the new statistics curriculum initiatives.

Focus of Paper and Background to the Research Approach

This paper focuses on primary teachers' views about their knowledge and confidence related to teaching primary school statistics The work reported was undertaken as part of a larger study which adopted a broad approach to teachers' ideas subsuming knowledge, beliefs, attitudes, concepts, perceptions and understandings under the umbrella term of 'ideas'. Teachers' 'ideas' here refers to the thought structures of the teacher; the knowledge, beliefs and attitudes stored as schemas in the mind of the teacher (Ernest, 1989).

The larger study used a number of research techniques including semi-structured and clinical interview approaches, concept maps and survey scales. This report, however, focuses on only a part of the data and considers just the responses to the 'survey scales' and some of the interview data. Interviews were used as they are generally regarded as a useful means of exploring someone else's ideas or thinking about something (Merriam, 1988). The principal reason for including survey scales in the larger study was for triangulation and comparison purposes and not for their efficacy as research instruments. One of the main drawbacks in using these survey scales is that one has little idea about what each rating actually means. Other researchers (eg. Gal & Ginsburg, 1994) currently working in the statistics education area advocate a more open-ended approach when using such 'scales' in the belief that the additional information gained will provide insights into how the scales have been used and thus help explain why people think what they do. Hence, in this study respondents were also asked to 'talk aloud' as they completed the scales. In the tradition of research on teachers 'beliefs, attitudes and conceptions, the study reported here is generally interpretive in nature and predominantly qualitative.

Method

Sample

Twenty two inservice primary teachers were surveyed using a semi-structured interview format. The sample is best described as an 'opportunistic' sample. Under this approach friends or colleagues were asked to provide contacts with primary teachers and these teachers in turn provided other contacts. This method of selecting the sample was a deliberate choice as initial efforts at more formal approaches in schools captured only teachers with an interest in mathematics. The 'opportunity' approach largely eliminated this self-selection bias. Care was taken in selecting the sample to ensure a spread of primary teachers from Years 1 to 8. Despite the small sample size it is reasonably representative of the New Zealand primary teaching work force in that it favours female teachers with a significant amount of teaching experience. Background data collected from the teachers indicated that they were fairly well qualified with more than half holding at least a Bachelors degree or its equivalent. Very few however had studied any mathematics or statistics at a tertiary level beyond what was required in their teacher training programmes. More than half of the group had dropped school mathematics by year 10 with most of this group reporting difficulties with mathematics at school. In addition, only four of the group gave mathematics as their principal area of teaching interest while more than half indicated language and arts areas.

Procedure

Participants were each interviewed using a semi-structured interview format. Each interview took approximately 60 minutes and focussed on getting the teachers to talk about their views of statistics and the teaching of statistics. The survey scales were introduced about 35 minutes into the interview and teachers were encouraged to 'talk aloud as they completed the 'scales'.

Instruments

In addition to the semi-structured interview two forms of scale and rating were used to collect the information presented in this study. The first of the scales, a beliefs and attitude scale (BAS) assessed levels of agreement with statements about statistics and its teaching while the second, a confidence rating scale (CRS), asked teachers to rate on a numerical scale, their confidence in teaching particular topics or concepts in statistics. The statements included in the BAS, as well as the structure of the larger study itself, were organised along the lines of Ernest's (1989) model for exploring teachers beliefs, knowledge and attitudes about mathematics and its teaching. This model casts a wide net and includes beliefs, attitudes and knowledge (including both subject content knowledge and pedagogic content knowledge), rather than trying to deal with them as separate and distinct constructs. The BAS was made up of 37 statements about statistics education, including statements on mathematics. Most of the statements can be grouped into subscales along the lines of the components in Ernest's model. Examples of subscales include; beliefs about the nature of statistics and mathematics, beliefs about how to teach statistics and mathematics, knowledge of teaching mathematics and statistics with an emphasis on pedagogic content knowledge, and attitudes towards statistics and the teaching of statistics. The statements themselves came from a variety of sources within the mathematics and statistics education literature (eg. Meredith, 1993 and Callingham et al., 1995). Teachers were asked to rate on a continuous scale their level of agreement or disagreement with each of the 37 statements. The scale is similar to that used in Callingham et al's. (1995) study. A sample scale is shown below.

Strongly Agree SA |_____|_____|_____|_____|_____|_____| SD Strongly Disagree

The confidence rating scale (CRS) was adapted from a similar survey by Bright et al.(1993) and used terms or ideas from levels 1 - 4 in the 'Mathematics in the New Zealand Curriculum' (Ministry of Education, 1992). Teachers were presented with a scenario of having to teach a statistics unit to a standard 2 (Yr 4) or a Form 2 (Yr 8) class and were asked to indicate their level of confidence in dealing with each of the terms or ideas at a particular class level. Teachers were given a choice of three categories to indicate their level of confidence (1=Not confident, 2=Somewhat confident and 3=Very confident).

Analysis

Data from the interview transcripts and survey scales was summarised according to emergent themes relating to the guiding questions of the study. As analysis work is still in progress on the larger study the BAS scale data presented here is in a non-numerical form so that the original data is not lost. Instead a visual format, a stem and leaf approach, is used. Each teachers' response to a statement is placed on the one scale with the lower numbers representing teachers working lower down the school.

Results and Discussion

As the results discussed here come from a larger study, currently in progress, the findings are preliminary in nature. In discussing the results I first present some general findings from the survey scale (BAS and CRS) data. Second, the two interrelated themes of teacher confidence and essential knowledge for teaching statistics are explored by augmenting the survey scale data with data from the interview and 'talk aloud' approaches.

Some Preliminary Findings

- Most of the teachers view teaching statistics as the same as teaching mathematics. However, they do acknowledge that some aspects are different (eg.. the hands-on, pervasive nature of statistics). In addition, while the teaching of statistics is generally located in mathematics it is not always taught there. Its interdisciplinary character is recognised by most.

- Most do not see ability in mathematics as a constraint to understanding basic statistical concepts. On the contrary most see statistics as providing students who have had difficulties with mathematics as giving them another 'chance'.
- Most see their role as teachers as being to facilitate student learning. Teaching for them is not a case of communicating their understanding of statistics or giving answers to help students learn. They largely see their role as being to explore ideas with the children.
- While nearly all teachers were sceptical of the way statistics is often used, particularly by politicians, they acknowledge its value and usefulness to society.
- Statistics is generally regarded as an integral and necessary part of the primary school curriculum.
- While many teachers expressed doubts about their own statistical knowledge, particularly of more 'formal, traditional' statistics, nearly all said they were comfortable with everyday statistical notions as are found in the media.
- Most teachers expressed considerable confidence about teaching statistics as long it was at their usual class teaching level. 'Anxieties' usually only became apparent when teachers were asked to move out of their particular class level or to deal with unfamiliar material.
- Although many of the teachers reported they would like to learn more statistics this is not seen as a priority for their teaching.
- Most of the teachers indicated that group work was a common practice when teaching statistics.

The use of technology was generally considered advantageous but not essential for teaching statistics.

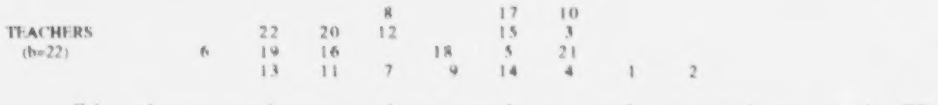
- While most of the teachers regard statistical knowledge as being important they do not necessarily see this as being the most important component in their teaching. Having appropriate statistical activities, for example, is seen as just as important or more important than statistical knowledge 'per se'.
- The teachers also reported that their own understanding of mathematics and statistics had been transformed in the process of teaching it.

Teacher Confidence

In this section I first consider teacher background and self concept with respect to statistics and then discuss their confidence with respect to teaching statistics.

Teacher background and self concept with respect to statistics. I begin this section on teacher confidence by first reiterating a few pertinent background characteristics of the sample. Firstly, more than half the teachers surveyed have at least a degree or its equivalent, with only one specialising in mathematics. Secondly, more than half of the group reported having difficulties with mathematics or statistics during their education. Of the rest only four indicated success in mathematics or statistics during their education. Lastly, more than half said their key area of teaching interest was in language and arts. Only a few gave mathematics as their key area of teaching interest.

Considering the teachers backgrounds my initial expectation was that they would have a low opinion of their own mathematical and statistical abilities. Results however revealed quite a range of views. This can be seen in the response pattern, shown below, to the statement 'I am good at statistics'.



Further, most teachers feel comfortable with the everyday statistics found in the media. The response pattern to the statement 'I can easily read and understand the statistical terms, graphs and charts in the media (newspaper and television)' showed:

```
                        9                   2
  TEACHERS       19     18    22            21
   (b=22) 6      10     8           15
                 7      13    12            14                               20
                 11     16     1    17      5      3                          4

          SA      |------------|------------|------------|------------|------------|------------|      SD
```

Analysis of relevant interview data revealed other aspects with most teachers expressing reservations about their statistical knowledge based on either the class level they reached in their education or on how they performed in school or other courses. This can be seen in one teachers comment that 'I am not very good at maths because I actually dropped out of school in the fourth form (year 10).' Another factor that influenced self concept was their teaching experiences. All felt that their knowledge was sufficient for the levels they were teaching. This is clearly seen in a comment by the teacher just mentioned above: 'I feel comfortable because what I am teaching is really at a very simple level'. Many teachers, however, expressed doubts about their knowledge when asked to move away from their usual teaching level or in regard to more formal, traditional statistics. The same teacher, mentioned above, commented in this regard:

> 'I would totally freak out. If somebody suddenly said to me look, you either teach at an intermediate level (yrs 7 and 8) or you are out of a job, then I would be out of my tree. Literally, because I know or I am almost 90% sure that somewhere along the line I would be tripped up on statistics'.

To add a further dimension this particular teacher qualified her view by saying:

> No I guess I wouldn't (freak out) because I am not that sort of person. What would happen is every night I would be honing up on what I was to teach the next day. I would literally be going one step ahead of the kids and hoping that no kid asked me some curly question.

Confidence with respect to teaching statistics: Regardless of self concept nearly all teachers said they were confident or felt secure when teaching statistics. This can be seen in the response patterns to the following statements:

- Generally I feel secure about the idea of teaching statistics.

```
                        19
                        20
                        22           3
  TEACHERS              13     8     9
   (n=22)        6      14     5    21
                 7      16    12    15     11     17     4    10     18      2      1

          SA      |------------|------------|------------|------------|------------|------------|      SD
```

- I would find it difficult to explain to someone what an average is.

```
                                                                                  15
                              21                                                   8     20
  TEACHERS                    22                                            10     1     17
   (n=22)               6      2                                      3     14    18     19
                              12      4                  7            5      9    11     16    13

          SA      |------------|------------|------------|------------|------------|------------|      SD
```

A further indication of the confidence of these teachers with regard to their teaching of statistics comes from their views about whether they would give priority to inservice or professional development work in statistics when compared to the other strands in the mathematics curriculum. Even though most have had little training or inservice in statistics three quarters said it would either not be a priority or it would only be of low priority. Nearly all of this group said they felt confident with their teaching in statistics and instead would give priority to other strands in the mathematics curriculum, with algebra being mentioned most often.

These response patterns coupled with the interview data suggest that the teachers feel quite confident in respect to teaching statistics. Further analysis showed that as long as the teachers are working at their usual class teaching level their confidence is high. The view that teachers may be 'reluctant' (Gal, 1992) to teach statistics because of a lack of confidence gained only minimal support in this study.

Other concerns held by teachers with respect to confidence in teaching statistics relate to teachers lack of familiarity with statistical concepts and terms. Nearly every one of the teachers was unfamiliar with one or more of the terms taken from the curriculum for the CRS scale. Results from the CRS scales showed a clear pattern in terms of confidence. Teacher confidence is lowest on the newer ideas in the statistics curriculum such as, stem and leaf graphs and probability. Not surprisingly, teacher confidence is highest on the 'older' more familiar statistics curriculum topics such as, data collection, bar graphs and pictographs. This finding lends support to Callingham, Watson, Collis and Moritzs', (1995) suggestion that teacher confidence may be misplaced due to their understanding of statistics.

Knowledge Essential for Teaching

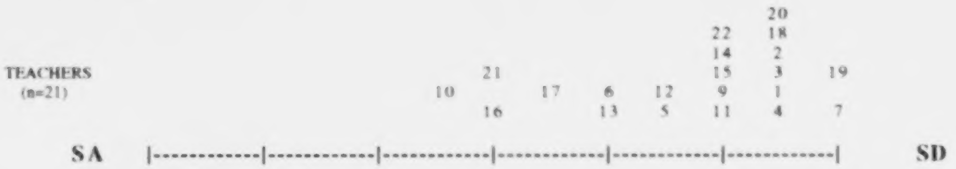
Some of the anomalies between teachers perceptions of their own statistical abilities(or lack of) and their high teaching confidence are illumined further when one looks at the teachers' views about whether statistical knowledge is essential for teaching. Much of the literature on teachers and statistics education points to concerns regarding teachers 'preparedness' for teaching statistics (eg. Greer & Ritson, 1993). These concerns about teachers 'preparedness ' appear to have their origins in teachers lack of statistical training. The jump from lack of statistical training to having difficulty teaching statistics is readily made, and assumed, by many of the researchers in statistics education. Hawkins (1990), for example, goes as far as portraying lack of statistical knowledge in medical disease terminology with the 'disease' being passed on to students. As was noted in the teacher confidence section the teachers themselves do not necessarily see their lack of statistical training as a problem for their teaching. The results in this study suggest that while teachers acknowledge the value of statistical knowledge 'per se' they are somewhat ambivalent, as a group, about how important it actually is. The teachers appear more concerned about the 'how' of teaching and consequently place more emphasis on finding appropriate ideas and activities than in gaining more statistical training. The response patterns below are indicative of these trends.

- A good understanding and knowledge of statistics is essential knowledge for teaching statistics.

```
                              22            19
                              17            15
 TEACHERS                     20            18
  (n=22)          13    3     21       1                     6    12
                   4    8     10   16  2     7     5        14     9    11

      SA    |-----------|-----------|-----------|-----------|----------|-----------|    SD
```

- Statistics knowledge is more important than knowing how to provide appropriate statistical activities for pupils.



This concern and interest in more appropriate activities is reinforced by the interview data. When the teachers were asked about what they would want in an inservice course on statistics nearly all said they would want more ideas and activities for use in their classrooms. Only a couple of the teachers said they would want further work on statistics itself. One teacher did not want any inservice on statistics at all as 'their school had, had enough on it already'.

Implications and Concluding Comment

These preliminary findings, if confirmed by the larger study, have a number of implications for curriculum development, teacher development, research and the teaching of statistics. The first implication is that the high teacher confidence ratings when coupled with the little interest shown in professional development in statistics do not bode well for reform efforts in regard to statistics education. Apart from finding good class activities the primary teachers surveyed are, by and large, not looking for additional statistical training input.

Secondly, the fact that the teachers lowest confidence ratings were recorded on the 'newer' areas of the statistics curriculum suggests their positive orientation towards the teaching of statistics may be based on teaching what is familiar to them. The newer curriculum initiatives in this area may have had only minimal impact.

Thirdly, while the considerable teacher confidence, coupled with low interest in gaining further statistical training could largely be due to the level (primary school) at which the study was conducted a number of other issues arise of a more philosophical nature. Perhaps, for example, teachers ideas about teaching and the nature of statistical knowledge itself already reflect, to some extent, the 'constructivist' philosophy behind the mathematics curriculum, or more so than do researchers who are calling for more statistical training. The apparent concern over teachers lack of statistical knowledge may be reflective of a view of teaching not necessarily consistent with the 'constructivist' philosophy underpinning the New Zealand mathematics curriculum. Concerns about the importance of the statistical knowledge of primary teachers may be more reflective of transmission models of teaching which emphasise the passing on of knowledge. The implication here being that you need to know something in order to be able to pass it on. It could be argued, on the other hand, that if we re conceptualise teaching along constructivist lines as more a process of negotiating meaning then teacher knowledge of statistical notions requires further examination. As such other aspects of teacher knowledge and understanding may be come more critical in the teaching / learning process.

Fourthly, the findings raise issues in regard to researching teachers ideas. I would argue that it is more useful to research from the perspective of exploring what teachers do understand and then how what they do understand impacts on the teaching and learning process rather than identifying what teachers don't understand or can't remember, as is often the case.

The primary teachers surveyed in this study appear positively oriented towards the teaching of statistics. While this may be good for statistics education generally it is not necessarily good news for teacher development and efforts at curriculum reform in statistics. This study reinforces the view that teacher's 'ideas' are an important element of the curriculum reform process and that it is more useful to start with where teachers' ideas are than where they are not..

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Newman Error Analysis: A Comparative Study Involving Year 7 Students in Malaysia and Australia

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After 206 Year 7 students—145 in 4 schools in Malaysian, and 61 in 2 schools in Australia—had answered 24 mathematics questions, they were interviewed, in accordance with the Newman interview technique. Types of errors made by the students in the two countries were analysed and compared. Data revealed that (a) around 70% of all errors were in one of the Comprehension, or Transformation, or Careless categories; and (b) strikingly different error patterns occurred for different questions.

In the mid-1970s an Australian language educator, M. Anne Newman (1977a,b) developed a systematic procedure for analysing errors made by students who had given answers to questions on written mathematics tests. According to Newman, any person confronted with a written mathematics task needs to go through a fixed sequence: Reading (or Decoding), Comprehension, Transformation (or Mathematising), Process Skills, and Encoding. Errors can also be the result of unknown factors, and Newman (1983) assigned these to a composite category, termed "Careless."

Over the past two decades the "Newman method" has been widely used throughout the Asia-Pacific region—in Australia (eg, Casey, 1978; Clarkson, 1980; Clements, 1980; Clements & Ellerton, 1992; Faulkner, 1992; Tuck, 1983; Watson, 1980); in Brunei (Mohidin, 1991); in India (Kaushil, Sajjin Singh & Clements, 1985); Indonesia (Ora, 1992); in Malaysia (Ellerton & Clements, 1992; Teoh Sooi Kim, 1991; Kownan, 1992; Marinas & Clements, 1990; Sulaiman & Remorin, 1993); in Papua New Guinea (Clarkson, 1983, 1991; Clements, 1982); Singapore (Kaur, 1995); in the Philippines (Jiminez, 1992); and in Thailand (Singhatat, 1991; Sobhachit, 1991; Thongtawat, 1992).

Analyses of data based on the Newman procedure have drawn special attention to the influence of language factors on mathematics learning, and by so doing have challenged mathematics educators and teachers to redefine what is "basic" in school mathematics. Newman research has generated a large amount of evidence pointing to the conclusion that far more children experience difficulty with the semantic structures, the vocabulary, and the symbolism of mathematics than with standard algorithms. Also, findings of Newman research have consistently pointed to the inappropriateness of many "remedial" mathematics programs in schools in which there has been an over-emphasis on the revision of standard algorithms, with hardly any attention being given to difficulties associated with Comprehension and Transformation.

In many Newman studies the proportion of errors first occurring at the "Transformation" stage has been great (in some studies, more than 50% of breakdowns first occurred at that stage).

Newman Research and Tests in Different Languages

Clarkson (1983) and Newman (1977a) reported Newman data for which 12 per cent of the initial errors were in the Reading category. With all other Newman error analysis studies, however, not more than 2% of initial errors have been in the Reading category (Clements & Ellerton, 1992). In case it is concluded that reading has less influence on students' responses to mathematics word problems than might be expected, it should be pointed out that by "Reading" Newman merely meant "Decoding," that is to say the ability to sound the words or symbols correctly. The second Newman Category, Comprehension, has proved to be of considerable importance in Newman research, and it is interesting that most language experts would regard comprehension as being an integral component of the act of reading.

A number of recent Newman error analysis studies carried out in schools in Southeast Asian nations have suggested that a high proportion of initial errors occurred at the Comprehension stage or at the Transformation stage. Thus, for example, Marinas and Clements (1990), working with Year 7 students in Malaysia, found that 71% of errors made on typical Year 7 mathematics questions were in the Comprehension or Transformation categories. Singhatat (1991), working with Thai students, found that 68% of errors were in either of these two categories. Neither Marinas and Clements nor Singhatat found that students in their samples made any Reading (i.e. Decoding) errors.

However, as far as we know there have been no Newman error studies carried out in which students with different language backgrounds took the "same" test in their own first language. In one sense this would impossible to achieve because different languages have different syntactic and semantic structures. However, it would be of interest to try to create two tests covering the same mathematics content but in different languages, with every attempt being made to preserve the semantic structures of the questions—that is to say, to match corresponding questions so far as semantic structure was concerned.

This paper describes an attempt to construct two forms of a mathematics test: one written in *Bahasa Melayu* and the other in the English language. Once the two forms of the test were constructed, Newman error patterns generated by Year 7 students in Malaysia and Australia responding to the form which represented their first language (*Bahasa Melayu* in Malaysia, English in Australia), were analysed and compared.

Comparing Error Patterns of Malaysian and Australian Year 7 Students on Comparable Forms of the Same Test

Methodology

It was decided to investigate the extent to which Newman error patterns for Australian children were similar to those of children in Malaysia, of the same age and grade level. A sample of Australian Year 7 students took an English-language form of a test comprising 24 pencil-and-paper questions, and a sample of Malaysian Year 7 students took the *Bahasa Melayu* form of the same test.

The instrument(s). In order to carry out this investigation it was necessary to establish two comparable tests, one in *Bahasa Melayu* (the language of instruction in Malaysia) and one in the English language (the language of instruction in Australia). After discussion with linguists and with experienced mathematics educators and teachers in Malaysia and Australian, a 24-question test was developed containing only items which were regarded as representing essential knowledge and skills for Year 7 students in both nations.

The test was first written in English. It was then translated into *Bahasa Melayu* by Malaysian mathematics educators. Back translation methods (Brislin, 1970) were used to check that each question in a pair of corresponding questions was as much like its pair as possible—with particular attention being paid to content, vocabulary, and semantic structure (Riley, Greeno, & Heller, 1983). Every teacher of the pupils involved in the study agreed that the questions were well worded, and covered material which they had taught to their students.

The questions were carefully chosen to include basic skills questions, standard word problems, and unfamiliar word problems for which an appropriate method was not obvious. The three types of questions are illustrated by Questions 2, 13, and 17, which were all concerned with subtraction:

Question 13:

(English): Find the value of 940 – 586.

(Bahasa Melayu): Selesaikan 940 – 586.

Question 17:

(English): There are 950 students in a school, but one day only 587 of them were present. How many students were absent that day?

(Bahasa Melayu): Terdapat 950 orang pelajar di sebuah sekolah, tetapi hanya 587 orang pelajar hadir pada satu hari tertentu. Berapa ramaikah pelajar yang tidak hadir padar hari itu?

Question 2:

(English): There are 20 boys and girls altogether in a room. If there are 4 more girls than boys, how many boys are there in the room?

(Bahasa Melayu): 20 orang budak lelaki dan perempuan berada di dalam sebuah bilik. Jika terdapat 4 orang budak perempuan lebih daripada budak lelaki, berapa orangkah budak lelaki di dalam bilik itu?

Some of the questions tested understanding of measurement concepts (time, area, volume, money, mass, distance/speed relationships), some were concerned with spatial ideas, and many involved important aspects of number (the four operations on whole numbers, ratio, fractions, percentages).

The sample. The location of samples was determined by the availability of trained interviewers. In Malaysia, 145 Year 7 students in four Year 7 classes (in three different schools) participated in the study; in Australia, 61 Year 7 students in two Year 7 classes (in different schools) were involved. One of the Malaysian classes was a top-stream group in a school located in a middle-class suburb in Kuala Lumpur; another of the classes was a non-streamed group in a highly regarded school in Penang; the other two classes were from a school located in a rural part of Malaysia. Both of the Australian classes were in schools in Perth: one was a top-stream group in a school in a middle-class suburb (referred to as m/c), the other was a heterogeneous group in a school in a working-class suburb (referred to as w/c)

The interviewers. All interviewers were senior and experienced educators who had been trained by the authors to carry out Newman interviews. The interviews were conducted, using the normal Newman interview method, in the students' language of instruction (which, in all cases, was the first language of the interviewers)—*Bahasa Melayu* in Malaysia, and English in Australia. All Newman error classifications made by the interviewers during the interviews were subsequently checked by the authors.

Results

Table 1 summarises the results, and it can be seen that the overall error profiles for the Malaysian and Australian samples were fairly similar. However the following within- and between-sample differences should be noted:

1. In each of the Malaysian subsamples, the sum of the percentages for Reading, Comprehension, and Transformation was at least 70%, but this was not the case for the two Australian groups.
2. Higher proportions of the errors made by Australian students than of errors made by the Malaysian students were classified as being of the "Process Skills" variety (14% for Australian children, but only 3% for Malaysian children).
3. In all 6 schools, only a small proportion of errors were classified as Reading errors.
4. A relatively high proportion of errors made by high achieving students (especially in the Perth m/c school and in the Kuala Lumpur group) were "Careless." This is in line with results obtained in an early study carried out in Papua New Guinea by Clements (1982).

Table 1

Results of Newman Interviews of Year 7 Students in Malaysia and Australia

Nation	*Location*	*N*	*No. of Errors Analysed*	*% of Total Errors in Category*					
				R	*C*	*T*	*PS*	*E*	*X*
Malaysia	Kuala Lumpur	31	210	0%	12%	62%	2%	3%	21%
	Penang	44	473	2%	26%	42%	5%	5%	19%
	Rural	70	780	4%	28%	42%	3%	5%	18%
TOTAL	(Malaysia)	145	1463	3%	25%	46%	3%	5%	18%
Australia	Perth (m/c)	28	143	1%	13%	28%	15%	10%	33%
	Perth (w/c)	33	350	1%	12%	51%	14%	1%	20%
TOTAL	(Australia)	61	493	1%	12%	44%	14%	4%	24%

Code: R = Reading; C = Comprehension; T = Transformation; PS = Process Skills;
E = Encoding; X = Careless

Different Error Profiles for Different Students

Previous Newman error analysis researchers have often noted that different students can generate quite different error profiles, even though they respond to the same set of questions. This phenomenon was again evident in the study being reported in this paper. Thus, for example, one student in the Malaysian rural subsample made 9 Careless errors out of 17 errors, whereas three other students in the same subsample did not make a Careless error, even though they made 23, 24, and 24 errors. It would seem to be the case that, over the past two decades, Newman error analysis research has established that differences in error profiles are to be expected among students in all classrooms. No further consideration of the phenomenon will be given in this paper.

Different Error Patterns for Different Questions

Curiously, given the recognition that different students are likely to generate different error profiles, there has been little recognition of the possibility that different questions might generate different error patterns. Consider, for example, the different error patterns associated with Questions 13, 17, and 2 (see Table 2).

From Table 2, it can be concluded that the Australian students did not subtract as well as the Malaysian students. A similar result was found on all questions requiring straightforward applications of numerical skills. Yet the Australian Year 7 students had a higher mean score on the 24-question test than the Malaysian students. (The mean number of questions correct (out of a possible 24) for the 61 Australian students was 15.9, and the mean for the 145 Malaysian students was 13.8.)

Data for Question 2 (in Table 2) indicate that Australian students did better than their Malaysian counterparts on word problems which demanded careful analysis of the semantic structure. This pattern was generally evident in the overall analysis. These results are similar to those reported by Kaushil et al. (1985), who found that students in Delhi (India) did much better than Australian students of the same age on questions requiring straightforward applications of numerical skills, but much worse on word problems with non-trivial semantic structures.

Table 2 indicates that although two questions on the same test can generate similar error patterns, they can also generate quite different error patterns. For example, both Questions 13 and 17 generated a large proportion of Careless errors, in both their Bahasa Melayu and English forms. In Australia, but not so much in Malaysia, there were also many Process Skills errors.

By contrast, for Question 2, most of the errors made by both Malaysian and Australian students were in the Transformation category. Neither Australian nor Malaysian Year 7 students made Process Skills errors on Question 2.

Table 2
Errors on Three Questions by Year 7 Students in Malaysia and Australia

*Question**	*Nation*	*% Responses Which Were Errors*	*% of Total Errors in Category*					
			R	*C*	*T*	*PS*	*E*	*X*
13. Find the value of 940 – 586.	Malaysia (145 students)	13%	11%	16%	5%	16%	0%	53%
	Australia (61 students)	33%	15%	10%	0%	35%	0%	40%
17. 950 students, 587 present. How many absent?	Malaysia	12%	6%	6%	6%	6%	6%	71%
	Australia	33%	0%	0%	0%	45%	0%	55%
2. 20 boys & girls. Four more girls. How many boys?	Malaysia	90%	0%	21%	78%	0%	0%	1%
	Australia	59%	0%	6%	86%	0%	0%	9%

* Note: The wording given under "Question" in Table 2 provides a summary only of each question.

If one analyses the wording and structure of Questions 13, 17, and 2, the above results are hardly surprising. Most experienced teachers of mathematics would predict that with Year 7 students, Questions 13 and 17 would generate mainly Process Skills and/or Careless errors, and Question 2, mainly Transformation and/or Comprehension errors.

One might agree to label Questions 13 and 17 as "Careless" questions in the Malaysian education context, and "Careless & Process Skills" questions in the Australian education context. From a similar perspective, Question 2 might be labelled a "Transformation" question in both the Malaysian and Australian contexts. A convention might be developed that if more than 25% (say) of errors for a question are in the same category, then the name of that category would be included in the label for the question. If two (or even three) categories are such that each of the categories contained more than 25% of the errors for a question, then the category associated with the highest proportion of errors will be named first in the label for a question.

However, it is not always easy to predict which categories of errors will be associated with which questions. The reader is invited to consider the following two questions (which were Questions 20 and 23 among the 24 questions used in the study reported in this paper), and to predict which type(s) of errors would be most common if the questions were answered by (a) the Australian, and (b) the Malaysian Year 7 students in the study. Predictions can be checked against pertinent data from the Malaysian and Australian samples which are given in an Appendix to this paper (immediately before the Reference list). The English versions only of the questions are given below:

> *Question 20:* There are 12 apples on a table. If you picked up one-third of the apples, how many apples would remain on the table?
>
> *Question 23:* Bella takes exactly 3 hours to walk 15 km. How long would it take her (if she walks at the same speed) to walk 20 km?

Some Spearman Rank-order Correlations

In order to check the extent of agreement between relative difficulties of corresponding questions on the two forms of the test, the 24 questions were ranked in difficulty—one rank being based on the Malaysian sample, and the other on the Australian sample. A between-country Spearman rank-order correlation co-efficient was then calculated (Lumsden, 1969). The correlation obtained was a moderate 0.30.

However, when in-country rank-order correlations between the ranks for Kuala Lumpur-Penang, Kuala Lumpur-rural (Malaysia), Penang-rural (Malaysia), and Perth (m/c)-Perth (w/c) were calculated, these all ranged between 0.70 and 0.80. It would appear to be the case that there are strong in-country influences on the difficulty of mathematics questions. The extent to which these arise from linguistic differences or from approaches to teaching and learning mathematics needs to be further investigated.

Concluding Comments

Although Newman research has invariably appealed to teachers whose students have been involved in Newman studies because it generates important, authentic, and interesting data, the implications of this research for teachers and policy makers have not been clear. Can results of Newman studies based on data generated from interactions between English-speaking students and tests written in the English, be legitimately generalised to predict likely error patterns when non-English-speaking students take the "same" tests—with the tests *not* in English but in the students' own first languages?

From reports of previous Newman studies, and from the data presented in this present paper, it would appear to be the case that the answer to the question in the last paragraph is a qualified "Yes." Newman research data from a range of Asia-Pacific nations suggest that the most basic components of a mathematics education should be "learning to read, write, comprehend, and mathematise." In fact, the four operations on natural numbers seemed to have been well handled by most pupils in upper primary and secondary classes in all nations in which Newman research has been conducted thus far.

Findings from Newman research indicate that many lower secondary students in many Asia-Pacific nations have little understanding of, or ability to apply, percentages and decimal and vulgar fraction concepts. It also seems to be the case that many Australian students do not have their numerical skills as well developed as their age- and/or grade-counterparts in other Asia-Pacific nations. However, as data in this paper reveals, that does not imply that the Australian students are better (or worse) off so far as school mathematics is concerned. Many Australian students do better than corresponding students in other Asia-Pacific nations at comprehending and "mathematising."

Analyses presented in this paper indicate that not only are the interactions of particular questions with particular groups of students likely to generate predictable Newman error patterns, but also the patterns for one question are likely to be quite different from those for other questions. This would appear to be relevant to those involved in test development, or who are concerned with best-test design. In particular, it is likely that psychometric considerations (such as reliability, validity, and item discrimination) need to be complemented by findings of Newman research.

Findings from Newman research raise the difficult issue of what educators can do to improve learners' comprehension of mathematical text and their ability to transform (that is to say, to identify an appropriate sequence of operations which will solve a given word problem). At present, little progress has been made on this issue, and it should be an important focus of mathematics education researchers in Asia-Pacific nations over the next decade. At issue is the extent to which this is related to linguistic features, as opposed to teaching and learning patterns found in mathematics classrooms.

Appendix

Table 3 shows data for Questions 20 and 23. Entries in Table 3 suggest that Question 20 is a "Comprehension & Transformation" question for both Malaysian and Australian Year 7 students. Question 23 is a "Transformation" question for Australian Year 7 students (and a "Transformation & Comprehension" question for Malaysian Year 7 students).

Table 3

Errors on Two Questions by Year 7 Students in Malaysia and Australia

Question	*Nation*	*% of Responses Which Were Errors*	*% of Total Errors in Category*					
			R	*C*	*T*	*PS*	*E*	*X*
20. 12 apples on a table; 1/3 are removed. How many remain?.	Malaysia (145 students)	56%	1%	40%	44%	0%	0%	15%
	Australia (61 students)	26%	0%	56%	25%	0%	0%	19%
23. 3 hours to walk 15 km. How many hours for 20 km?	Malaysia	46%	1%	37%	46%	5%	0%	10%
	Australia	44%	0%	4%	89%	0%	0%	7%

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Gender Issues In Tertiary Mathematics Education

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Fewer females than males opt to study mathematics at the tertiary level, particularly at postgraduate levels. Factors influencing students' decisions to pursue tertiary studies have been identified. Less is known about the factors associated with studying mathematics in particular. Reported here are results from the first phase of a three year study aiming to uncover relevant contributing factors. Enrolment data for 1995 were gathered from La Trobe University (Bundoora campus) and several students taking undergraduate mathematics courses that year were interviewed. While the results are not generalisable, interesting gender-related trends were apparent. The implications of the findings are discussed.

Introduction

More Australian females than males now complete secondary school (Ministerial Council on Education, Employment, Training and Youth Affairs, and Curriculum Corporation, 1994) and women comprise at least 50% of the tertiary intake (Ministerial Advisory Committee on Women and Girls [MACWAG], 1990; Powles, 1986). Yet 'traditional' gender divisions in tertiary course and subject enrolments and in postgraduate studies are still evident. Indeed, Graetz (1991) suggested that gender divisions in higher education were more marked than at school levels. Overall, proportionally more males than females are enrolled in engineering courses and in the physical sciences (Lumley, 1992; MACWAG, 1990). Moses (1990) identified the 'maleness' of disciplines and the lack of female role models as gender-specific barriers to postgraduate participation.

Social, cultural and affective factors have been implicated in students' beliefs about tertiary studies (including mathematics), their decisions to enrol in tertiary courses, and their achievements. Carpenter and Fleishman (1987) found that attitudes towards higher education in Australia were related to academic achievement, school type attended and gender. Encouragement by parents, peers and teachers, was important for attitude formation. Social background factors, followed by ability, were the most important determinants of tertiary attainment reported by Graetz (1991). The academic achievements of females were "much more constrained by social origins" (p.7) than males'. Wilson and Boldizar (1990) found that gender differences in high school aspirations, more than mathematics achievement and the income potential of courses, accounted for gender-segregation in US college enrolments. School attended, but not school type, Hunter (1987) suggested, was related to the gender divisions in New Zealand tertiary course enrolments. At Monash University in 1992, Lumley (1992) found attendance at single-sex schools was not related to the patterns of mathematics and science-based subject enrolments of first year females from private school backgrounds.

The literature suggests that tertiary students' mathematics course decisions can be influenced by a range of background and affective factors. The pedagogical approach adopted by tertiary teachers has also been implicated (eg. Rogers, 1990). At the school level, classroom learning environment factors can affect students' attitudes and beliefs about mathematics and about themselves as learners of the subject (Forgasz, 1995). The research design of the present study incorporated many of these previously identified variables. The study set out to explore the extent to which these and other factors might influence Australian students' decisions to pursue tertiary-level mathematics courses. Reported here are the findings from the first phase of this three year project.

The Study

Identifying variables that might be related to students' decision-making about pursuing tertiary mathematics courses was the main aim of this first phase of the three year study. The findings would serve to inform the design of a large scale survey instrument to be administered in subsequent phases of the project. Variables for which gender differences were found in the pilot study are discussed in this paper.

Methods

Semi-structured interviews with a representative sample of students enrolled in mathematics courses at La Trobe University in 1995 were the main data source. Questions asked covered a range of factors derived from the literature:

* *Affective factors* eg., Did you enjoy mathematics at school? How good are you at mathematics?

* *Socio-cultural factors*. eg., Why are you studying mathematics? Did you receive encouragement and/or advice to do so?, and

* *Learning environment factors* eg., How would you compare the teaching of mathematics at school and at university? Do you feel comfortable in your university mathematics 'classroom'?

Demographic and other background information about the interviewees (e.g., age, school attended at grade 12, language background, parents' educational levels) were gathered prior to the commencement of the interview.

In order to gauge the representativeness of the sample interviewed, enrolment data were gathered from the university as a whole and from the School of Mathematics in particular.

The sample

Seventy students were randomly selected from the class lists of all undergraduate mathematics subjects offered by the School of Mathematics in 1995 and sent invitations to participate in the interviews. Male and female students were invited in proportion to their numbers in the various courses. A response rate of 30%-40% was anticipated. However, only seven students agreed to be interviewed. A successful personal approach for volunteers at tutorial and lecture sessions followed. In all, twenty three students (16M, 7F) were interviewed. Of these, 12 (9M, 3F) were mature-aged (at least 21 years of age at the commencement of their current undergraduate degrees). In Table 1 is shown the break-up, by year of study and gender, of the 23 interviewees.

Table 1: Interviewees by year of study and gender

	1st year	2nd year	3rd year	4th year (Hons)
Males	4	6	4	2
Females	3	3	1	-

Results and Discussion

Representativeness of the sample

In 1995 15,886 students were enrolled at La Trobe University (Bundoora campus), 64% were female. However, there were more males than females enrolled in mathematics courses. For example, there were 135 (62%) males and 82 (38%) females enrolled in the core first year subject, Maths 1A (NB. this subject does not involve the majority of students studying engineering). This ratio of males to females (about 2:1) was approximately the same for all core second and third year mathematics subjects with the exception of 3rd year Pure Mathematics in which the ratio was reversed: 20 females (62%) and 12 males (38%). At 3rd year, the cohort was less than 50% of the first year enrolment numbers. At the Honours level (4th year) there were 10 students; 7 were male.

The ratio of males to females among the interviewees was also approximately 2:1. That is, the gender composition of the sample was fairly representative of the entire cohort enrolled in mathematics. These enrolment figures also showed that as students progressed through the year levels, the drop-out rate for males and females was similar. The major contributor to the gender difference in mathematics course enrolments at La Trobe University would appear to result from the subject choices made by first year students.

Gender differences

The small sample size precludes any generalisations being drawn from the findings. However, the gender differences apparent in the trends that were found are reported below. They invite further investigation.

The factors that seemed to have the most noticeable differences by gender included: school attended, grade 12 mathematics background, and parents' educational levels. Students' perceptions of aspects of the tertiary learning environment showed that gender-stereotyped attitudes are still evident.

School background: The schools attended for grade 12 studies are shown separately for males on Figure 1 and for females on Figure 2.

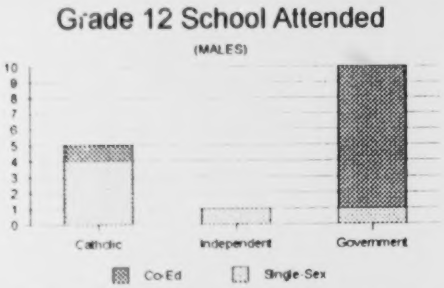

Figure 1 School type attended by males when in grade 12

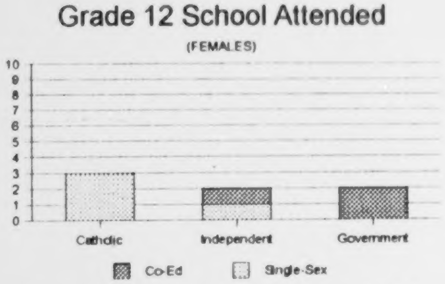

Figure 2 School type attended by females when in grade 12

Figures 1 and 2 indicate that a larger proportion of males (10/16 = 63%) than females (2/10 = 20%) had attended government schools. With one exception, the government schools attended were co-educational. With two exceptions the non-government schools were single-sex. It should be noted that about 62% of all students in Australia in 1993 were enrolled in the government sector (Ministerial Council on Education, Employment, Training and Youth Affairs, 1994). The trends revealed by Figures 1 and 2 suggest that:

- females attending government co-educational secondary schools seem less likely than their male counterparts to study mathematics at the tertiary level
- females at single-sex non-government schools may be more likely than their female counterparts in the government co-education sector to pursue tertiary mathematics courses

Grade 12 mathematics background: The Victorian Certificate of Education [VCE] (or equivalent) mathematical backgrounds of the interviewees by gender are shown in Figure 3.

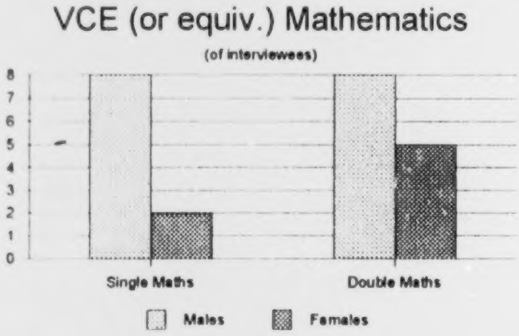

Figure 3 VCE (or equivalent) mathematics background of interviewees

It can be seen from Figure 3 that a larger proportion of the females (5/7 = 71%) than of the males (8/16 = 50%) had taken double mathematics in grade 12. The data suggest that females taking tertiary mathematics have broader and stronger mathematical backgrounds than their male counterparts. In part, this may support frequently reported findings that females are less confident about their mathematical abilities than are males (see Leder, 1992). It may be that some females who are capable of studying tertiary mathematics are not choosing to do so. On the other hand, some males who are enrolled in these courses may well have been better advised to consider alternatives.

Parents' educational levels: The educational levels of the interviewees' parents are shown in Table 2.

Table 2: Parents' educational levels (percentages shown in brackets)

		Tertiary	Gr 12	Secondary	Primary	Unknown
Males (16+1*)	Mother	6 * (35)	-	8 (47)	1 (6)	2* (12)
	Father	8 (50)	1(6)	4(2 5)	2 (13)	1 (6)
Females (7)	Mother	1 (14)	3 (43)	3 (43)	-	-
	Father	5 (71)	1 (14)	1 (14)	-	-

* One male's natural mother was tertiary-educated. The educational level of his step-mother (with whom he'd lived since a toddler) was unknown

The data in Table 2 indicate that the educational backgrounds of the males' parents were more disparate than the females' parents, particularly at the lower end. While all of the females' parents had completed primary school, several of the males' parents had not. For both the males and the females, a higher proportion of fathers than mothers had completed tertiary education. While a higher proportion of the females' than of the males' fathers were tertiary qualified, the pattern was reversed for the mothers. From the data it might be inferred that:

- parents hold higher expectations for their sons than for their daughters with respect to mathematics and other science-related careers.
- compared with males, females gain less support from less educated fathers to pursue studies in mathematics-related fields.

These inferences would not be inconsistent with some of the findings on parents' aspirations for their children reported by Kelly (1986). For example, Kelly found that parents rated "the jobs of engineer, electrician and draughtsman... much more suitable for boys than for girls" (p.98). Although fewer fathers than mothers completed the questionnaires, and while those who did were more likely to be working class and to have daughters, they "expressed much more sexist ideas than the mothers who replied" (p.102).

The tertiary mathematics learning environment - lecturers and fellow students:

The students were asked whether they felt that any groups of students were treated differently in the School of Mathematics, and whether they had personally experienced or observed any discrimination. Many students believed that all students were treated fairly. A greater proportion of female than male students commented that female students might experience some discrimination. Karen, for example, believed that some lecturers held prejudiced views.

> ... some tutors and lecturers will pay more attention to the women and some don't think women can do maths so then they actually pay, they are

> more reluctant to help you and there is also, and some of them are racist, arrogant. They don't give much time to some students. But I think, yes, the maths department here they seem to have a fairly good balance of people whereby if you strike one who is a bit funny towards you, you can find one who is the opposite way.

She also commented on the effects of a female lecturer on male students with sexist attitudes:

> ... apart from some female tutors, this semester, second semester of second year, is the first time we have had a female lecturer. And yes I think that affects some of the guys' attitudes because they can see that she does a good job and perhaps even better than some of the other male lecturers... I think there are a lot of entrenched beliefs that women cannot do maths as well as men... I mean I enjoy seeing a woman teacher at university and having women teachers because, I guess there is a bit of a difference.

Jan had strong feelings about her personal experiences of the overt sexism of one of her lecturers:

> Like first semester I had a tutor who was a male chauvinist pig, he was rude but I could not handle him. If you asked him a question he would respond to the male partner who you were working with. ... [T]hat made learning really difficult... his belief is that women should not be in maths at all.... [that] women are not capable of maths and don't have the learning capability to keep up with the work later on.

While not having experienced discrimination personally, Jack was prepared to concede that females (and overseas students) might be disadvantaged:

> Well I can't really [perceive disadvantage] from my perspective, you know what I mean. If I was female or from another country it would be much more easy for me to be aware of such things.

Phil said that he had heard that females had experienced discrimination:

> I have heard that some of the male teachers are a little uncomfortable around some of the female students but again I have not seen that and the women that are studying it do quite enjoy it so, no I don't know...Um one of the comments was that women should not be studying mathematics but if they are they will get the help. But they [the females] got the impression that the teachers felt that they should not be studying.

Some male students exhibited stereotyped views of females' mathematical abilities. When asked to explain why fewer females than males studied the most demanding mathematics courses at grade 12, three males believed that genetic factors were involved. Colin, for example, said:

> Just genetics. Males and females are different. I am sure there are subjects that more females do. I am not talking about sewing and cooking, I am not being so blind as that but there are just subjects that appeal to males more than they appeal to females and visa versa, there are subjects that appeal to females more than males.

It should be noted that the students generally spoke very highly of the teaching program in the School of Mathematics and praised the support and assistance provided by the majority of lecturers and tutors. The examples discussed above, however, suggest that some of the females interviewed may have experienced gender-based discrimination. The females' perceptions of the sexist attitudes of some male lecturers were supported by a few male students. It would appear that it is not only some lecturers, but also some male students, who hold ill-informed, gender-stereotyped views that the genetic make-up of females 'naturally' inclines them away from mathematics and towards the humanities. At present, it seems that females who persist with mathematics at the tertiary level have to be highly motivated and prepared to face and overcome the obstacles and adversity they may face.

Final Words

Several gender issues pertinent to tertiary mathematics education emerged from the analysis of the data gathered from undergraduate students enrolled in mathematics courses at La Trobe University in 1995. The factors identified were: school attended, grade 12 mathematics backgrounds, parental educational levels, and the beliefs and attitudes of academic staff and students. Further investigation is needed to determine whether the trends discussed here extend to larger cohorts of students whose personal and educational background profiles may differ and to other tertiary institutions. The influence of female role models within an apparently 'masculine culture' is also worth examining in more depth.

Much more needs to be known about the factors implicated in students' decisions to pursue tertiary mathematics. There are longer-term benefits to the nation to be gained from identifying and addressing gender inequities in mathematics and related fields. As well as the variables identified in the relevant literature, the factors discussed in this paper have been incorporated into a survey instrument to be administered to students at three Victorian universities during 1996 and 1997.

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Using the Internet in mathematics education research

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The Internet is a new and exciting technology. It offers enticing and seductive possibilities as a research tool. Simultaneously, it presents a number of challenges to current accepted research practices. Access to its capabilities and the amount of information available have expanded rapidly. In this paper we discuss several issues in the use of the Internet to conduct research. We also outline some of the difficulties we encountered in our initial explorations of the Internet's potential as a source of gathering survey data.

The Internet: Background

The Internet refers to a collection of thousands of interconnected computer networks that function as a single, large virtual network. Data can be stored in a variety of forms, eg., text, graphics, audio, video/movie, and computer language (software) and are found on different types of sites eg., Gopher servers, World Wide Web (WWW) sites, FTP (File Transfer Protocol) sites, and Telnet sites. Information and other data can be transferred, downloaded (saved), viewed and printed via Web browsers (eg., Netscape) and associated software. Usernet news groups, listservers (electronic mailing lists) and e-mail are communication options available via the Internet. IRCs (Instant Relay Chats) allow real-time 'conversations' and interactive games can be played through MUDs (Multi User Dimensions) and MOOs (Object Oriented MUDs).

Access to the Internet has expanded rapidly in the last decade. De A'Morelli (1995/6) notes that the number of computers connected to the Internet has grown from 5,816 in 1985 to 1.2 million in 1995. The number of people sending and receiving e-mail is purported to have increased from 27.5 million in October 1994 to 35 million in July 1995 (The Internet group, 1995). Most Australian academics now communicate via e-mail. It may prove frustrating at times (eg., decoding attachments), yet the ability to send and receive messages quickly and to transfer large amounts of data over vast distances spur us on. The Internet is also beginning to be exploited as a research tool. This presents several challenges and invites cautions.

In this paper we explore the Internet's potential as a research tool and discuss some of the methodological and ethical issues raised. We also describe a study in which the Internet was used as a data gathering source. Some of the difficulties encountered are outlined.

The Research Potential of the Internet

The Internet offers numerous research and publication possibilities which we discuss briefly.

Information sources

Gathering background information for a research project is fairly simple once Internet search techniques have been mastered. Access is available to library catalogues around the world, to a range of databases (eg., ERIC and CARL), to government statistics, and to a plethora of other material stored on the vast number of national and international Gopher and WWW sites. For mathematics education, the number of WWW sites continues to grow. The accessible material shows variety both in content and in quality. Research articles, teaching resources and relevant downloadable software can be found. Electronic mailing lists provide avenues for discussions and information sharing.

For research purposes, the authenticity and reliability of much of the material found on the Internet are dubious and subject to individual judgment. As yet, no hierarchy of electronic journals and WWW sites or guidelines to direct the researcher have been established. MacLennan (cited in Okerson, 1995) advocated the establishment of standards for electronic serials collections and improved search mechanisms to exploit their value.

Citing Internet sources is also problematic. The most recent edition of the American Psychological Association [APA] style manual (APA, 1994) contains a section on the citation of on-line information: "as with any published reference, the goals of an electronic reference are to credit the author and to enable the reader to find the material" (p.28). At this stage, the guidelines provided are indicative, rather than prescriptive. Li and Crane (1995) suggest a number of bibliographic formats in line with APA guidelines, but they do not suggest how to cite within the text. In this paper, we have followed their recommendations and were guided by the APA manual for in-text references.

As noted by Minchew (1996), there are other difficulties associated with electronic source citations: "there is an increasing need for a standard way of citing on-line research. On-line sites may change addresses, move to new locations, be revised, and/or disappear". Thus, article references to Internet addresses may be obsolete before the toner has adhered to the paper. We have followed Li and Crane's (1995) suggestion that access dates be shown in square brackets at the end of the reference. At the time the WWW pages were accessed, hard copies were printed and filed.

Publishing on the Internet

Loading and 'publishing' material on a WWW site is a straightforward process. While some electronic journals claim that articles are subjected to the usual refereeing procedures associated with scholarly publications, many papers located on the Internet do not seem to have passed through any peer review process. McEldowney (1995) claims that "definitions of 'electronic scholarly journals' are still in transition" and that no agreement on 'scholarly' exists.

Copyright and other legal implications associated with loading, reproducing and downloading material to and from Internet sites have been discussed on the Internet (eg., Bailey, 1996; Godwin, 1993; Templeton, ND) and should keep lawyers busy for some time. According to Alexander (1995)

> ...people with access to these networks have a novel inducement to act contrary to the property laws - especially copyright laws - which allocate rights over intellectual goods, and it seems likely that in fact there is widespread flouting of such laws. (p.1)

In an US Office of Technology Assessment study, *Intellectual property rights in an age of electronics and information*, Garcia (1990) noted that:

> Just as the technologies of printing gave rise to the need for a copyright system in the first place, so too the new technologies were creating major problems for the system - problems that might be solved only by significant changes in the intellectual property system itself. (Garcia, 1990, p.1)

Professional safeguards form another related area of concern. Is the same protection enjoyed by academics with respect to traditional publications guaranteed for Internet publications that might subsequently be subject to litigation? It may take test cases to come before the courts for this issue to be resolved. Meanwhile, the Australian Vice-Chancellors' Committee (1995) has noted that electronic data bases are among the categories for which "confusion may result" (p.29) with respect to ownership and intellectual property issues. The ease of publishing on the Internet poses an additional dilemma pertinent to tertiary funding and as an indicator of worthwhile academic pursuits. To what extent will (or should) Internet publications be counted in the research quantum? Recent information on the 1996 research quantum (Smith, 1996) makes no mention of electronic publications.

Research involving the Internet

A distinction needs to be drawn between the types of research involving the Internet. Some work has been reported on the effects of the Internet on student learning. For example, McGill and Jessup (1995) explored the impact of the part delivery of a university education course via e-mail. Traditional survey and interview techniques were used. In addition, the flow of e-mail exchanges between students and their tutors were archived. Despite the students' initial unfamiliarity with the technology, it was concluded that e-mail could be used successfully for course delivery.

Arguably, the research methods used by McGill and Jessup (1995) did not challenge conventional research practice. The participants were a closed and known group. Established ethical considerations with respect to confidentiality of data would have been under the researchers' control. However, when the Internet is used as a substitute for established data gathering methods, sampling concerns and ethical issues arise that challenge accepted research practice.

The Internet as source for data gathering

The Internet appeals as a data gathering tool. E-mail, IRCs and electronic mailing lists are potentially promising sources of participants for research projects and offer the means to gather survey responses and conduct 'pseudo'-interviews. These Internet facilities appear to offer distinct advantages over traditional research methods eg., the costs and time factors associated with producing, distributing, administering and returning surveys. Since e-mail messages are accompanied by the sender's 'address', incomplete surveys or unclear item responses can be readily followed up, thus reducing missing and unusable data.

On the other hand, the operational aspects of the technology raise interesting challenges about conventional sampling techniques and question the practicality of some of the concerns of Human Ethics Committees.

Sampling issues. For results to be generalisable, samples need to be representative of the population and "this generally requires some aspect of random selection" (Wiersma, 1986, p.263). Can a random sample be obtained on the Internet? Consider mounting a survey via e-mail. Sampling is immediately limited to individuals who access the technology. Questionnaires, distributed through closed distribution lists or mounted on one or more listservers, can easily be forwarded beyond the groups initially targeted. In these circumstances, the researcher loses control over the sampling process.

A recognised advantage of questionnaires is that they permit anonymity, thus increasing the chances of receiving genuine responses (Henerson, Morris, & Fitz-Gibbon, 1978). Is anonymity possible on the Internet? Since e-mail responses automatically provide the sender's 'address', respondents are clearly identifiable and true anonymity cannot be assured. Even with a specifically targeted population, the researcher cannot be confident of the authenticity of respondents. McEldowney (1995) discusses the possible use of "password protections to verify and maintain identity of respondents".

Ethical considerations. Those familiar with the procedures of Human Ethics Committees will be aware of issues of confidentiality and informed consent. While Ethics applications are often tedious to complete, the principles underpinning current recommended procedures aim to protect the rights of the participant. A research project involving school children, for example, requires the permission of the relevant school authorities and school principals, and informed consent is required from all participants including the parents of students. If school children were the target population of a survey or 'pseudo'-interviews conducted via the Internet, could the usual means to obtain the required permissions and informed consent be assured? Have Human Ethics Committees responded to the challenges posed by research designs of this kind?

Gathering data via the Internet does not seem to conform fully to traditional research methods. The Internet cannot and will not be ignored, however. With time, the challenges posed will have to be resolved. At this stage, the prudent researcher would wisely be cautious in the gathering of data from the Internet and in their interpretation.

In the next section a case study in mathematics education is described in which data were gathered from the Internet.

Using the Internet as a research tool: A case study

In a recent study, the facilities of the Internet were used to gather data. Sampling, ethical and copyright concerns arose, and difficulties with the technology itself were experienced. A brief overview of the aims of the study and of the methods is presented first.

Aims

The study had two inter-related aims:

To determine: (1) how mathematics and related areas are portrayed in the print media, and
(2) whether individual sex-role orientations appear to affect the 'reading' and interpretation of media reports

Method

It is inappropriate to describe the study's methodology, instruments, or results in detail here. Instead, the information provided focuses on the aspects most relevant to the e-mail component of the study and the special issues that arose from using this method of data gathering.

Instruments

(i) Questionnaire. A questionnaire to determine gender role orientations was devised. After extensive trialing of new items and those modified from other relevant surveys and materials, 35 items were retained. In addition to the five-point ratings (from Strongly Agree to Strongly Disagree) elicited from respondents, participants were asked to give reasons for their answers. A sample item is shown below:

1. Some jobs are unsuitable for women. SA A U D SD

(ii) Print media extracts. Three newspaper articles dealing with gender-related educational and career issues were identified. The first was concerned with increasing enrolments for women at Australian universities, the second with negative social outcomes of feminism, and the third with strategies to attract women to engineering. Respondents were asked to read the articles and answer a series of questions about their content. Thus a link could be made between individuals' questionnaire responses and their reactions to the media reports.

(iii) Biographical information. Respondents were also asked to complete a small number of biographical items (e.g., age, sex, nationality, educational and occupational levels attained) and to indicate where they found the survey.

Procedure

The biographical items, questionnaire and print media materials were e-mailed to five electronic mailing lists: AERA-G, a list for the discussion of the social context of education, GENDER-SET, a list for the discussion of gender, science and technology, XTAR, a teacher research list, EDSTYLE, a learning styles theory and research list, and GRANITE, a gender and information technology list. The following initial information and instructions were given:

> Hello! This questionnaire has been posted to mailing-lists concerned with women, gender, feminism, education or a combination of these topics. Sorry for any multiple posts. If you do not wish to take part, please delete this message. If you know of anyone who would be interested in taking part, feel free to circulate. For those taking part, thank you in advance.
>
> Because of the nature of e-mail, responses are not anonymous. They are, however, confidential. If you feel uneasy about responding via e-mail, feel free to use the conventional post. My postal address is attached.
>
> The questionnaire is designed so that you can use the 'forward' or the 'reply' command of your mailer to allow you to fill in your responses and send them back to me. If you have any trouble with the process or your particular mailer won't allow you to enter text into a forwarded message please contact me. My e-mail address is... Please use 'survey' in the subject line...
>
> Please enter your answers directly after each question.

Internet issues

A number of special issues arose. Not all had been anticipated.

Getting approval

Rigid steps to satisfy copyright requirements were added to conditions normally dictated by the University's Ethics committee. (Perhaps our decision to consult the University's solicitor was too cautious. But given the novelty of using the Internet to gather data we felt this to be appropriate.) We required written permission from the three publications in which the articles we used had appeared, as well as from the writers of the three articles. Considerable follow-up was required before six positive responses were received.

The sample

From the outset we acknowledged that our sample was not random. Rather, we anticipated getting responses from self-selected groups with interests in gender, education, research, science and technology. Our mailing-lists had been chosen accordingly. However, by using e-mail as the medium for dissemination of the questionnaire and associated materials, we expected to reach a geographically much more diverse group than would otherwise have been possible with the limited financial resources available. Given that the first 40 responses we received (within two days of 'posting' the material) came from 10 different countries, we can claim to have been successful on that score. What we had not anticipated was the extent to which users of the mailing-lists we had selected would take up the suggestion to circulate the material. The first 40 responses received had travelled via 12 different mailing-lists. Our plans of carefully targeting certain mailing-lists were thus contaminated.

Beyond the questionnaire responses

The ease and speed of communicating via e-mail led to several advantages not usually afforded by questionnaire-based research. Not only were participants quick to respond, but incomplete responses could be followed-up immediately and, in most cases, participants were happy to provide the missing data.

This efficiency was complemented by the flexibility of the e-mail page when compared to a hard-copy questionnaire. In addition to their multiple-choice responses, participants were encouraged to make comments at any stage throughout the questionnaire. The flexibility of an on-line document facilitated this. There was no restriction on the length of comments for those participants with a lot to say, nor any compulsion for those who might otherwise not have completed the rather long questionnaire. Thus, a rich body of data was collected, with volunteered comments available to confirm, clarify or problematise multiple-choice answers.

A further advantage of this electronic medium was that the data did not require transcription. Completed questionnaires received via e-mail were easily transferred to a local hard-disk using FTP. This saved the time and tedium associated with more traditional questionnaire data, particularly because comments and extended answers were collected.

Final comments

In the first part of the paper we indicated how information technology is challenging the boundaries of traditional research. Conventions about random sampling, anonymity, informed consent, citations, copyright and intellectual property, for example, need to be reconsidered.

In the case study described some potential hazards were anticipated. Others emerged as the study progressed. The scope of the study, and the efficiency of its implementation, were clearly affected by the characteristics of the e-mail technology used. Given the undoubted promise of the Internet as a research tool, academic institutions and their staff should address relevant research, educational, legal, and ethical issues, such as those raised in this paper, as a matter of urgency. This will ensure the maintenance of the high research standards to which we have become accustomed.

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Perceived Competence in the Transition to Secondary School Mathematics

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Transition to secondary school mathematics often poses problems for particular students. This paper reports on the initial stage of a larger longitudinal study that examines students' beliefs about strategies for success in mathematics and whether they are personally capable of succeeding. The aim is to identify students in grade 6 who are at risk of failure in secondary school mathematics. Students will be followed from grade 6 into year 7 to see what effect transition has on their belief systems.

Introduction

Many secondary schools have set up transition networks and specific transition programs for students entering from primary school. The change from primary to secondary systems is frequently seen by teachers, students and parents as problematic, as the primary system is perceived to be close and nurturing while the secondary system requires a great deal more independence of the student and is more often characterised as impersonal and subject oriented. Clarke (1989) argued "if transition creates problems in a student's mathematics education it seems equally true that mathematics contributes significantly to the difficulties of the student in transition" (p. 1).

Students may transfer to secondary school with many others from their own school, in small groups, or alone. Each situation poses a different transition problem. Cockcroft (1982), in a report on mathematics education in Great Britain, suggested that "the greatest problems exist in the transfer to secondary or upper school" (p.125). In the popular press too, the problems of transition have also been highlighted. For example;

> Peter, now 14, enjoyed primary school because he knew the teacher and felt he belonged. At high school, he found the classwork hard and the environment bewildering. The only way to get a teacher's attention, it seemed, was to act up.
>
> The Sunday Age, 5.2.95, p. 4

Some children certainly make the transition to secondary school successfully and happily. Ellerton & Clements (1988) found that many of the students they surveyed coped well with secondary school, some seeing it as an opportunity to "start again".

This research has been prompted by the following concerns: Do transition programs target the problems that can cause students to disengage from the learning process in mathematics? Are there indicators in late primary school that may point to students at risk of failure in the early years of mathematics at secondary school?

This study will focus on the development of motivational behaviours across the transition from primary to secondary school, considering whether aspects such as students' beliefs about their own ability level and perceived competence in mathematics effect the individual's engagement and performance. Data will be collected while students are in grade six and year seven. Entire cohorts have been surveyed at grade six level, and as many students as possible will be followed into year seven and surveyed again.

Description of constructs used in this study

Conceptual Framework

The research questions are to be explored using a model based on that developed by Connell and Wellborn (1991), who presented a global representation of how attitudes have an indirect effect on achievement in the classroom through the construct of engagement. This model suggests that student engagement is optimised when the social context fulfils student's basic psychological requirements. These include relatedness, which refers to a need to have some involvement on a personal level with other people, a need to feel competence in the context in question, and a need to feel autonomous (Connell & Wellborn, 1991). This paper will report on the issue of perceived competence and autonomy at the grade six level.

Perceived Competence

White (1959) presented evidence that an innate characteristic of humans is an intrinsic 'need' to feel competent, and that behaviours such as exploration and mastery attempts are best explained by this innate motivational force. Students' needs for competence are satisfied if they feel they can achieve their desired ends and if they have the required amount of structure in their environment. Structure refers to the amount of information available within the classroom about how to effectively achieve desired outcomes. Skinner and Belmont (1993) explained that teachers provide structure by "clearly communicating their expectations, by responding consistently, predictably, and contingently, by offering instrumental support and help, and by adjusting teaching strategies to the level of the child" (p. 572).

The scale that assessed students' perceptions of competence in this study measured the degree to which students feel they are in control of their learning, the extent to which they believe that outcomes in mathematics are due to effort, ability, powerful others, luck, or unknown causes (*strategies*), and the extent to which they feel they are able to enact these causes (*capacities*). Examples of items are: "*Trying hard is the best way for me to do well in maths*" (effort strategy), and "*I can work really hard in maths*" (effort capacity).

Engagement

Cambourne (1994) claimed that while learners are exposed to thousands of demonstrations of techniques, methods and other learning practices during their lives, many of these are ignored by students, and therefore learning does not occur. He argued that teaching can only result in learning if the student is engaged with the learning process. Skinner, Wellborn & Connell (1990) carried out a study on 220 students in grades three to six, measuring, among other things, engagement, grades and achievement test scores. They found that students who are more engaged earn higher grades, score better on standardised tests of achievement, and show better personal adjustment to school.

Skinner & Belmont (1993) contended that children who are engaged are positive emotionally and show sustained behavioural involvement. "They select tasks at the border of their competencies, initiate action when given the opportunity, and exert intense effort and concentration in the implementation of learning tasks; they show ... enthusiasm, optimism, curiosity and interest" (p. 572). Disaffected students, however, can be bored, anxious, depressed or even angry about their presence in the classroom, they can withdraw from learning opportunities or be rebellious towards teachers and their peers.

The engagement questionnaire assessed teachers' perceptions of students' cognitive engagement (ie. flexible vs rigid problem solving, independent vs dependent work styles), behavioural engagement (i.e. class participation vs uninvolvement, on task vs off task behaviour) and emotional engagement (i.e. nervousness, happiness, excitement). Cronbach alpha for this scale was found to be .93.

Skinner, Wellborn & Connell (1990) found that particular combinations of competence-related beliefs were particularly relevant for undermining or promoting engagement in school. They found that engagement was undermined by beliefs in nonaction strategies (ability, powerful others and luck) and particularly by not knowing what strategies are effective. Highest levels of engagement were found for students who held high strategy and capacity beliefs for effort, while the lowest levels of engagement were found among those students who reported high strategy and low capacity for ability, powerful others and luck.

Autonomy

Autonomy is described by Deci & Ryan (1991) as "referring to a sense of endorsement and initiation with regard to one's behaviour" (p. 272), while Patrick, Skinner & Connell (1993) described autonomy as "the extent to which a person feels free to show the behaviours of his choice" (p. 782). Deci & Ryan (1991) further claim that "contexts in which others are both autonomy supportive and involved allow satisfaction of the individual's basic needs and are thus optimal for development" (p. 273). Research has indicated that when the classroom climate is experienced as autonomy supportive rather than controlling, it has been associated with greater intrinsic motivation, trust, self-worth and satisfaction (Deci, Schwartz, Sheinman & Ryan, 1981).

A manifestation of autonomy can be found in the reasons that students engage in various activities, such as mathematics. The least autonomous reasons are *external* reasons, such as expectations of punishment or reward, followed by *introjected*, which are more internal but still pressured. Next along the continuum lies *identified* reasons, which are freely chosen but not quite intrinsic, and lastly *intrinsic* reasons, which focus on learning for it's own sake rather than for particular outcomes. In this study, autonomy was measured by nine items that tapped into the reasons for students participating in mathematics (ie *Why do I work in maths classes?*). These reasons could include: external (*because I'll get into trouble if I don't*, $\alpha = .69$), introjected (*because I'll feel bad about myself if I don't*, $\alpha = .60$), identified (*because I want to learn new things*, $\alpha = .71$) and intrinsic (*because it's fun*, $\alpha = .82$).

Method

November December 1995- Grade 6 students

In November and December 1995 ten schools were visited, and the entire cohort of grade 6 students present on the day were asked to complete the questionnaire. Only about 30 students overall refused to participate, and so data from a total sample of 510 students, 231 females and 279 males, were collected. The researcher administered each of the instruments, and the questionnaire was read to the students to obviate any particular reading problems. In general, the teachers completed the engagement questionnaires on each student, while the students were finishing these tasks.

Students' Questionnaire

This instrument consisted of eleven questions about students' general attitudes towards maths, 116 questions about perceived involvement, autonomy support, structure, relatedness, competence, engagement, coping and classroom environment, six questions about perceived ability and perceptions about others' beliefs, and six open ended questions. Unless otherwise stated, students rated their responses to statements on a four point Likert-type scale ranging from *not at all true* to *very true*. All of the items discussed in this paper were adapted from the Rochester Assessment Package for Schools (RAPS, Wellborn & Connell, 1987).

Teachers' Questionnaire

Teachers were asked to complete a 30 item engagement questionnaire for each student present on the day of testing. Teachers rated the behaviours listed on the questionnaire on a Likert- type scale ranging from *almost never* to *very often*. Scoring of the engagement questionnaire produced a combined score for engagement that could range from -4 to +4, with positive values indicating engagement and negative values reflecting disaffection.

Some preliminary findings

Of major interest in this study are students whose belief structures are optimal for engagement, and those whose belief structures are those most likely to undermine engagement. To this end, three approximately equal groups were formed using the teacher rating of student engagement. The lowest third thus relates to low teacher rated engagement, the middle third to average rated engagement, and the upper third to high levels of rated engagement. Table 1 shows the number of male and female students from each of the schools in each group. The percentage shown in brackets in the "Totals" row is the percentage of that gender in the particular category. Twenty engagement questionnaires were not completed by teachers, which causes the slight discrepancy in the data. From these data it can be seen that more males (36%) than females (28%) are rated by their teachers as having low engagement, while more females (36%) than males (29%) are seen as being highly engaged.

Table 1
Number of male and female students from each level of engagement group in each of the primary schools

School (n= Females, Males)	**Low engagement (n=166)**		**Average engagement (n=160)**		**High engagement (n=164)**	
	F	M	F	M	F	M
1 (n=32,33)	6	11	12	10	10	10
2 (n=8,13)	2	5	2	6	4	2
3 (n=48,38)	13	11	14	15	18	11
4 (n=7,13)	2	7	2	3	3	3
5 (n=17,14)	2	5	7	3	8	6
6 (n=16,41)	7	17	5	12	2	10
7 (n=10,27)	1	7	4	13	5	7
8 (n=24,27)	11	8	3	6	10	13
9 (n=46,52)	17	25	15	15	12	12
10 (n=23,21)	4	5	5	8	10	8
Totals (F=231, M=279)	65 (28%)	101 (36%)	69 (30%)	91 (33%)	82 (36%)	82 (29%)

Perceived Competence

Table 2 shows the means and standard deviations for all perceived competence constructs for each of the three groups defined by teacher rated student engagement, F values and significance levels obtained from one way analysis of variance. It can be seen from this table that perceived control and all capacity beliefs are highest for those students in the high engagement group, lowest for those students in the low engagement group. Strategy beliefs in general follow the opposite pattern, with particular strategies becoming less important as engagement increases. Strategy effort beliefs show a slightly different pattern. The trend is increasing from low engagement to high engagement, except for a slight downturn in the average engagement group. Thus for lower engaged students, effort is not as important a strategy as it is for highly engaged students, but both feel it more important than the average engagement group.

Table 2

Means and standard deviations for perceived competence variables for the three engagement groups

	Low engagement		**Average engagement**		**High engagement**		**F**	**p**
	Mean	**SD**	**Mean**	**SD**	**Mean**	**SD**		
Control	3.07	.78	3.44	.63	3.71	.50	40.31	***
Strategy								
Ability	2.38	.85	2.12	.79	1.91	.71	14.98	***
Effort	3.34	.59	3.26	.63	3.47	.51	5.33	**
Luck	2.60	.81	2.30	.66	2.02	.67	27.01	***
Unknown	2.21	.82	1.88	.76	1.52	.56	38.43	***
Powerful others	1.81	.91	1.57	.70	1.47	.67	8.56	***
Capacity								
Ability	2.55	.86	3.00	.76	3.42	.63	55.30	***
Effort	3.32	.66	3.56	.53	3.72	.50	20.36	***
Luck	2.64	.70	2.90	.58	3.10	.60	21.95	***
Powerful others	2.89	.80	2.92	.74	2.97	.73	.42	
Teacher rating of student ability	2.42	.74	3.10	.65	3.51	.41	218.33	***
Student rating of their own ability	3.14	.79	3.57	.75	4.05	.70	61.72	***
Student rating of their own engagement	3.14	.51	3.32	.43	4.04	.74	26.98	***

** p < .01 *** p < .001 *df* = 2, 487

The data were analysed for differences using one way analysis of variance followed by the Tukey HSD test to discover where any differences lay. All but three variables, strategy effort, strategy powerful others and capacity powerful others, showed highly significant differences between all three groups of students. Strategy powerful others only showed differences between the lowest engagement group and the other two groups, with no differences between the average and high engagement groups. There were significant differences in the importance placed on effort as a strategy only between the average and high engagement groups, and capacity powerful others showed no differences between any of the groups, indicating that all groups of students felt that the teacher would provide assistance when required.

Effort was perceived to be the most important strategy for success, followed by luck. This latter strategy reflects the endorsement by students of statements such as "I have to be lucky (by getting asked the right questions) to do well in maths". Problems occur with strategy and capacity beliefs when strategy beliefs are high and the corresponding capacity beliefs are low; for example when students believe that it takes luck to do well in maths but that they're not lucky.

The findings from this study confirm those of Skinner et al. (1990). Engagement was undermined by beliefs in nonaction strategies (ability, powerful others and luck) and by not knowing what strategies are effective. Engagement was promoted by students' high strategy and capacity beliefs for effort, while the lowest levels of engagement were found among those students who reported high strategy and low capacity for ability, powerful others and luck. Teachers' ratings of student ability indicate that the highly engaged students are those most likely to be rated as the most able.

Although the low engaged students believe that luck and ability are important strategies, it is apparent from Table 2 that their capacity beliefs are lowest for these particular causes, "Ability is important in maths and I'm not very smart". For the highly engaged group, ability is rated as a fairly unimportant strategy, however the students feel that they are smart at maths "Ability isn't important, but I am smart". Some students aren't sure what it takes to achieve success and avoid failure in maths. These students endorse those statements "I don't know what it takes to do well in maths". It is much more common for students in the low engagement group to endorse these statements than it is for those students in the high engagement group.

Autonomy

Table 3 shows the means and standard deviations for the autonomy constructs for each of the three engagement groups, F values and significance levels from one way analysis of variance (ANOVA). Most students work for identified reasons; "Because I think it's important", while Tukey's HSD test found that external motivation for engagement in learning was found to be significantly higher among students in the low engagement group.

Table 3
Means and Standard Deviations for Autonomy Constructs

	Low engagement		Average engagement		High engagement		F	p
Self-regulation	**Mean**	**SD**	**Mean**	**SD**	**Mean**	**SD**		
External	2.41	.94	2.13	.90	1.97	.89	9.87	***
Introjected	2.06	.89	2.13	.90	2.08	.94	.32	
Identified	3.45	.69	3.60	.58	3.75	.45	11.07	***
Intrinsic	2.65	.81	2.83	.76	3.11	.77	14.60	***

*** p < .001 *df* = 2, 487

Identified and intrinsic reasons for working increased significantly from low to high engagement levels. Tukey's HSD test found significant differences between the low engagement group and the average and high engagement groups for identified reasons, with no significant differences between the average and high groups, and between the high engagement group and the average and low engagement groups for intrinsic reasons, with no significant differences between the average and low groups. Introjected reasons for working were the lowest for all groups, and showed no significant differences between groups.

Discussion

The long-term aim of this study is to identify functional and dysfunctional patterns of beliefs about learning mathematics. It is clear from the results discussed that the instruments described serve this aim well. Students were able to be categorised using the engagement questionnaire into low, average and high engagement groups. Subsequent analysis based on these groups found that students who hold beliefs deemed to promote engagement feel more in control, understand more about strategies for learning, feel more capable of succeeding and are more intrinsically motivated. Students at the other end of the spectrum, in the low engagement group, have the lowest levels of control beliefs, believe that luck and ability are important strategies for succeeding in mathematics but they aren't particularly lucky or smart, and have higher levels of external motivation.

The next phase of this study follows many of the students in the transition to secondary school. Students will be surveyed in terms two and four of year seven, and analysis of data pertaining to the particular engagement groups described in this paper will be carried out to investigate patterns of change over transition. There are also different patterns of transition; students moving on with a large group of peers, with a few peers only, and on their own. Some investigation towards effect of type of transition is also being planned.

The students involved in the study have already foreshadowed some of the problems they anticipate with secondary school mathematics.. For example, when asked whether maths classes would be different in secondary school, some answers were: *"Definitely. [We do] algebra, the really, really, really, really, really hard maths"*, *"Much harder maths. I think I will fale* (sic)*"*, *"Yes they will be harder. The teachers will not help me"*. It remains to be seen what effect these perceptions will have on students' engagement and on subsequent learning in mathematics.

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Testing to a Purpose: Assessing the Mathematical Knowledge of Entering Undergraduates

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Two circumstances of major significance are presently impacting on undergraduate mathematics courses. These are respectively increased participation resulting in a wider spread of abilities among entering students, and the increasing use of symbolic algebra software in course delivery. This paper reports on preliminary work in projects at two Universities. Responses to sample questions are discussed in the context of their purpose which is motivated by the need to address the dual circumstances indicated above.

Studies addressing concerns about the mathematics knowledge base of entering undergraduates have been reported from time to time (Buckland, 1969; Gray, 1975; Clement, Lochead, and Soloway, 1980; Galbraith, 1982; Tall and Razali, 1993). Gray, for example, wrote of misconceptions, misguided and underdeveloped methods, and unrefined intuition that appeared to survive assignments, corrections, solutions, tutorials, and examinations. A common thread in all studies was the negative influence of fragmented learning and the apparent absence of cognitive strategies to co-ordinate and test the consistency and validity of accumulating knowledge.

The entering characteristics of undergraduates have become an even greater matter of interest because of contemporary changes in the intake and delivery of tertiary courses. Firstly the increased intake into universities means that entering students are being drawn from lower performance bands than previously. In Queensland for example selection is based on an overall position rating (OP). Whereas in the recent past undergraduates at the University of Queensland have been drawn from bands OP1 to OP9, in 1996 students from bands as low as OP15 have been awarded quota places. Hence the disparity in entering performance level has been greatly exacerbated with flow on implications for first year teaching.

The second contextual event of significance is the increasing use of symbolic algebra software (Derive, Maple, Mathematica) in undergraduate teaching. Fey (1989) captures a significant point when he notes that effective use of technology in learning involves more than projecting from a pen and paper medium into a computer based context on the basis of our intuitions. New elements are introduced into the teaching-learning situation by the use of computer technology and their effects need to be researched, not assumed.

With respect to the first challenge it becomes necessary to establish entering knowledge levels from a wider perspective than formal Year 12 test results. Many students taking undergraduate mathematics bring a background that is both diverse in its scope and variable in its recency (mature age entry). Hence screening programs need to provide for a much wider spread than the lowest common denominator previously associated with formal year 12 testing procedures. It is with the purpose of distinguishing lower levels of intake proficiency that we are partly concerned in this paper. (It is also acknowledged that some universities have been grappling with this issue for some time.)

With respect to the second issue (readiness for technology based instruction) one particular attribute stands out as central to learning based around symbolic algebra software. This is the capacity to understand, co-ordinate and apply alternative symbolic representations, in particular the algebraic and graphical representations of functionality.

Two recent studies provide evidence supporting the importance of algebraic/graphical inter-relations. Tall and Razali (1993) report on the performance of students comprising a diagnostic and remedial group of undergraduates at the Universiti Teknologi, Malaysia. Among a variety of findings they reported that students generally had difficulty in co-ordinating graphical and algebraic representations when slight variations were introduced into the context. Boers and Jones (1994) describe the voluntary use of graphics calculators in a traditional calculus examination. While the calculator was used extensively throughout the course teaching, and its use was built in to the instructional materials, analysis of the students' solutions indicated that, contrary to expectations, the graphics calculator was under-utilized by most students. The capability of the students to deal simultaneously with graphical and algebraic information from two independent sources was identified as the main obstacle to effective use. The authors make the significant point that the ability to integrate algebraic and graphical information will not manifest itself just because a student has the technology. 'It is a learnt skill which needs to be consciously nurtured.'

Knowledge base of new undergraduates

To address the problem outlined as the first issue above a multiple choice entry test was designed for administration during the first tutorial hour of the subject MP105 *Mathematical Methods and Applications* at the University of Queensland. This subject draws students from at least seven faculties and in 1996 has an enrolment of 252. It is therefore a subject for which the clientele forms a representative sample of the variety of students undertaking tertiary mathematics.

The test comprised 20 multiple choice questions designed in four clusters viz algebra, co-ordinate geometry, indices and logarithms, and simple calculus. In the main the items addressed knowledge from junior mathematics apart from several simple one-step questions devised around calculus operations. The alternatives were chosen so that the distribution of responses would provide information as to the extent of popular but incorrect approaches.

Some examples are provided below together with the percentage of correct responses, the most popular incorrect response, and an indication of the flawed approach most likely to be responsible for the incorrect response.

1. $(5a^3)^2$ was correctly given as $25a^6$ by 67% of students. The most popular incorrect response was $25a^5$ given by 15% of students. The probable reasoning behind the incorrect response involves adding indices.
2. Solutions to the equation $x(x+2)=3$ were given correctly by 64% of students. One of two equally popular incorrect responses was $x=1, 3$ given by 10% of students. The probable reasoning behind this response is that either $x=3$ or $x+2=3$.

3. The equation to a circle centre (1,2) and radius 3 was correctly given by 43% of students. The most popular incorrect response was $(x+1)^2 + (y+2)^2 = 9$ given by 23% of students. The most probable reason behind this response is flawed rote learning.
4. The form $\ln y = \ln 4 + \ln x$ was correctly equated with $y = 4x$ by 30% of students. The most popular incorrect response was $y = 4+x$ given by 25% of students. The most probable reason involves cancelling ln.
5. The derivative of $e^{2x} + 1$ was correctly obtained by 53% of students. The most popular incorrect response was $2xe^{2x-1}$ (23%) which represents a grotesque attempt to make use of the form nx^{n-1}.
6. The value of $\int_1^2 x^2\,dx$ was given correctly by 50% of students. The most popular incorrect response was 3 given by 21% of students. This result is achieved by substitution of the terminals directly into the integrand.

These illustrations verify that broken knowledge has fundamental roots beyond senior school mathematics courses. A question of interest is the extent to which symbolic algebra software, by performing many routine tasks, provides remedial instruction to the learner, or whether its use produces, for these students, an even wider disparity between their mathematical world and the world of professional mathematics.

Assessing algebra and graphical linkages

In designing items to probe algebraic and graphical linkages we accept the view that knowledge may be usefully classified in terms of concepts and procedures (Anderson, 1990; Hiebert and Lefevre, 1986) and that retrieval of knowledge from memory storage may be understood similarly.

Conceptual knowledge is assumed stored as a linked network of individual units, where the more elaborate the network the more points of access there are for activation to occur. Inadequate conceptual knowledge means that needed information cannot be located (blank response) or that some inappropriate version will be activated (1/A written as the inverse of matrix A).

Procedural knowledge involves the execution of a routine (rule) in response to an activating condition.

e.g. If $ax_1 = ax_2$ and $a \neq 0$ (condition) then $x_1 = x_2$ (rule)

Flawed procedural knowledge may be a consequence of either a mis-applied or wrongly remembered condition (losing a solution from $x(x-1) = x(2x-3)$ through cancelling x) or an incorrect rule $(x - a)^2 = x^2 - a^2$.

In performing mathematics students may simply need to complete a *mechanical* routine. At a higher level they may need to *interpret* information to facilitate a conceptually based conclusion. At a higher level again they may need to *construct* a solution that involves the creation of links and integrations between concepts and procedures that must be themselves generated by the solution process.

Consequently we believe that mathematical knowledge can usefully be considered in terms of these three conceptions 1. mechanical 2. interpretive 3. constructive.

In using this classification resulting test items can play several roles:

- as a measure of entering mathematical competence. In this role they serve a *diagnostic* purpose in which performance patterns identify areas of strengths and weaknesses on entry.
- as measures (pre and post) to assess the overall impact of a teaching program in deepening the understandings and capacities of students on basic concepts and procedures that underpin its structure.
- when designed (as in the present case) to address concepts and procedures that are specific targets of computer based learning sequences, to assess the initial state of knowledge with respect to these concepts and procedures.

Sample test items

Mechanical: the items in this group require the performance of a standard procedure that is cued in the wording of the question.

M. The graph with equation $y = 2x^2 - bx^3$ cuts the X-axis at X=4. The value of b is

A 4 B 2 C 0 D 1/2 E none of these.

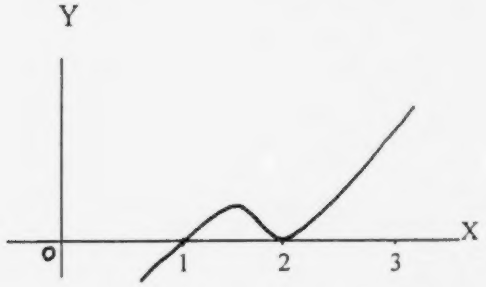
The specific cue to 'find b' is provided. It will be recognised that the items discussed previously from the University of Queensland test are mechanical items.

Interpretive: the items in this group require the retrieval of conceptual knowledge and its application to identify a correct alternative. Procedures as such are not involved.

I. Which of the following could be the equation of the graph shown (Figure 1)?

A. $y = (x-2)^2 (1-x)$ B. $y = (2-x)^2 (1-x)$
C. $y = (x-2)^2 (x-1)$ D. $y = (x-1)^2 (x-2)^2$
E. none of these

Figure 1. Diagram for item I



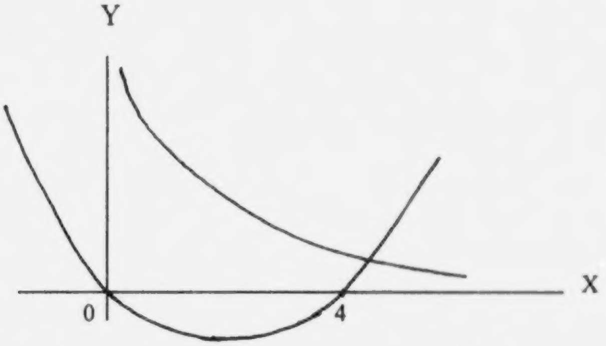
Reasoning such as the following is required. Since a double root occurs at x=2 and a single root at x=1 the equation is either $y = (x-2)^2 (1-x)$ or $y = (x-2)^2 (x-1)$ noting the equivalence of A and B. Since for large x the graph behaves like $y = x^3$ (not $y = -x^3$) alternative C is selected.

Constructive: the items in this group involve the use of both conceptual and procedural knowledge but in this case necessary procedures must be identified and introduced by the student - they are not cued. Responses involve the construction of a solution rather than the selection of an alternative.

C. The equations of two graphs are $y = 3/x$ and $y = x^2-4x$. Obtain a cubic equation where solution gives the x-coordinate of the point(s) of intersection of these two graphs. How many positive roots does this equation have?

The solution involves: recalling that the required equation is obtained by equating $3/x$ and x^2-4 (concept), simplifying $3/x = x^2-4x$ to provide a cubic equation in some form (procedure), sketching a graph such as Figure 2 (procedure), and recognizing that one intersection to the right of 0 means one positive root (concept).

Figure 2. Diagram for item C



More examples of all three item types are provided in Galbraith and Haines (1995). All items used content from preparatory secondary mathematics courses.

Preliminary Outcomes

Results from 244 entering mathematics undergraduates at City University, London were obtained for six questions of each type on an entry text. It was predicted that student performance would follow the pattern score (mechanical) > score (interpretive) > score (constructive). The results are summarized in Table 1 below after removing one a-typical outlier from each set.

Table 1. Summary response data (N = 244)

	Mechanical (5 questions)	Interpretive (5 questions)	Constructive (5 questions)
Mean score	0.59	0.39	0.28
Standard deviation	0.15	0.09	0.15

The table entry 0.59 means that on average 59% of students obtained the correct answer to mechanical questions and similarly for the others. Looked at another way we can say that overall an entering undergraduate student has a probability of 0.59 of scoring correctly on the mechanical questions etc.

Given that the challenge increases from mechanical through constructive as predicted, the greater demands on the capacity to relate graphical and algebraic representations found in the interpretive and constructive questions suggest that this type of item development will be useful in identifying knowledge factors important when teaching occurs per medium of symbolic algebra software.

The test items discussed in this paper have been developed during the preliminary stages of two projects. The items from the Queensland test will be recognised as *mechanical* in terms of the definitions above. Design, refinement, and testing will continue in the course of efforts to address the two issues facing undergraduate teaching described in this paper.

So far the design of *mechanical, interpretive,* and *constructive* questions in terms of constructs of conceptual and procedural knowledge have proved useful, both for the targeting of specific attributes involved in the co-ordination of algebraic and graphical representations, and by way of explaining the choice of incorrect solutions in terms of inadequate knowledge networks, improperly learned connections, or faulty production rules. These themes will be elaborated further in the presentation.

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Evaluating Mathematical Knowledge Elements

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This paper is a brief description of a larger study conducted to examine mathematical responses from 90 first year university students. The main part of the study was concerned with identifying and describing the quality of existing mathematical knowledge that these students brought with them to university. The SOLO taxonomy (Biggs & Collis, 1982) technique was use d initially as the evaluation tool. However, certain limitations of the SOLO technique were identified which led to the development of a knowledge element coding technique. The focus of this paper is a brief description of this coding technique, its use and value in evaluating mathematical knowledge elements.

Several researchers, including Ball (1990, 1991), Eisenhart et al. (1993), Even (1993), Leinhardt (1988), Hiebert and Leferve (1988) and Gagne (1985) have generally agreed that mathematical knowledge has two components. The procedural component is that which consists of knowledge of algorithms, mathematical rules and procedures. The conceptual component is mathematical understanding. Other researchers, however, do not distinguish between the two components but define the growth of understanding mathematics as a continuing process of organising one's knowledge structures (Pirie & Kieren, 1994). Mathematical understanding from this perspective is defined as a whole dynamic process and not as a single or multi-valued acquisition, nor as a linear combination of knowledge categories. Regardless of whether mathematical knowledge is the acquisition of components or a growth process, a significant problem is its evaluation in order to determine whether or not the person has acquired it. Such an evaluation is particularly important in identifying potential secondary teachers of mathematics, since competency to teach mathematics requires proficiency in both mathematical knowledge and pedagogical knowledge (Ball, 1990, 1991; Eisenhart et al., 1993; Berliner et al., 1988).

Methodology

The sample consisted of 90 first-year university mathematics students. Fifty percent of the sample had received pre-tertiary education in Tasmania, forty percent from other Australian states and ten percent from overseas. These 90 students were enrolled in the pre-requisite mathematics unit for the Engineering, Science and Technology, and Mathematics and Science Education programmes. The students were classified into three groups according to their pre-tertiary mathematical background. Group A completed high level mathematics, group B the middle level and group C the low level mathematics. It was hypothesised that the quality of the students existing mathematical knowledge would be related to the differences in their pre-tertiary backgrounds. The data collection instrument was a questionnaire consisting of four mathematical items (see Appendix B). The instrument identified misconceptions relating to the product of two negative numbers, trigonometric and logarithmic functions and statistical variance. All the items required responses concerning processes in order to explain the procedures provided as cues.

To determine the types of mathematical knowledge elements that would facilitate and promote mathematical competence, a modified form of the SOLO taxonomy (Collis & Watson, 1991) task analysis technique was used for qualitative analysis. Several difficulties were encountered during the mapping of the students' responses using the SOLO technique (Collis & Watson, 1991). One of the main difficulties was categorising the responses into SOLO levels under the characteristics being specified for each level. For example, a uni-structural level was to be characterised by the presence of a single correct aspect of the specified mode. However, more than one type of 'single correct aspect' was observed during evaluation of the responses. This created classification and analysis problems. Therefore, the Knowledge Elements (KnE) task analysis technique was developed primarily to strengthen the validity of these SOLO level measures. The KnE is specifically for mathematical learned outcomes. It is a coding system to facilitate the categorisation of responses into complexity levels and to provide codes for descriptive measures. KnE takes into account procedural and conceptual mathematical knowledge. The codes are in the form of six broad bands of knowledge elements. Five bands of 'procedural' knowledge and the sixth band is for 'conceptual' knowledge elements. These bands are described in Table 1.

KnE Procedural Knowledge

The KnE procedural knowledge is based on Gagne's (1985) description of procedural knowledge as pattern-recognition and action-sequence. Pattern-recognition refers to the process of classifying information so that the person is able to recognise specific examples of concepts by relating them to general patterns. Action-sequence refers to the persons ability to carry out sequences of symbolic operations, that is, recognising patterns specified by the conditions whilst carrying out a sequence of actions. These actions are either covert, mental, actions or both covert and overt, physical, actions consisting of a 'series of steps in their correct sequence' (Gagne, 1985, p.103). Gagne suggested that these two forms of procedural knowledge are linked in performance. For example, consider the Trigonometric item: solve for x in Cos(2x+1)=0. Here, the pattern-recognition is knowing that this 'given cue' is not in the same form as similar cues such as y(2x+1) that can be expanded to 2xy + y. The action-sequence type knowledge for the 'given cue' follows on from pattern-recognition. That is, after recognising, covertly, that Cos(2x+1)is a trigonometric function, the person could proceed to the next action of giving meaning to the symbols '(2x+1)'. Knowing that (2x+1) represents an angle measurement corresponding to the Cosine value of zero, would lead to the next action of recalling what the angle size is. Once this is retrieved, the computation procedures would follow resulting in a solution for x. Note, however, that pattern-recognition can occur simultaneously during the process of action-sequence. For instance, knowing what angle size to recall requires the ability to recognise, compare and to eliminate angle sizes of other related trigonometric functions, e.g. sine and tangent. Anderson (1981) and Gagne (1985) have maintained that pattern-recognition and action-sequence procedures are closely connected and that the relationship between them is similar to that between concepts and rules. In addition, mastery of pattern-recognition is through generalisation and discrimination processes. Generalisation explains how a person can classify concepts they have never seen before. Therefore, generalisation increases the range of situations to which a procedure applies. For example, an individual who learned that km + 2m is the same as m(k+2) could generalise this pattern incorrectly to others of similar form as in Cos(2x+1) and log(x-1), seeing them, respectively, as Cos2x + Cos1 and logx - log1.

Discrimination, on the other hand, restricts the range of conditions to which a procedure can be applied. The statistic, trigonometric and logarithmic items are examples of situations with restricted conditions, see Appendix B. For instance, the evaluation of log(2x+1)=0 is restricted to procedures that meet the conditions for this situation only. Other procedures would be required for the evaluation of, say, log 2x=1.

Computation is a function or a dependent process of procedural knowledge. Computation involves speed and accuracy. These two components are dependent on the pattern-recognition processes of generalisation and discrimination. Computation is also part of the action-sequence procedures and in some cases it is the link between pattern-recognition and action-sequence. For example, if a person was asked; find the area of the largest circle inside a square of 2cm length, then the procedure that connects pattern-recognition or discriminating between shapes and area, to action-sequence, is computation of the required area. Accuracy and the rate in which the computation is carried out are essential features of this element of procedural knowledge. According to Collis (1990, p.5), computation in mathematics illustrates mastery of 'the link between the symbol systems themselves and the world'.

Procedural knowledge, therefore, involves more than knowing and recognising symbols, patterns, rules and algorithms. It also involves processes of generalisation, discrimination and computation, all of which are based on understanding and prior knowledge. Eisenhart et al. (1993, p.9) have defined procedural knowledge as 'mastery of computational skills and knowledge procedures for identifying mathematical components, algorithms, and definitions'. These authors have further described procedural knowledge of mathematics as consisting of two parts: (a) knowledge of format and syntax of the symbol representation system and (b) knowledge of rules and algorithms.

KnE Conceptual Knowledge

Knowledge comprising facts has been described by Anderson (1980, p. 222) as 'declarative knowledge'. According to Collis (1990), learning in the concrete symbolic mode leads to declarative knowledge which is demonstrated by 'an ability to make symbolic descriptions of the experienced world' (p.5). Declarative knowledge is knowledge that is expressed through 'the medium of a symbol system in a way that is publicly understandable' (Biggs & Collis, 1991, p.59). From these perspectives it seems that mathematics, a symbol system, is 'purely' declarative knowledge. Such perspectives also imply a necessity for understanding the mathematical concepts represented by the symbols. In other words, the link or relationship between a mathematical concept and its symbolic representation is understanding. It follows that one way to determine an individual's mathematical understanding is to obtain a measure of the individual's declarative knowledge. Declarative knowledge in this sense encapsulates both procedural and conceptual knowledge.

Conceptual knowledge in mathematics is defined as fundamental knowledge which underlies the structures for mathematical competence (Gagne, 1985). These structures are the relationships and interconnections of ideas that explain and give meaning to mathematical procedures represented by symbols (Eisenhart et al. 1993). From Gagne's (1985) perspective, conceptual knowledge is understanding in the form of knowledge organisation or sets of propositions, sets of recognised patterns, or some mixture of these. In the KnE framework, conceptual knowledge is demonstrated by a person's ability to make the relationships between declarative knowledge, symbols and concepts (Anderson, 1980; Collis, 1990; Biggs & Collis, 1991), and knowledge organisation or sets of pattern-recognition (Gagne, 1985). Thus, the sixth broad band of knowledge elements is called 'relationships'.

Table 1: K1 to K5 are the broad bands for procedural knowledge elements and K6 is the band for conceptual knowledge

Code	Knowledge Band	Knowledge description
K1	**Recognition & Classification**	Knowledge of what the situation is or the ability to classify the situation into categories of known or familiar knowledge. [Generalisation & Discrimination]
K2	**Routine**	Knowledge of the problem in the most routine sense. For example, substitution of values into an equation; knowing the 'operation' for the terms such as a product, to increase, and knowing when to expand an expression or to factorise. [Generalisation]
K3	**Rules & Algorithms**	Knowledge of rules and algorithms to compute the task. [Discrimination]
K4	**Format & Syntax**	Knowledge of format and syntax of symbol representation. [Discrimination]
K5	**Computation**	Knowledge of how to solve or compute correctly. [Computation]
K6	**Relationships**	Knowledge of relationships and interconnections of ideas that explain and give meaning to mathematical procedures.

The processes stated in the 'brackets' are the main processes involved in generating the particular knowledge element.

Scale Values For The KnE Descriptors

Since the KnE method was essentially to provide codes for elements of procedural and conceptual knowledge. These codes, assumed to be associated with the SOLO levels: pre-structural, uni-structural, multi-structural, relational, and extended abstract, provided new descriptions for the SOLO. For example, instead of perceiving a uni-structural as representing a single aspect of a mode, the new description is represented by several KnE codes. Thus, each SOLO level can be perceived as having quantifiable elements. A justification of this approach is provided by Cryer and Miller (1991, p.108) who suggested that for categorical measurement, classification of elements according to a common attribute, new variables can be created through arithmetic or functional operations on existing variables and such creations would constitute a change in the measure of the magnitude of the new variable. Since the KnE descriptors are essentially new variables being created from existing SOLO classifications, a set of arbitrary numerical values was employed to indicate the change in the magnitude of the new variables. That is, pre-structural=1 (PRE), uni-structural=2 (UNI), multi-structural=3 (MULT), relational=4 (REL), and extended abstract=5. In addition, because the SOLO levels were assumed to have no clearly defined boundaries between the upper and lower levels, the values of +0.5 and -0.5 were also introduced. A value of +0.5 was added to represent response structures that were border-line towards an upper category and -0.5 to represent structures border-line towards a lower category. This addition of border-line values was also to reflect the continuous or 'growth' nature of knowledge. Thus, the numerical values associated with the KnE codes were; No-Attempt = 0, PRE- = 0.5, PRE = 1.0, UNI- = 1.5, UNI = 2.0, UNI+ or MULT- = 2.5, MULT = 3, MULT+ = 3.5, REL = 4.0, REL+ = 4.5, and Extended Abstract = 5.0.

KnE Descriptors

The KnE descriptors described here refer to the categories obtained from the re-classification of the study data using the KnE task analysis method. To distinguish between the SOLO levels and those described by the KnE technique, the term 'descriptor' rather than 'level' was adopted. The KnE descriptors or categories are the same as the SOLO taxonomy, these being pre-structural (PRE), uni-structural (UNI), multi-structural (MULT), and relational (REL). However, the criterion for classification was different. That was, the KnE descriptors were classified in terms of knowledge elements rather than 'relevant aspects' of the mode of intellectual functioning as assumed in the SOLO taxonomy (Biggs & Collis, 1982). The responses to the four items: NEG, TRIG, LOG, and STAT, were recoded as K1 - Recognition & Classification, K2 - Routine, K3 - Rules & algorithms, K4 - Format & Syntax, K5 - Computation, and K6 - Relationships. To differentiate between correct and incorrect use of the particular knowledge element, a subscript x was used. For example, K2x would mean that incorrect routine knowledge was demonstrated. Descriptions of each KnE descriptor are as follows as well as a sample coding of a multi-structural SOLO task analysis map.

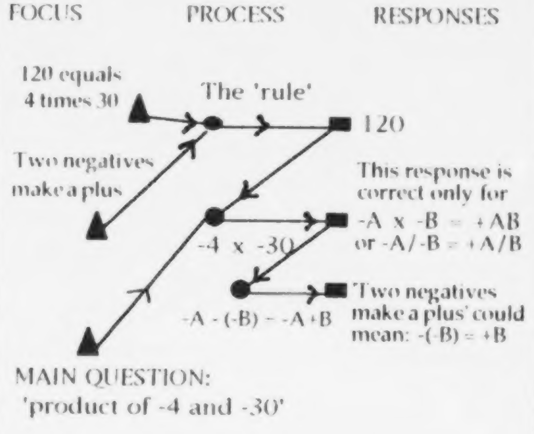
A pre-structural response (PRE) was characterised by failure to make correct generalisation and to carry out appropriate sequence of actions for the task. In other words, such a response reflected a lack of mastery of generalisation and discrimination processes which are required processes of pattern-recognition procedures.

A uni-structural response (UNI) was characterised by a sequence of K1, K2, K3, and K4 type knowledge elements. That is, K1 is an indication that the correct pattern-recognition was made for classifying the 'given cues' into known categories; followed by a covert or overt routine action (K2) while simultaneously recalling correct rules and/or algorithms (K3) and applying correct format and meaning to symbols (K4). A UNI+

response, on the other hand, would require an additional recognition of elements of the 'given cues' that could lead to K5 or computation.

A multi-structural response (MULT) was characterised by the presence of K5 or computation being correctly carried out. A MULT+ response was seen in attempts to utilise as much of the 'given cues' as appropriate to provide a 'justification' for the sequence of actions rather than for conceptual reasons. A sample coding of a multi-structural response is given below in Figure 1. The second column of Figure 1 is a task analysis using the SOLO mapping technique (Collis & Watson, 1991).

Figure 1: Sample coding of a multi-structural response using the KnE method

SOLO level	NEG item response structure Main Question: What is the product of -4 and -30	KnE
MULT	 FOCUS PROCESS RESPONSES 120 equals 4 times 30 The 'rule' 120 Two negatives make a plus This response is correct only for -A x -B = +AB or -A/-B = +A/B -4 x -30 -A - (-B) = -A+B Two negatives make a plus' could mean: -(-B) = +B MAIN QUESTION: 'product of -4 and -30'	**K1 and K3**: Recognised that the given 'rule' was for specific cases. **K3 and K4**: Acknowledged the differences or can discriminate between the different 'format & syntax of symbols'. **{K1,K3-K3,K4}** K5 was a 'covert' knowledge element in this response.

A relational response (REL) was characterised by evidence of 'justifications' for conceptual reasons. This process was classified as a K6 element, because it indicated an interconnection of the 'given cues' with the individual's conceptual understanding of the situation. It also provided an indication of mastery of the pattern-recognition and action-sequence processes. This process also appears to govern the action-sequence and the closure of the response. The REL+ response was characterised by references being made to a general or abstract form of the concept not represented in the 'given cues'.

A REL is distinguished from a MULT by the inclusion of K6. K6 could occur twice in REL responses, once in the beginning and once at the conclusion, usually as a statement to link the action-sequence as a whole unit of response. K6 acted to elaborate and organise knowledge. According to Gagne (1985, p.100), elaboration 'is the process of adding related knowledge to the new... [or given cues] ... knowledge' and organisation 'is the process of putting declarative knowledge into subsets and indicating the relationships among subsets'. She added that organisation enhances management of the 'limited-capacity of the working memory during retrieval' (p. 101).

To summarise the above descriptions, three distinct qualities of knowledge were found. These were: pre-mastery of pattern-recognition, concrete symbolisation and conceptual symbolisation. The pre-mastery of pattern-recognition type knowledge included responses classified as PRE-, PRE, and UNI-. The concrete symbolisation type knowledge comprised responses classified as UNI, UNI+, and MULT. The conceptual symbolisation type knowledge included responses classified as MULT+, REL, and REL+. There is a summary of the knowledge descriptors and their descriptions in Appendix A.

Results and Conclusion

The use of the KnE method in conjunction with the SOLO levels provided a clearer description of the quality of mathematical knowledge that constitute mathematical competence in the response structures . The SOLO technique alone was not sufficient in determining mathematical competence since SOLO is essentially for evaluating shifts in the cycle of learning within a developmental stage or mode (Biggs & Collis, 1991). Although mathematical competence has been described in SOLO terms as 'understanding in the concrete symbolic mode' (Collis & Romberg, 1991, p.96), it still begs the question of what exactly this understanding consists of. The KnE technique described above has been effectively used in describing mathematical competence for non-research mathematical items (Gates et al., 1995) as well as providing a mechanism for quantifying qualitative data.

It was hypothesised that the quality of the students' existing mathematical knowledge would be related to the differences in their pre-tertiary backgrounds. However, a Chi-square goodness-of-fit test was not significant ($p<0.01$) and it was concluded that there was no significant difference in the quality structure of responses from the three groups. Further analysis using a model to describe the interconnection of procedural and conceptual knowledge showed most of the responses from the three groups to be within the pre-mastery of pattern-recognition and concrete symbols, see Appendix A. The results strongly suggest that these first year university students enter university studies with mainly procedural knowledge of mathematics. Such findings could not have been achieved with the use of the SOLO taxonomy technique alone. The KnE technique has potential for evaluation and assessment of mathematical cognitive processes. Particularly evaluations and assessments of cognitive structures, knowledge elements, pertaining to mathematical competence in both the pre-tertiary and tertiary levels. This paper only provides a brief description of the KnE technique and a small part of a larger study. For a detail description of the research and methodology, the reader is encouraged to read the whole thesis which will be made available in the library at the University of Tasmania.

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APPENDIX A:

A summary of KnE descriptions

KnE Descriptors	Mathematical knowledge element description	KnE codes	Knowledge description
PRE$^-$	Predominantly incorrect pattern-recognition procedures presumably because of lack of knowledge - DON'T KNOW, or inability to recall learning - CAN'T REMEMBER or CAN'T RECALL.	$K1_x$, $K2_x$, $K3_x$	PRE MASTERY OF PATTERN-RECOGNITION
PRE	Failure to make correct generalisation and discrimination as well as providing inappropriate sequence of actions for the task	$K1_x$, $K2_x$, $K3_x$, $K4_x$, $K5_x$ [any combination of these]	
UNI$^-$	Showed the ability to make a correct pattern recognition This may be followed by a routine action or a statement of an incorrect rule or algorithm	K1, K2 or K1, K2, $K3_x$, K1, K3, $K4_x$	
UNI	Showed the ability to make correct pattern - recognition (K1); followed by routine action (K2) Demonstrated the ability to discriminate rules and algorithms(K3) and simultaneously apply correct format and meaning to symbols (K4)	K1, K2, K3, K4	CONCRETE SYMBOLISATION
UNI$^+$	Showed all the elements as in UNI with an additional indication of procedures that could lead to correct computation ->K5	K1, K2, K3, K4 -> K5	
MULT	Showed all the elements as in UNI$^+$ with the additional computation (K5) element.	K1, K2, K3, K4, K5	
MULT$^+$	Showed all the elements as in MULT plus a justification for the preceding sequence of actions A positive sign of a K6 element.	K1, K2, K3, K4, K5 -> K6	CONCEPTUAL SYMBOLISATION
REL	Provided evidence of conceptual links between the procedural knowledge elements Showed mastery of the processes involved in pattern-recognition and action-sequence.	[K6], K1, K2, K3, K4, K5, K6	
REL$^+$	In addition to the elements as in REL, evidence of reference being made to a general form, abstract concept, of the concept not represented in the given cues.	[K6], K1, K2, K3, K4, K5, K6$^+$	

APPENDIX B:

The Questionnaire Items

1 . **Statistics**

**Ten items were measured and four results were produced:
A class of year 11/12 students was given the information below to find the requested measures:**

$$\sum_{i=1}^{10} x_i = 40 \qquad \left(\sum_{i=1}^{10} x_i\right)^2 = 1600$$

$$\sum_{i=1}^{10} x_i^2 = 194 \qquad \sum_{i=1}^{9} \left(x_i - \bar{x}\right)^2 = 24$$

Use these results to find the *mean*, $\overline{X}$, and *variance*, S_x^2, for these 10 items.

The class responses:

Students produced 3 different values for the *variance*, S_x^2

I) 2.4 (II) 3.4 (III) 2.7

Q. Which *variance*, S_x^2, is the correct one? Please explain.

2. **Logarithmic function**

**Simplify and evaluate for x,
$\log_{10}(2x+1) = \log_{10}(x-1)$**

A student responded:

$\log(2x+1) = \log(x-1)$

$\log 2x + \log 1 = \log x - \log 1 \qquad (\log 1 = 0)$
$\log 2x - \log x = 0, \qquad (\log_{10} x = 0)$

$x = 10^0 \qquad \therefore x = 1$

Q1. Is the student's response correct?

Q2. Please explain and show why you answered yes/no to Q1.

3. **Negative Number**

**A high school student was asked the following question:
What is the product of -4 and -30?**
A student responded:

**"120 equals 4 times 30
Two negatives make a plus"**

**Q1. Would you say this student understands how to explain the products of negative numbers?
Q2. How would you explain the result of 120?**

4 . **Trigonometric Function**

Evaluate for x, $\cos(2x+1) = 0$
A Student responded:

$\cos(2x + 1) = 0$
$\cos 2x + \cos 1 = 0$
$\cos 2x = -\cos 1$
$2x = -1$
$\therefore x = -1/2$

**Q1. Is the student response correct?
Q2. Please explain and show why you answered yes/no to Q1.**

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Number Plugging or Problem Solving?
Using Technology to Support Collaborative Learning

Vince Geiger — **Hillbrook Anglican School**
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This paper reports on a study that examined student interaction and discussion while working on computer-based tasks in a senior secondary school classroom. Analysis of verbal and observational data suggested that the task itself was an important variable influencing the degree of collaboration between students, and that the teacher's intervention could change students' engagement with the task. The findings have implications for specifying the teacher's role in relation to the use of computer technology in the mathematics classroom.

Background

The current push for the incorporation of mathematically enabled technology and applications (META) into the pedagogical practice of school mathematics classrooms is mirrored in curriculum documents (e.g. Australian Education Council, 1990; National Council of Teachers of Mathematics, 1989), school mathematics syllabuses (e.g. Board of Senior Secondary School Studies, 1992), and classroom support materials (e.g. Faragher et al., 1994; Sjöstrand, 1994). This stands in stark contrast to the degree to which META have been adopted and adapted into the instructional design of school mathematics programs where "it almost seems that computing technology is to be held at bay for as long as possible perhaps in the vain hope that it will go away" (Willis & Kissane, 1989, p. 57).

Significant changes in the affordability of these technologies, the development of new hypermedia technologies and the convergence and packaging of mathematically important technologies into "super-calculators" or hand-held "Personal Mathematical Assistances" (Kissane, 1995), as well the ever increasing dependence of society on technological infrastructure means that "to expect that schools and teachers can continue to exist apart from serious technological support is hopelessly myopic" (Kaput & Thompson, 1994, p. 682).

At the same time little research has been conducted into the influence of META on the learning and teaching of mathematics (Kaput & Thompson, 1994) although a number of authors have attempted to define the territory (e.g. Kaput, 1992; Taylor, 1981).

Taylor (1981) has suggested three ways in which computers are used in education:

- as a tutor—in which the computer environment is programmed in such as way as to provide instruction on some topic within a program of study. Examples of this category include Computer Assisted Instruction (CAI), Interactive Tuition and Electronic page turning.
- as a tutee—in which the learner takes an active part in the programming of the computer environment and learns something about specific non-computer oriented content domains as a consequence. For example, learning to program in LOGO demands students deal with problem situations which are mathematical by nature.
- as a tool—in which students make use of the capabilities of a computer to perform mathematical activities they would otherwise have conducted in some other way or to perform tasks that may have been beyond their capabilities without the assistance of computer technology.

Willis and Kissane (1989), building on Taylor's commentary, have also added the category of The Computer as a Catalyst. In this mode the computing environment is used as a means of provoking mathematical explorations and discussion or to invoke the use of problem solving skills.

The purpose of this study is to examine the effect of using META (specifically spreadsheets), as both a tool and a catalyst, on the collaborative learning practices already established within a particular classroom environment (see Goos & Geiger 1995). We begin by presenting a theoretical rationale for using computer technology to promote peer interaction and discussion.

Collaboration and Discussion in Computer Environments

In the past, educational theory has been largely concerned with individual modes of thinking, and educational practice has been based on the prevailing view that learning is a solitary enterprise. However, individualistic models of learning fail to take into account the social and communicative processes through which students are initiated into the intellectual life of their culture. Recently, interest in Vygotsky's (1978) ideas on the interactive nature of learning has led to the development of a new theoretical framework for understanding social aspects of cognition. Sociocultural theory shifts attention from individual to social modes of thinking, and emphasises the role of language in learning, both as a tool for thinking and as a medium for communication. A corresponding shift from an individualistic to a communicative approach to teaching and learning requires some consideration of classroom practices that encourage social interaction and sharing of knowledge. Collaborative learning arrangements favour such interaction by offering students opportunities to think together through the medium of language.

It has been claimed that collaborative peer interaction is facilitated by *computer environments*, since the computer provides a concrete and public focus for students' joint mathematical activity (Kennewell, 1994). Observational studies in the US and UK have confirmed that computer tasks provoke spontaneous interaction and discussion between students (Light, 1993), and have identified some patterns of interaction that promote different types of computer-based learning. For example, Hoyles and her co-workers (e.g. Hoyles, Sutherland & Healy, 1991) analysed the way in which pairs of secondary school students generated hypotheses about mathematical patterns while working in EXCEL and LOGO environments. Examination of the students' verbal exchanges and the video record of the computer screen revealed critical points in the discussion that provided the bridge between specific cases and formal generalisations.

Hoyles et al. (1991) also found that choice of software and choice of task were important decisions affecting students' style of interaction. The role of *software* is emphasised in the growing body of research concerned with the design of computer environments to support collaborative learning. Much of this work makes use of network communication tools or explicitly collaborative software to help students solve problems together in domains such as physics or engineering (e.g. Narayanan et al., 1995). However, collaboration may also be fostered by software not designed for that purpose, if the system possesses structural properties that encourage students to share information, ideas and program tools (e.g. diSessa, 1995).

In this paper we focus on both the social organisation of computer-based activity and the material basis for mediating collaboration. In contrast with the research mentioned above, the students who participated in our study worked collaboratively while using commercially available software whose structure neither explicitly nor implicitly encouraged collaboration. Our interest therefore centres on the second aspect of students' material environment—the nature of *tasks* that promote collaborative interaction at the computer.

Context

Participants in the study consisted of fifteen students, seven female and eight male, aged between 16 and 17 years of age, studying the second year (Year 12) of a two year senior secondary school course, their teacher (first author), and an observer (second author). The students were participating in the subject "Mathematics C" which formed part of their secondary accreditation and also contributed credit towards entrance into tertiary studies. The subject is considered an advanced course, being based on content areas applicable to students intending to pursue careers in engineering or the physical sciences. Students enrolled in Mathematics C are also required to study a parallel subject, Mathematics B, an introductory calculus and statistics course. Thus, 40% of their timetabled course time is devoted to the study of mathematics. Mathematics C students are typically able and highly motivated.

Instruction was based on written materials produced by the teacher. The materials defined the mathematical phenomena to be investigated over a series of lessons and offered guidance as to how to proceed during the course of the exploration. Students were strongly encouraged to interact with the teacher and each other when working through the materials and appeared to respond equally well to peer interaction as direct interaction with the teacher.

The topic being examined by students was part of an unit of work based on Chaos theory. This was a school option chosen and developed by the class teacher. The unit was chosen as a vehicle to:

- introduce students to an area of recently developed mathematics demonstrating the developmental nature of the discipline
- provide students with a learning experience based on the study of a topic within discrete mathematics, a branch of applicable mathematics of developing importance
- act as a vehicle for the natural use of computer technology as a means of exploring mathematical ideas.

Classes were conducted in the students' regular classroom. Fifteen Apple Powerbook 150 mobile computers were used, providing each student with access to the spreadsheet module of the software package Clarisworks. Students had gained familiarity with using these machines in junior secondary mathematics studies, Year 11 Mathematics C, and a number of topics in Year 11 Mathematics B. Thus they were relatively experienced users of spreadsheet software although it must be said they had had little experience in Year 12 in either subject.

This group of students had also participated in an earlier study during Year 11, when they worked on a similarly structured set of computer based activities on applications of Chaos theory (see Goos & Geiger, 1995). During the Year 11 observation period the students had displayed highly collaborative behaviour as they shared their ideas, sought help from their peers, and checked their understanding with each other. However, in the second series of lessons in Year 12 the students appeared to be working individually, with minimal interaction and discussion. The present study was conducted in order to investigate this difference in behaviour.

Data Analysis

Several lessons were observed during each unit of work on Chaos theory in Years 11 and 12. Here we present observations from one lesson in each year, to illustrate the differences in the students' interaction mentioned earlier. A small audio tape recorder was placed near one group of students during each lesson, and their talk was later transcribed. An observer also kept field notes that included a written record of the students' computer screens. While the discussion of results refers specifically to these target students, we consider that their behaviour was typical of that displayed by the whole class.

In our analysis of the verbal and observational data we examine three variables: the function of students' *talk*, the structure and focus of their *interaction*, and the type of *task* on which they worked. Student talk was classified according to the following functions (based on Kumpulainen, 1994):

Informative	Providing information
Organisational	Organising the task or the learning process, or controlling behaviour
Argumentational	Seeking and providing clarification, explanation and justification
Exploratory	Speculating, predicting, discovering, hypothesising
Metacognitive	Planning, monitoring progress, evaluating outcomes.

Patterns of student-student interaction were analysed with the aid of a two-dimensional framework developed by Granott (1993). The first dimension refers to relative expertise between students, while the second dimension represents the degree of collaboration. As the target students were of similar expertise, it is the second dimension that is of interest. Highly collaborative interactions are characterised by shared activity and continuous communication, while in less collaborative situations the activity is mostly independent, with occasional interaction to observe another participant or exchange information. Although both the relative expertise and collaboration dimensions are continuous, for our purpose it is helpful to label high and low collaborative interactions as *mutual collaboration* and *parallel activity* respectively.

Similarly, tasks may be conceptually defined at two levels according to the quality of interaction they elicit (Hertz-Lazarowitz, 1989). In *low-collaborative* tasks students interact about means (such as how to use materials), or about products (for example, the results of a calculation). In *high-collaborative* tasks, students interact about the process of doing the task, and their discussion focuses on planning or decision making. Task structure also influences the level of verbal elaboration in peer interaction, with high-collaborative tasks producing higher levels of elaboration (evaluation, application) than low-collaborative tasks (characterised by information exchange). These levels of elaboration can be matched with the functional categories of talk outlined earlier, so that high-level elaboration is heard in argumentational, exploratory and metacognitive talk, and low-level elaboration in informative and organisational talk. Table 1 shows the relationship between talk, interaction and task that guided our investigation of students' computer-based activity.

Table 1. Framework for Analysis of Talk, Interaction, and Task

Type of Task	**Focus of Interaction**	**Structure of Interaction**	**Type of Talk**
Low-collaborative	Products (e.g. results)	Parallel activity	Informative
	Means (e.g. materials)		Organisational
High-collaborative	Process (e.g. planning or decision making)	Mutual collaboration	Argumentational Exploratory Metacognitive

As well as observing peer interaction and discussion during lessons we also interviewed the class towards the end of Year 12, to seek their views on the benefits of collaboration. The next section presents our analysis of students' activity during the two sample lessons referred to earlier, together with the students' perceptions of the differences we observed.

Results

Mutual Collaboration on a High-Collaborative Task (Year 11)

In Year 11 the students were introduced to iterative processes as a means of examining exponential growth and decay, for example, in compound interest or animal populations. As explained earlier, a high degree of interaction and discussion was observed during this initial work on computer-based tasks involving chaos theory, even though the students had individual access to computers. In the lesson we have chosen to illustrate this type of activity the target students (given the pseudonyms Belinda, Rob and Louise) were working on the following task.

Nga is offered the following terms with two different financial institutions:
Institution 1 12.5% compounded annually
Institution 2 12% compounded monthly.
Nga knew that in the short term the conditions offered by Institution 1 were superior but suspected that in the long term Institution 2 might be the best proposition. For what period of time would she need to invest with Institution 2 before she realised a better return on her investment?

Students were expected to create two spreadsheets in order to compare returns from the institutions.

The following excerpt from the audio tape transcript shows how the task elicited the kind of continuous communication and sharing of ideas that characterises mutually collaborative interaction. Rob has incorrectly used Institution 2's annual rate of interest instead of converting to the monthly rate of 1%. Belinda points out the error and then explains her own spreadsheet strategy (which is also incorrect). The interaction focuses on the *process* of justification, and the talk has both *argumentational* and *metacognitive* functions as the students challenge and evaluate each others' reasoning.

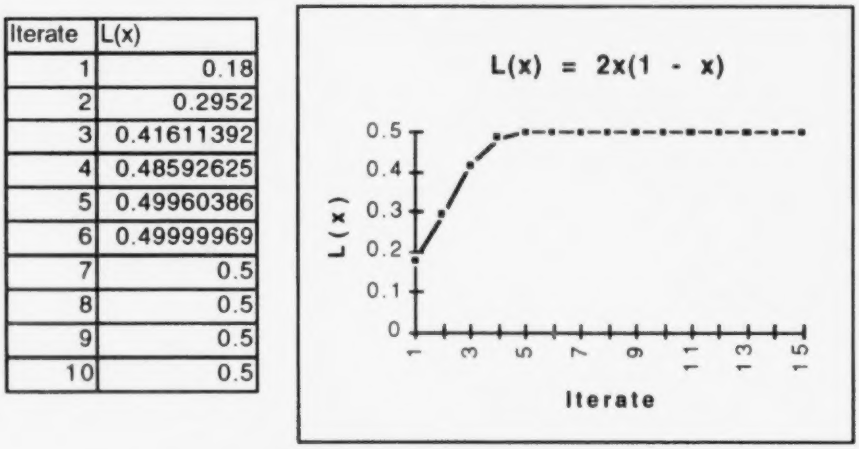
R: OK, how come if you're working the interest out, for three years compounding *annually*, for 12.5, you get that? (pointing to computer screen) That's my problem. And at two years, that's all you get? Or am I doing it wrong?

B: Because it's compounding interest. [inaudible] That's years and that's months. (confidently) *This* is compounding annually so it doesn't matter. *This* one is compounding monthly so you have to take into account the number of times you compound it in a year.
R: (doubtfully) Yeah ... (flash of understanding) Oh, this is interest rate per *year!*
B: (simultaneously) —*Year!* So you've got to work it out with your calculator then do the interest—
R: —So you divide 12 by 12—
B: No you've got to do point one two divided by 12 which is point zero one, and then that's—
L: It's not years, it's 12 months.
B: Twelve months?
L: If I do two years, that's 24 spaces (referring to rows in spreadsheet), and then—
B: I did 20 years and I had 240 spaces.
R: But why?
B: (to R) Why? Because the only reason you didn't have to do any calculations for this one is because [inaudible] when it's only calculated annually.
R: ... So then put the zero one—no problem. (after a pause) What interest rate did you use for the second one?
B: Point zero one. It's not actually point zero one, it's [inaudible].
R: How did you work that *out*?
B: See, compound interest, you've got to add one to it.
R: Is that it? That it, one? What for?

This style of interaction continued after Belinda recognised that the answer her spreadsheet produced for Institution 2 did not make sense. She clarified her thinking through discussion with others, rejected explanations she did not understand, and eventually succeeded in resolving the impasse caused by her flawed strategy.

Parallel Activity on a Low-Collaborative Task (Year 12)

The topic of growth and decay was extended in Year 12 to include mathematical models for dealing with fluctuating populations. The task was centered around an investigation of the behaviour of the logistic equation, $L(x) = bx(1 - x)$, when the parameter b is varied. Students were asked by examine how the equation behaved graphically when iterated (generating an orbit) for a range of values of b. This results in either stability after an initial increase (Figure 1) or decrease (Figure 2) in the values of L(x), periodic behaviour (Figure 3), or chaotic behaviour.



Iterate	L(x)
1	0.18
2	0.2952
3	0.41611392
4	0.48592625
5	0.49960386
6	0.49999969
7	0.5
8	0.5
9	0.5
10	0.5

Figure 1. Spreadsheet and Chart for b = 2

The materials guided students through a sequence of activities aimed at assisting them to discover a relationship(s) between the values of b and the iterative behaviour of L(x). By using a spreadsheet to calculate and chart successive iterates of L(x) from an initial value, it was anticipated that students would find different behaviours manifest themselves between specific ranges of values of b. It was also hoped that students would discover the phenomenon of period doubling, in which a two cycle orbit splits, or bifurcates, into a four cycle oscillation. This in turn can also bifurcate into an eight cycle orbit, then into a sixteen cycle, and so on. However, instead of following instructions to compare the resulting graphs and search for patterns, the students became so immersed in the process of generating the data that there was very little interaction and talk.

Iterate	L(x)
1	0.045
2	0.0214875
3	0.01051289
4	0.00520119
5	0.00258707
6	0.00129019
7	0.00064426
8	0.00032192
9	0.00016091
10	8.0442E-05
11	4.0218E-05
12	2.0108E-05
13	1.0054E-05
14	5.0269E-06

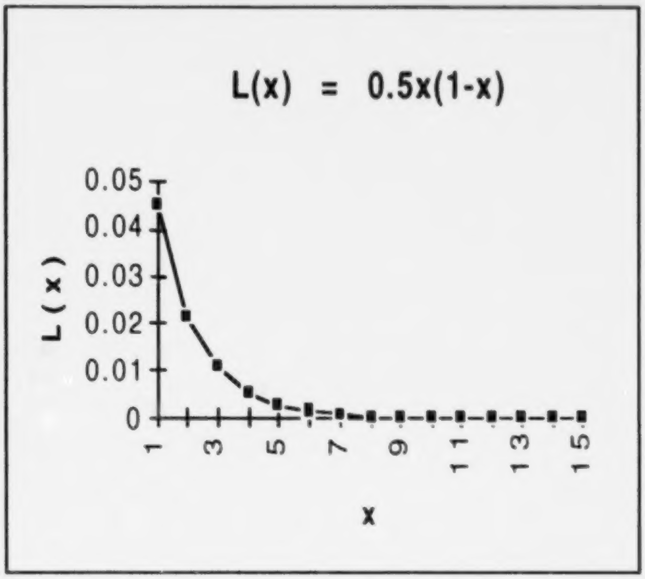


Figure 2. Spreadsheet and Chart for b = 0.5

The target students for the lesson discussed here, Rob, Ben and Duncan (pseudonyms), were chosen for observation because they usually worked collaboratively on classroom tasks. However, although there was a small amount of exploratory talk (e.g. speculating whether the graph would reach the same limit if a different initial value had been chosen), the students worked in parallel for most of the lesson and interacted to exchange information about *products*, such as spreadsheet cell values and the general shape of the graphs, and *means*, for example, procedures for managing the software. Thus the two main functions of their talk were *informative* and *organisational* respectively (see examples in Figure 4).

Examples of Informative Talk

1. R: Is that what you got?
2. B: Did you write anything on these graphs?
3. R: Did you get 0.32 for the initial value?
4. B: I get the same graph for 0.81 as for 0.8. But x_0=1.2 gives a different graph.

Examples of Organisational Talk

1. R tells B how to use keyboard shortcut -M to make a chart.
2. B asks how many rows are available in a spreadsheet.
3. Discussion of how to use the copy-paste-edit commands to make new charts.

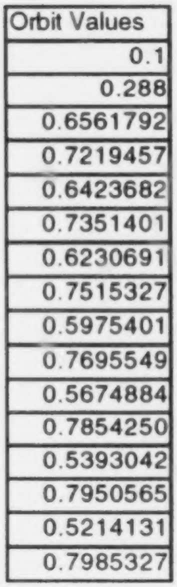
Figure 4. Examples of Informative and Organisational Talk

Teacher Follow Up—Changing the Task

After confirming with the observer that there appeared to be a change in the quality of student interaction, the teacher decided to change the task to one that had a clearly defined goal at the beginning of the activity, but was less directive; that is, the desired outcome was made public but not the means of achieving it. The chosen task (below) was a natural extension of the ideas on which students had already been working, and thus was accessible to all students, but was not accompanied by detailed directions on procedure, as was the case in earlier tasks.

In your investigation of periodic and chaotic points in the growth equation f(x)=bx(1-x), you examined transition values for period doubling between 1 cycle, 2 cycle, 4 cycle, and 8 cycle behaviour. Can you find the point at which the 3 cycle begins? Does this begin another series of period doubling behaviour?

Procedural uncertainty made the new task more challenging for the students, and their interaction and discussion once again took on a collaborative character.



Orbit Values
0.1
0.288
0.6561792
0.7219457
0.6423682
0.7351401
0.6230691
0.7515327
0.5975401
0.7695549
0.5674884
0.7854250
0.5393042
0.7950565
0.5214131
0.7985327

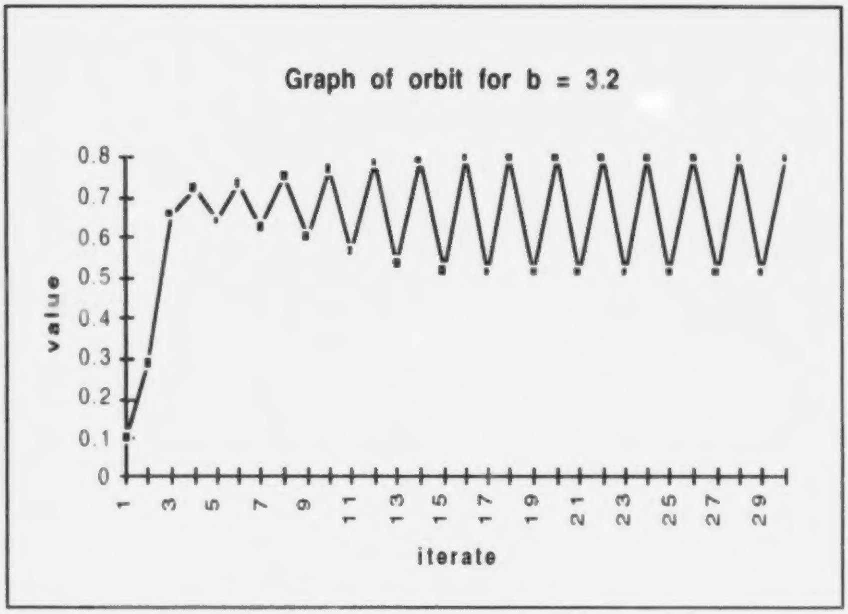


Figure 3. Spreadsheet and Chart for b = 3.2

Student Perceptions of the Tasks

Towards the end of Year 12, the students were interviewed on the general topic of how peer discussion "fits in with learning". When they were reminded about their lack of discussion during the second series of lessons on Chaos theory, they explained that the task was boring and repetitive because it had merely involved "number plugging":

D: We did talk, but it was about "what did you get?"
B: We only talked at the end when you said we hadn't been writing things about the graphs.
R: Yeah. And maybe we all knew what we were doing, so we didn't have to talk.

When the teacher explained how the task was then modified to make it more problematic, Rob responded:

R: Well, then we didn't know what to do! So we had to talk.

These comments seem to confirm that the task is a critical variable affecting peer interaction.

Discussion

This study has investigated the social and material mediation of computer-based learning. The computer was intended to act as both a tool, in enabling students to generate and manipulate data, and as a catalyst, in provoking exploration of the patterns that emerged from the data. However, the extent to which such exploration occurred depended on the type of task the students were given. Differences in the social organisation of students' work, identified in the function of their talk and the structure of their interaction, were associated with differences in task focus, with a focus on process, rather than products or means, producing collaborative discussion. Our results show that computer environments do not automatically facilitate peer interaction, and that tasks need to be carefully structured if they are to elicit high level verbal reasoning.

These findings have implications for teachers wishing to integrate META into school mathematics programs. Task design is clearly an issue of importance: the purpose of the computer task is for students to achieve new mathematical understanding, not simply procedural facility with the hardware or software. However, as students will need to generate some data in order to begin exploring ideas, there is a danger that the initial process of "number plugging" will conceal the problematic nature of the task. Hence, the teacher's intervention is needed to contextualise the task, focus students on the conceptual goal, and steer their discussion away from information exchange towards argumentation and hypothesising. Once students have completed the task, the teacher can stimulate further discussion and reflection on the solution process by conducting a whole class review in which students are encouraged to share their learning.

The role of the computer itself as a partner in the learning process should not be overlooked. When used as a tool or catalyst, computer technology has the potential to reorganise interactions between students and enrich their understanding of mathematics; but the computer does not replace the teacher. The extent to which computer environments allow students to articulate and reflect on their growing understanding depends on the teacher's expertise in designing appropriate tasks, and in orchestrating students' interaction with the computer and each other.

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When Does Student Talk Become Collaborative Mathematical Discussion?

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The potential role of student discussion in developing mathematical knowledge continues to interest researchers and teachers. This paper reports on a study that investigated patterns of student discussion and interaction in senior secondary school mathematics classrooms. Observations of one group of students are used to illustrate three factors that appear to influence the collaborative quality of mathematical discussion: students' orientation towards the task; their relative task-specific expertise; and the degree of challenge the task presents.

Introduction

Recent research in mathematics education has increasingly focussed on new ways of conceptualising mathematics teaching and learning. Informed by theories concerning the social context of learning and development, this research argues that learning mathematics involves participating in the activities of a community of practice, whose identity is forged by the adoption of the language conventions and ways of thinking valued by the wider community of mathematicians (Forman, in press; Lampert, 1990) .

In "community of practice" classrooms, students take on new roles that contrast sharply with those allowed within more traditional settings (Nickson, 1992). In traditional classrooms, mathematics is presented as abstract knowledge already discovered by experts, with the teacher as the sole source of knowledge and authority. Students are passive consumers of knowledge, their participation limited by the structure of the conventional recitation script. By comparison, in community of practice classrooms mathematical knowledge is constructed and tested by students as they work collaboratively under the guidance of the teacher. Student participation is therefore expanded to include discussion with peers in order to solve problems and assess their growing understanding.

This paper reports on the preliminary outcomes of a two year research project whose general aim is to investigate patterns of classroom social interactions that improve senior secondary school students' mathematical understanding, and facilitate a more accurate perception of the communal nature of mathematical knowledge. The specific purpose of the paper is to identify factors that influence the collaborative quality of student discussion.

Collaborative Mathematical Discussion

The role of peer discussion and collaboration in developing mathematical knowledge has been an issue of interest to researchers for some time. For example, previous studies have investigated the effectiveness, characteristics and occurrence of student-student talk (Hoyles, 1985; Pimm, 1987; Pirie, 1991; Pirie & Schwarzenberger, 1988) and the development of classroom norms for collaborative dialogue in small group interactions (Wood & Yackel, 1990). However, some care is needed in clarifying what is meant by *mathematical discussion*, and what makes the discussion *collaborative*. We find useful Pirie and Schwarzenberger's (1988) definition of mathematical discussion as purposeful talk on a mathematical subject, with genuine student contributions (whose input assists the talk or thinking to move forward) and interaction (indicating that ideas have been picked up by other participants). Further, the discussion is collaborative if the students explore each other's reasoning and viewpoints while working on a common activity, so that shared understanding evolves simultaneously for all participants (Granott, 1993).

The question then arises as to how mathematics teachers might create conditions that encourage students to engage in this kind of collaborative dialogue. One factor that appears to influence the quality of students' interaction and talk concerns the structure of the task on which they are working. Tasks that require students to interact about processes, such as planning and decision making, elicit more elaborated reasoning than tasks that only require students to interact about products or means, for example, answers, materials, learned procedures and algorithms (Hertz-Lazarowitz, 1989).

However, our study has identified a set of interacting factors, related to both task and student characteristics, that influence the quality of student discussion. These factors will be illustrated by reference to three lessons involving the same group of students.

The Study

Four mathematics classes (three Year 11 and one Year 12) in four secondary schools have participated in the first year of the study. All schools are co-educational; two are government high schools and two are independent schools. At the beginning of the year questionnaires and other written tasks were administered to obtain information on students' beliefs about mathematics, perceptions of classroom practices, and metacognitive knowledge (see Goos, 1995, for details). From March until September, one mathematics lesson per week was observed in each classroom in order to gather data on patterns of teacher-student and student-student interactions. At least ten lessons were videotaped for each class.

The initial purpose of our observations was to select in each classroom a group of students who were in the habit of working together and discussing their ideas in mathematics lessons. These groups became the focus of our observation and videotaping in subsequent classroom visits. We also used video-stimulated recall techniques (e.g. Frid & Malone, 1995) to interview each group of students on at least one occasion, to seek their interpretations of their own interactions with each other.

Although we observed and recorded many student discussions in the four classrooms, in this paper we have chosen to illustrate our findings by focussing on one group of students and their social interactions during three separate lessons.

The Students

Lincoln, Gary, Paul and Christopher (pseudonyms) were students in a Year 11 Mathematics B class in one of the participating schools. (Mathematics B is a prerequisite subject for students seeking entry to science based tertiary courses.) The boys were good friends, and their interaction and helping behaviour were similar for most of the lessons observed.

Lincoln was a high achieving student who was confident in the knowledge of his own ability, but anxious that his performance should match his high expectations for himself. When working on mathematics problems with his friends he tried to maintain organisational control by eliciting their input and assigning them different subtasks. (However, it was our observation that the other students usually ignored him and continued to contribute in any way they chose.) Gary was also a capable student, with a more relaxed approach to his work. Both Lincoln and Gary also took Mathematics C, a more specialised subject that prepares students for further study of mathematics at tertiary level. This subject combination conferred a privileged status that was not shared by Paul and Christopher, whose additional enrolment in Mathematics A (which concentrates on mathematical applications for daily living) further marked them as lower status members of the group. Consequently, Paul's useful ideas and suggestions were often ignored by Lincoln and Gary. Christopher was the least able student in the group, and he rarely contributed to the discussion. Yet he appeared to follow his friends' reasoning as they talked, and kept track of progress with his calculator. When interviewed, he explained the benefit of his participation in the group as being able to learn from the others.

These students was chosen as a target group for a number of reasons. First, questionnaire data showed that the students themselves reported using discussion as a means of learning mathematics. In one questionnaire students were asked to indicate frequency of participation (always, often, seldom, never) in seven different learning activities. The four target students claimed that they were *always* or *often* likely to be *talking about maths to other students* or *listening to other students*, with none of them expressing a greater preference for any other learning activity (e.g. *listening to the teacher, working alone, copying notes from the blackboard*). Written responses were also sought to the open-ended question *How do you know when you understand something in maths?* Unlike the majority of students who measured their understanding in terms of their ability to obtain the correct answer to a problem, Lincoln and Gary reported that they knew they understood a mathematical idea when they could ***explain it to other people***. The students' self-reports were confirmed by our observation that they worked and talked together during mathematics lessons, more so than other students in the class. Taken together, this evidence seemed to suggest a commitment to learning through peer discussion.

The Lessons

Describing Interaction and Discussion

The collaborative quality of the students' mathematical discussion is described in terms of two indices: the structure of their social interactions, and their use of explanations. Social interactions are structured according to the extent to which the activity is shared, and the degree to which students take direction from each other (Damon & Phelps, 1989; Granott, 1993). Thus we may distinguish between the following interaction structures:

Parallel activity	Students work separately on the same task, with little or no sharing of materials or ideas.
Peer tutoring	Students work on a common activity, but one student with superior competence controls the instructional agenda.
Collaboration	Students work jointly on a problem that none could solve alone.

Differences in the structure of interaction are associated with corresponding differences in the quality of the students' discussion, especially the kinds of explanations they use. During parallel activity students interact only intermittently, perhaps to check results with each other, and elaborated explanations are not heard. In peer tutoring, the more knowledgeable student teaches a partner how to carry out a task, and the student-tutor explains in order to impart knowledge that the partner does not yet possess. However, when students work collaboratively they share their ideas with each other, and it is through a process of mutual explanation that a common understanding is reached.

The differences between no-explaining (parallel activity), explaining-to-teach (peer tutoring) and explaining-to-understand (collaboration) are illustrated in Lessons #1 and #3. A further contrast is provided by Lesson #2, in which the students avoided interaction altogether.

Lesson #1. The Observatory Problem

> An observatory is in the shape of a vertical circular cylinder, of diameter 30 metres, surmounted by a hemispherical dome of equal diameter. It stands on horizontal ground. From a point on the ground 50 metres from the wall of the observatory, the angle of elevation of the apparent top of the building is 30°. Find how high the building is centrally.

The Observatory Problem was presented as a practice exercise to prepare students for an assessment task to be given the following week. Students were allowed to work together in class to clarify the task and discuss strategies, as they would for the assessment task, but were required to produce individual written solutions. However, in this lesson the four target students worked through the entire problem and solved it as a group.

After clarifying the aim of the problem the students began working with an incorrect diagram, in which the tangent to the dome extended only to the point of contact. (The correct diagram is shown in Figure 1.) Assuming that the tangent nevertheless represented the hypotenuse of a right angled triangle with sides of 65 and $(x + 15)$ metres, the boys "solved" the problem by calculating the height of the building as 65 tan 30°, or 37.53 metres. Although they were suspicious of the apparent ease with which they had reached a solution, the boys called on the teacher to verify their answer. His suggestion that they should extend the tangent alerted the students to their error, and they quickly constructed the correct diagram.

Realising that the height of the building was $(37.53 - y)$, the students then explored various ideas for finding the unknown distance y (see Figure 1). Their discussion centred on ways of modifying the diagram in response to Paul's suggestion that "there's a triangle here, in here somewhere". The following excerpt from the videotape transcript illustrates the collaborative quality of their interaction as they shared and tested ideas.

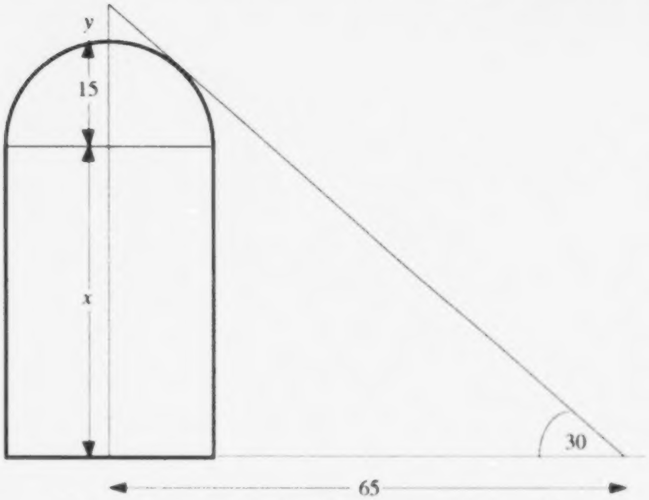

Figure 1. Diagram for the Observatory Problem

L: So where are you saying we draw the triangle, Paul?
P: Here could be good.
L: But how does that help us?
P: Oh no ...... draw it in there, I guess.
G: If you calculate x (the height of the cylinder) it'll help you.
L: X?
G: Yeah.
L: How?
G: If you calculate x you should be able to —
P: But we can't calculate x if we don't know the—
L: You can't draw a triangle as a tangent down there, can you? And if that is a right angle—
P: How will that help us? We don't want to know that.

Although progress was slow at first, a method for constructing similar triangles was eventually recognised in the following exchanges, and a solution was found soon afterwards:

L: OK well lets have a look. What's that there? (i.e. the distance marked as $x + 15 + y$ in Figure 1)
P: That's ... 37.5.
L: So if we know that up to there is 37.5 ... Does that help us?
P: And we've got this triangle here (constructs a radius perpendicular to the tangent) and we know that's 30 degrees (angle subtended by the radii).
L: (looking at diagram) How?
P: Because it's ... that's 90 degrees (i.e. radius is perpendicular to tangent) , and—
G: Similar triangles.
L: And that's 60 degrees because that there's 90, OK?
P: (pushing L's hand out of the way) That's 30, so that there has to be 30.
L: Why?
G: Because that there is a similar triangle.
P: Because isn't that 60—
L: (responding to G) Is it?
G: And that's 60. (P and G both explaining at once.) See that whole angle there's 60—
P: And that's 30, and that's 30 60.
L: OK. But how does that help us?
G: Well we can work that out with similar triangles ...

The collaborative interaction between Lincoln, Gary, Paul and Christopher was similar to that observed in previous lessons when group tasks were given. They tackled the task, and the obstacles it presented, by generating ideas that were tested, challenged and justified in discussion with each other. In doing so, they explained their way towards understanding and solving the problem.

Lesson #2. The Theatre Design Problem

One week later the students were given the assessment task for which their group work on the Observatory Problem had prepared them. They were provided with a diagram representing a small picture theatre in cross section, together with information on the height of the screen, its distance from the floor, and the height of the audience's eye level. The students were to determine the horizontal distance x from the screen that would give the viewer the maximum viewing angle θ. As the class had not yet studied calculus, they were given a table of x values for which to calculate corresponding values of θ. The second part of the task asked them to investigate the effect of a sloping floor on the viewing angle.

The teacher again explained the procedure, which was the same as that for the earlier lesson: sharing of ideas about possible strategies, followed by production of individual written solutions. However, with assessment now at stake, the target students were unwilling to discuss the problem in the way observed in previous lessons. Lincoln, in particular, was reluctant to share his thoughts, and busied himself with his own calculations until he had completed the first part of the problem. Towards the end of the lesson Paul attempted to initiate some interaction:

P: So what'd you get for [x =] four?
L: (Covers his working.) Noo!
P: Oh you can show me! ...
L: You've got to write it up yourself.
P: Was that right, what I had?
L: I don't know.
P: (Smiling) Show me! (L grins and places his fist firmly on his closed book. P grabs book from him and inspects L's working. L doesn't protest. P looks disappointed when he sees L's results.) So it goes ... (L retrieves his book) So three metres is the best, you reckon? (L puts his hand over his results; P tries to lift it.)
L: Then you do it each ten centimetres. (referring to values of x to use in the table)
P: Do you? Do you have to do that all the time? (L nods.) Geez! Are you going to do it like that?
L: I've done it.
P: (Impressed) Have you worked it out?
L: Yeah.
P: You're joking!
L: No. Now I've got to do the second half when I get home.

Despite the similar task demands and teacher instructions, collaborative interaction was not observed during this lesson.

Lesson #3. Binomial Expansions

In this lesson, which occurred about one month after Lesson #2, students were instructed to work in groups to complete a worksheet on Binomial Expansions. They were to explain the number patterns in Pascal's Triangle, and compare them with the patterns of coefficients and powers obtained by using the distributive law to carry out expansions from $(x+y)^0$ to $(x+y)^4$. The students were then asked to use the patterns they had discovered to write instructions for expanding an expression in the general form $(x+y)^n$. The teacher emphasised that the most important part of the task was for the students to be able to explain and communicate their ideas to each other.

Although this task appeared to present an ideal opportunity for collaborative discussion and explanation, the teacher's intentions were foiled by Lincoln and Gary, who had already covered this work in Mathematics C. Ignoring Paul and Christopher, they worked in parallel through the first part of the task, and used Pascal's Triangle instead of the distributive law to quickly expand the given expressions. Because explanations were unnecessary, their talk focussed on checking their results with each other:

L: What's x to the third?
G: X cubed—
L: Yeah.

G: —plus three ... stop, stop. (reaches for eraser)
L: Plus what?
G: (erasing) Um ...
L: Three x squared y ...
G: Yup, and then—
L: Plus three, y squared x?
G: (writing) Plus three x y squared. (to L) In the first one you have x squared, so in the second one you have y squared—
L: Yeah yeah yeah, plus y cubed.
G: Yep.
L: Too easy.

The structure of the interaction changed when Lincoln and Gary reached the part of the worksheet that required written explanations of the patterns of coefficients and powers. Because they already understood the connection with Pascal's Triangle, Lincoln and Gary formed peer tutoring pairs with Paul and Christopher respectively, in order to explain to the less knowledgeable students how the patterns worked. However, the tutors were more interested in demonstrating their expertise with the number patterns, than in helping the other two boys work through the preliminary algebraic expansions in order to discover the pattern for themselves. The tutors' explanations were purely procedural in that they emphasised what to do, rather than why.

The peer tutoring interaction precipitated by the Binomial Expansions worksheet did not provide opportunities for the students to exchange ideas, or to use mutual explanations to generate an understanding of the mathematical ideas around which the task was designed. The inadequacy of Lincoln's "expert" explanation is shown in the following exchange:

L: You see, the main numbers are just those down here, 1 3 3 1 (pointing to Pascal's Triangle), 1 3 3 1 (pointing to coefficients). And then the powers have to add up to, what the term is. So you've got 3, then you've got 2 plus 1, then 1 plus 2, and then there's 3. Do you get it?
P: No. I don't understand.

In the three lessons described above we have shown how differences in the structure of students' social interactions gave rise to differences in the quality of their discussion (Table 1). We now consider some factors that may be implicated in these differences.

Table 1. Summary of Interaction Structures and Discussion

Lesson	Interaction Structure	Explanations
#1 Observatory	Collaboration	Explain-to-understand
#2 Theatre Design	Interaction avoided	No discussion
#3 Binomial Expansions	Parallel activity	No explanations
	Peer tutoring	Explain-to-teach

Factors Promoting Mathematical Discussion

From our observations of students' talk and social interactions, we have identified three related factors that influence the occurrence of collaborative mathematical discussion: students' perception of the purpose of the task, the relative expertise of the students, and the degree of challenge the task presents (Figure 2).

Students' Orientation towards the Task: Learning vs Performance

We observed the operation of collaborative social norms within the target group of students when the task on which they worked had a learning purpose (e.g. Observatory Problem). However, these norms were suppressed when they were given a task that was to be individually assessed (Theatre Design Problem). The students' orientation towards the task—learning versus performance—appears to influence how they respond to it and to each other.

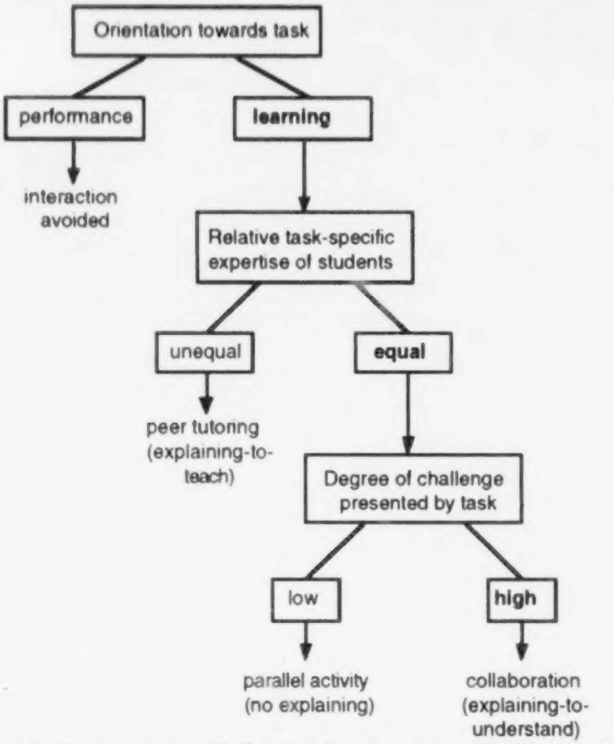
Orientation towards task
performance
learning
interaction avoided
Relative task-specific expertise of students
unequal
equal
peer tutoring (explaining-to-teach)
Degree of challenge presented by task
low
high
parallel activity (no explaining)
collaboration (explaining-to-understand)

Figure 2. Factors Influencing the Collaborative Quality of Mathematical Discussion

Relative Expertise of the Students: Equal vs Unequal

Given that students approach the task with the purpose of learning, the next factor that influences the quality of their interaction is their relative expertise with respect to the task itself. Here, expertise depends not on previous mathematics achievement, but on relevant task-specific prior knowledge. If some students know more than others, then the interaction becomes one of peer tutoring rather than collaboration. This kind of interaction was observed during the Binomial Expansion activity, where Lincoln and Gary's previous experience with the topic gave them an advantage over the other members of the group. The purpose of the more expert students' explanations was to demonstrate procedures, rather than to share their understanding.

Degree of Challenge: High vs Low

If students come to a task with similar background knowledge, then the degree of challenge the task presents will also have an effect on their interaction. Collaborative interaction occurred when no one in the group knew how to attack a task, or when an obstacle prevented progress (e.g. Observatory Problem). Here, explaining was a mutual process arising from the need to clarify and justify ideas to the satisfaction of one's peers.

If the task was less challenging, the structure of the interaction changed to parallel activity, as observed during the Binomial Expansion lesson. Because the first part of the task proved to be a routine exercise for Lincoln and Gary, there was no need for them to test their understanding by explaining their thinking to each other. They merely exchanged information on results.

Conclusion

In this paper we have presented a sample of our extensive observations of senior secondary mathematics classrooms, made over the course of one school year. As a result of analysing patterns of student interaction and talk, we have identified three conditions that seem to favour collaborative mathematical discussion:

1. students have a *learning orientation* towards the task;
2. students have roughly *equal task-specific expertise*;

3. the task presents a *challenge*.

These findings are tentative and subject to confirmation through observation of additional classrooms in the second year of the study. Nevertheless, they have implications for mathematics teachers who wish to encourage collaboration and discussion in their classrooms.

Previous research has found that task design is an important variable in peer learning, but our study suggests that it is the interaction between student and task that influences the character of their talk. A task that is challenging for one student may be less so for another, so that the question of who works on which task with whom has a bearing on the form of social interaction, and the kind of talk, that will be observed. In addition, students whose task goals are related to performance, rather than learning, may be unwilling to work in collaboration with their peers. For some students, learning goals are further compromised if collaborative work on a task is to be individually assessed. Careful thought is needed in designing assessment programs that encourage students to tackle tasks with their classmates, yet allow teachers to measure the understanding that individuals gain as a result of collaborative effort. The problem of satisfying the individualised assessment requirements of senior secondary schooling while developing a collaborative classroom ethos is one which deserves further attention if the "community of practice" ideal is to be realised.

Acknowledgments

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Investigating the Teaching and Learning of Negative Number Concepts and Operations

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A report of progress in a Doctor of Education research project supervised by Professor Kaye Stacey

The paper provides an overview of some of the outcomes of an investigation comparing the teaching of negative number at junior secondary level using tiles, as discrete integer entities, with other teaching approaches more commonly used. The experimental approach seems to have facilitated better performance for average ability level students. For more able mathematics students the topic does not appear to be difficult and such students, in both experimental and control groups, indicated good levels of general topic mastery.

Introduction and background

Much has been written and critical comment made concerning suitable and effective negative number teaching strategies (eg. Fary (1980), Kuchemann (1981), Freudenthal (1983), Streefland (1993), White (1994), Lytle (1994), Gates (1995)). It continues to be one of the ongoing sagas in mathematics education. However improved learning and understanding for many students is still not being achieved. Recently, at the beginning of a workshop on the teaching of negative number, not one of the twelve fourth year trainee mathematics teachers present could prove or give an explanation for either '0 - -6 = 6' or '-2 x -3 = 6'. They appeared to merely know or accept them to be true. None could recall how (or if) such facts had been taught or justified. A large proportion of students emerge from secondary school with a seriously flawed and incomplete understanding of the real number system. Any area or application of mathematics requiring the use of negative numbers and related concepts is likely to produce difficulties. Ongoing development in mathematics and mathematics related fields is therefore adversely affected.

Two years ago at MERGA in Lismore a paper was presented describing the, then beginning, project to study the teaching and learning of negative number concepts and operations (Hayes, 1994). Also outlined were the experimental teaching methods to be investigated. This paper will provide details of what has occurred since and offer some preliminary findings. In particular it discusses some of the outcomes of school-based teaching and testing activities that took place in 1995 and early 1996. The data analysis process has not yet been completed. It is hoped that the project will be completed by the end of this year.

Experimental Details

In the period mid-1994 to term one 1996, the study has been conducted in three secondary schools involving students in years seven, eight and nine, comparing the effectiveness of the annihilation/creation method of teaching negative number (see Freudenthal, 1983) as the initial strategy, with other commonly used strategies. The contention is that the method provides a superior introduction and operational model for the teaching of integer operations and will result in improved long term learning and understanding, particularly for students who have difficulty in mathematics. The experimental teaching groups used reversible two centimetre square tiles labelled [+1], [-1] and [0] in conjunction with especially prepared self-paced student workbooks covering the four basic integer operations. It was intended that the topic would fit into the normal time devoted to it in the syllabus (about three weeks) at either late year seven or early year eight depending on usual school policy. An aim of using the tiles in conjunction with the workbooks was for the students to discover and formulate for themselves the integer operation rules. Teaching methods in comparison groups were those traditionally used by the teachers of the control classes involved in the study. In general number-line interpretations and applications tended to feature strongly as the preferred initial teaching strategy used. Consistency patterns were commonly used to justify (establish) the multiplication rules. The major difference in strategy between the experimental and control groups was that the experimental groups started with the tiles. By the end of the topic the experimental group students had also used the number line in the context of ordering and 2D point plotting. Across the three schools there were four experimental groups and four control/comparison groups involving a total of 150 students and seven teachers. One of the teachers took both an experimental group and a control group.

Participants

Three schools were involved. School A provided two year 8 classes one of which was the experimental group and the other the control group. Both were taught by the same teacher. The experimental group was allowed to work in self-paced mode, using the materials provided (tiles and workbooks) throughout the three weeks devoted to the topic early in term one. Students were allowed to take the workbooks away for the purpose of homework. The control group was taught by the usual class-lesson methods favoured and described by the teacher as a multi-embodiment approach. The number-line was used as the introductory strategy for teaching the processes of integer addition and subtraction. The textbook used in the control group classes was *Mathematics Today Year 8*.

School B provided two year 8 classes and the teaching also took place early in term one. In this case different teachers taught the experimental and control groups. The experimental group worked with the tiles and workbooks provided in self-paced fashion. Workbooks were collected for correction following each lesson and not taken away by the students for homework In this case the class teacher appeared to provide the students with more detailed individual written feedback comments and corrections than those given by the teacher in School A above. The control group teacher appeared to generally follow the topic treatment adopted by the class textbook used (*Mathematics Today, Year 8*). In the latter half of 1995 one of the teachers involved (who was also the mathematics coordinator) was transferred to another school. This caused some problems and organisational disruptions which affected access to the control group for the purpose of retention testing later in the year.

School C initially provided three year 7 classes. The teaching took place in term three, 1995. Initially there were two experimental groups and one control group. Each class was taken by a different teacher. In this school the experimental group teachers monitored student progress very closely and tended to keep the classes more together (ie. a minimum pace was set) but students were also allowed to take their workbooks home and work ahead. Class reviewing and discussion of the content material was most apparent. One of the experimental group teachers regularly used a process of page by page class corrections for some sets of workbook examples. (eg. "David, please read out your answers for the exercises on page 10.") The control group teacher used the number line approach and textbook exercises from Lynch et al. *Maths 8*, supplemented by additional prepared review worksheets. Retention-testing was done in February, 1996. Because classes had been regrouped for the transition to year 8 it was decided to minimize class disruption by retention-testing all year 8 students. This included additional students (an accelerated group) to those who had previously been involved in the study in 1995 - effectively providing an extra control (comparison) group.

Testing

Students were pre-tested (usually at the start of the first lesson in the topic), post-tested (usually the next maths period following the topic completion) and then retention-tested (four to six months later). In 1995 the pre-test covered basic positive number facts in the form of addition and multiplication scramble tables and simple positive and negative number operation examples (eg. $3 + {}^{-}3$, $5 + {}^{-}2$, ${}^{-}3 + {}^{-}2$, ${}^{-}4 \times {}^{-}2$ etc.). The scramble tables were intended as indicators of readiness to begin the topic. (None of the students were found to be grossly unready - all had at least reasonable knowledge of basic simple number facts.) The integer operation examples were intended to provide an indication of naturally acquired knowledge and feeling for the topic. At that stage it was expected that few, if any, of the students would have experienced formal teaching in the topic. However most students, at this level, seem to be aware of at least one example of practical usage (ie. temperature scale). The pre-test showed that most students seemed to know that $3 + {}^{-}3 = 0$ and ${}^{-}3 + {}^{-}2 = {}^{-}5$. Many students performed quite well on the simpler examples included (eg. $7 + {}^{-}4$) but most did not know (or correctly guess) that ${}^{-}4 \times {}^{-}2 = 8$ (the common response was ${}^{-}8$). Five students (one in each of Schools A and C initial groups) achieved full marks (14/14) on the integer operations part of the pre-test.

Both the post-test and the retention-test included one common large question (Q1.) containing 30 items covering the four processes, using small integers and intended to test knowledge of and competence in basic operations. The numerals in the items were changed between the tests. The post-test also covered some additional skills and integer applications traditional at this level (eg. ordering, temperature scale, above and below sea-level questions). Apart from Q1. the content of the post-test was modified for School C. The intention was to make comparisons within schools rather than across schools at the post-test stage. However the retention-test used was common to all 1995 participants. In the retention-test Q2. contained 20 items testing knowledge of the operation rules. Using large numbers the task involved deciding whether the answer in each case would be positive (P), negative (N) or zero (Z) for each of the binary operation examples. The 12 items in Q3. involved substituting small integers into simple algebraic expressions and then evaluating each.

Early in 1996 a selection of 30 students of varying ability, involved in the study in 1995, were interviewed and asked to think aloud and provide explanations and reasons whilst they responded to a selection of items similar to those contained in questions 1 and 2 of the retention-test. One of the intentions was to unravel reasons for wrong answers. Working and voices, but not faces, were video-recorded. (At the time of writing this paper the data (video tapes and response sheets) from the interviews were still being analysed.)

Results and discussion

To facilitate data analysis and comparisons for parallel groups in each school the responses of all students to all items on the tests were entered into EXCEL spreadsheets. (Tables 2, 3 and 4 which are discussed later show sample extracts.) Due to the diverse conditions existing between the three schools involved (eg. nature of student intake, staffing stability, policy) only a minor attempt is being made to directly compare student performances between schools. For the purpose of this study each school has been regarded as somewhat of a separate case study and major comparisons made within schools rather than across schools. In fact, as this study proceeds, it is becoming quite evident that the reactions of the individual students appear to provide the most interesting features and indicators of teaching strategy effectiveness. Detailed analysis of workbooks is also still in progress.

Apart from the School C accelerated group, who emerged, as mentioned above, at the retention-testing stage, each of the schools regarded the respective parallel groups participating in the study as containing generally similar ability ranges. It would not have been possible to randomly allocate students to groups without causing major disruptions in any of the schools.

Table 1. provides mean scores and standard deviations for the testing program for the 1995 teaching groups. (Ttests have been used to provide possible indicators of within school group differences.)

Table 1. Test results for students in the 1995 teaching groups

TEST RESULTS								
School	**Group**	**Pre-test**	**Post-test**		**Retention-test**			
		Basic op.	Basic op.	Ov%score	Basic op.	Rules	Substitns.	Ov%score
		/14	/30		/30	/20	/12	
A	Exptl.	9.3(m)	22.0	73.8	21.8	16.1	6.7	72.0
Year 8	N=22	3.0(sd)	5.6	16.0	6.5	3.0	3.7	19.7
Term 1	Control.	6.9	18.4	60.7	17.2	13.2	5.5	57.8
	N=22	4.4	8.2	23.2	7.2	3.3	3.3	20.1
		p<.05	ns	p<.05	p<.05	p<.05	ns	p<.05
B	Exptl.	5.1	18.5	62.5	23.3	13.4	6.7	70.1
Year 8	N=17	3.7	7.7	21.1	7.4	7.9	4.9	20.1
Term 1	Control.	8.5	18.9	60.4				
	N=19	3.5	7.7	27.1				
		p<.05	ns	ns				
C	Exptl.1	8.4	25.6	74.2	26.8	18.0	10.1	88.4
Year 7	N=17	2.1	4.3	17.3	3.3	2.4	1.4	11.4
Term 3	Exptl.2	8.7	26.0	75.6	26.4	17.4	10.0	86.4
	N=18	2.6	4.4	19.0	5.8	3.8	2.0	17.9
	Contrl.1	6.4	24.6	65.7	25.9	17.2	8.6	83.5
	N=13	2.7	5.5	21.0	4.5	2.5	3.0	14.3
	Contrl.2				26.9	17.8	9.4	87.2
	N=22				5.4	2.9	2.4	16.7
		p<.05	ns	p<.05	ns	ns	ns	ns

School A (Both classes taught by the same teacher): Prior to commencement of teaching the topic, pre-test scores, (mean 9.3) obtained on the 14 basic integer items by the experimental group, appeared to indicate a reasonably good intuitive idea for the correct answers. This was significantly higher (Ttest, p<.05) than the control group mean (6.9). However whilst all members of the experimental group attempted (perhaps sometimes guessed) these items four of the control group did not attempt any and two others attempted only a few. (Four zero and two other very low scores considerably reduced the class mean.) One such student included a note; "I haven't done these yet and I don't know them." Another provided the comment, "I don't know what the dashes in front of the numbers mean." The experimental group performed significantly better on the post and retention-tests. Overall post-test mean scores were 73.8% and 60.7% respectively (p<.05) and the overall retention-test scores 72.0% and 57.8% (p<.05). Before the teaching commenced the teacher, whilst deciding which would be the experimental and control group, considered the classes to be of about the same ability levels and, following teaching, predicted that the control group would have performed better on the post-test than the experimental group when asked, "Which group do you think did better?". The teacher was surprised by the difference in measured performance levels between the two groups on the post-test.

School B (Classes taught by different teachers): In this case the control group (mean 8.5) scored better than the experimental group (mean 5.1) ($p<.05$) on the pre-test integer items. However the gap had closed by the post-test and the overall mean score (62.5%) by the experimental group was slightly higher than that of the control group (60.4%) (ns). Unfortunately the retention-test was not administered to the control group by the school. (It is hoped that both groups can be re-tested simultaneously again soon.)

School C (All classes taught by different teachers): On the pre-test the experimental groups performed better than the initial control group. The overall post-test mean scores of the experimental groups (74.2% and 75.6%) were significantly better than those achieved by the initial control group (65.7%). However mean scores on the 30 basic operations items were similar: 25.6, 26.0 and 24.6 respectively. On the retention-test the initial control group scored slightly lower (ns) than the other three groups in each of the aspects of the test however the overall results indicate a high level of general mastery in the topic for each of the classes involved. The two experimental groups performed as well

Table 2. Item analysis retention-test spreadsheet extract (School A experimental group)

Q1. .	q	r	s	t	u	v	w	x
Content .	-27÷[ ]=-9	2 x (7-8)	-3 x (2--4)	-15 ÷(1-4)	0 - -2	-4 -[ ]= -8	-3 x 6	-4 x -2
Corr.Resp	3	-2	-18	5	2	4	-18	8
Annie	1	2	-1	3	-2	*	18	1
Bert	1	-1	6	1	-2	1	1	1
Claude	1	1	6	1	1	12	1	1
Dot	1	2	*	*	-2	-4	-9	-8
Eddie	1	1	1	1	0	-12	1	1
Fay	1	2	-6	-5	-2	-4	1	-8
Gladys	1	1	18	1	-2	1	18	1
Harry	1	2	18	-5	-2	-4	18	1
Irma	1	1	18	1	-2	1	1	1
Jay	-3	1	18	1	1	1	18	1
Kiran	1	1	1	1	1	1	1	1
Les	1	2	1	1	1	1	1	1
Mavis	1	2	5	-5	-2	-4	1	1
Norm	x	2	1	1	1	x	1	1
Ossie	1	1	1	1	1	1	1	1
Pete	1	1	+6	-5	-2	1	1	1
Quaid	1	1	1	1	1	1	1	1
Rex	1	1	1	1	1	1	1	1
Sam	1	1	18	-5	1	1	1	1
Thelma	1	1	18	-5	1	1	1	1
Una	1	1	1	1	1	1	1	1
Viv	1	1	1	1	1	1	1	1
#correct	20	14	9	14	12	14	17	19
#wrong	2	8	13	8	10	8	5	3
%correct	90.9	63.6	40.9	63.6	54.5	63.6	77.3	86.4
%wrong	9.1	36.4	59.1	36.4	45.5	36.4	22.7	13.6

as the accelerated group who had been selected to work above their level in mathematics and other subjects. In mathematics it appears that they covered the same syllabus as the other control class but at a faster pace and therefore covered integers earlier in the year. (They worked from Lynch et al. *Maths 8* during most of Year 7.)

Analysis of error patterns

Interesting data has emerged with regard to error patterns among students who made mistakes. In addition to experimental and control group comparisons the spreadsheets provide useful instruments for diagnostic purposes. It is suggested that such information could facilitate the selection of teaching strategies for overcoming and avoiding misunderstanding. Table 2 and Table 3 show selected and edited extracts from the School A experimental group and control group retention-test spreadsheets respectively. The tables show eight out of the overall total of 62 items contained in the test. The '1' on the right hand side of cells indicate that the students gave correct item responses. The numbers shown on the left hand side of cells indicate students' incorrect item responses. Asterisks (*) indicate non-attempts. Several non-attempts are apparent in the control group whilst none appear for the selected items for the experimental group. At the foot of the spreadsheets the item statistics for the class are obtained. The spreadsheets thus provide a rapid method of systematically analysing the class results. A major useful

Table3. Item analysis retention-test spreadsheet extract (School A control group)

Q1.	q	r	s	t	u	v	w	x
Content	-27÷[ ]=-9	2 x (7-8)	-3 x (2--4)	-15 ÷(1-4)	0 - -2	-4 -[ ]= -8	-3 x 6	-4 x -2
Corr.Resp	3	-2	-18	5	2	4	-18	8
Alf	-3	*	*	*	*	*	*	*
Bess	1	1	-6	1	-2	*	*	*
Connie	1	2	1	1	1	1	1	1
Dave	1	1	-12	15	1	1	1	-8
Edna	-3	2	-9	-5	-2	1	-9	1
Fred	*	*	*	*	*	*	*	*
Gough	-3	1	-6	-5	-2	1	1	-8
Helen	-3	1	18	1	1	1	1	1
Ivan	1	1	-12	15	-2	1	1	-8
Jill	*	1	*	*	*	*	*	*
Keith	1	1	1	1	1	1	1	1
Lora	1	1	1	1	1	-4	1	-8
Mick	1	1	18	-5	1	1	1	1
Ness	*	*	*	*	*	*	*	*
Olga	1	1	1	-5	1	1	1	*
Pam	-3	1	1	-5	1	1	1	1
Quinn	1	1	18	1	-2	1	1	1
Ray	1	1	*	*	*	*	*	*
Sally	1	1	1	1	1	1	1	1
Tommy	1	1	9	1	-2	*	1	1
Uno	1	1	6	1	0	1	18	1
Vida	-3	2	-9	-5	-2	1	-9	1
#correct	13	15	6	9	9	14	13	11
#wrong	9	7	16	13	13	8	9	11
%correct	59.1	68.2	27.3	40.9	40.9	63.6	59.1	50.0
%wrong	40.9	31.8	72.7	59.1	59.1	36.4	40.9	50.0

diagnostic feature is the range of incorrect responses for each item and the opportunity to analyse students' thinking. Item 1s, '-3 x (2 - -4)' was by far the most difficult (among this selection) for both groups. Several students in both groups provided '18'. These students have apparently evaluated the bracketed part incorrectly as -6 and then given -3 x -6 correctly as 18. In item 1r, '2 x (7-8)', the control group performed slightly better than the experimental group and for item v, '-4 - [ ] = -8', the same proportion of students in each group gave the correct response. Note, however, that seven control group students did not respond. For each of the other selected items the experimental group produced more correct responses. The 'x' shown for 'Norm' (experimental group) in both items q and v were actual responses (" 'x' is the unknown number?").

Across both the post and retention-tests the experimental group performed better than the control group on a large majority of items in the tests. On the retention-test the experimental group performed better on 56 out of the total of 62 items in the test, both groups provided the same proportion of correct answers on four of the items and the control group did better on only two items (Item 1r mentioned above and item 3l, 'evaluate ab ÷ c if a = 4, b = -3 and c = -2').

Table 4 shows the same portion of retention-test result spreadsheet for one of the School C experimental groups. A high level of mastery is apparent. The few error responses made are similar to some of those made by students shown in Table 2 and Table 3 above.

Table 4. Item analysis retention-test spreadsheet extract (School C experimental group)

Q1.	q	r	s	t	u	v	w	x
Content	-27÷[ ]=-9	2 x (7-8)	-3 x (2--4)	-15 ÷(1-4)	0 - -2	-4 -[ ]= -8	-3 x 6	-4 x -2
Corr.Resp	3	-2	-18	5	2	4	-18	8
Alice	1	1	1	1	1	1	1	1
Bill	1	1	1	1	1	1	1	1
Connie	1	1	1	1	1	1	1	1
Daisy	1	1	1	1	1	1	1	1
Elsie	1	1	1	1	1	12	+18	1
Foster	1	1	1	1	-2	1	1	1
Gill	1	1	1	1	1	1	1	1
Harry	1	1	-6	1	1	1	1	1
Ivan	1	1	1	1	1	1	1	1
Jack	1	1	1	1	1	1	1	1
Kylie	1	1	1	1	1	1	1	1
Lucy	1	1	18	1	1	1	1	1
Maisy	1	1	1	1	1	1	1	1
Nellie	1	1	1	1	1	12	1	1
Olive	1	1	1	1	1	-4	1	1
Peter	1	1	1	1	1	1	1	1
Quok	1	1	-12	1	1	1	1	1
#correct	17	17	14	17	16	14	16	17
#wrong	0	0	3	0	1	3	1	0
%correct	100.0	100.0	82.4	100.0	94.1	82.4	94.1	100.0
%wrong	0.0	0.0	17.6	0.0	5.9	17.6	5.9	0.0

Conclusions

The following preliminary findings have emerged at this stage.

- For students who are performing well in mathematics, the method of initially teaching the topic of negative numbers does not seem to affect their long-term performance. In general the topic is not difficult for such students. More able students in both experimental and control groups displayed similar levels of mastery on basic operations, knowledge of the rules and algebraic substitutions and evaluations.

- Lower ability experimental groups appeared to achieve better scores than similar ability control groups. There are some indications that the tile method may be a superior initial teaching method for average and below average students. A major strength and characteristic of the tile method of teaching the topic is the easy and natural way in which it models integer operations and facilitates classroom and small group discussion and interaction.
- By the end of the topic both experimental and control group students tended to perform required operations automatically using the rules. Only a few students used diagrams (eg. tile representations or number lines) when doing the tests. (As the topic progressed some experimental group students tended to draw small tile-like representations rather than use the actual tiles to do some exercises.)
- It was not possible to control for teacher effect and student attitude. Both factors may have affected outcomes. The influence of teachers (eg. control, provision of corrections and feedback, systematic monitoring of progress) and the attitudes of students (eg. reflected in the rate and thoroughness in which they tackled assigned tasks) appears to have been particularly evident in School C. Having the same teacher in charge of both groups in School A should have helped control for teacher effect, however there may have been attitude problems affecting test performance for some students. In both School A and School B there may have been a degree of test underachievement due to lack of effort by some students because some tests (pre and retention) didn't count toward assessment. Written comments on a few test papers included; "This doesn't count" and " I don't have to do this". This is an argument for attempting teaching and learning studies, on topics such as this, in the context of normal school classroom programs. All students involved, because they were required to give signed consent - an ethics committee condition, knew that this was part of an experiment.

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Mathematics Educators: Shaping the Curriculum?

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Two major developments in mathematics curricula have been the 'New Maths' of the 1960s and the recent changes of the 1990s. A framework developed by Howson (1979) is used to compare and contrast these two periods of mathematics curriculum change. What are the influences which shape curriculum development and to what extent are these changes under the influence of anything or anyone? It is suggested that mathematics educators have made a difference in curriculum development through placing constructivism on the curriculum agenda, countering the effects of current reforms in education. From this comparison, further research agendas are proposed.

Significant shifts in the emphases of mathematics curriculum documents occurred in the 1960s and the 1990s, with consequences for the teaching and learning of mathematics. This paper focuses on curriculum change and development at the national level and does not address the working out of curriculum at the level of the school or the individual teacher.

Curriculum change in mathematics in the 1960s and 1990s occurred in a number of nations. Of particular relevance to the New Zealand context are developments in the United Kingdom and the United States of America. The focus in this paper is the New Zealand context, although it is necessary to acknowledge how this is informed/influenced by international pressures. Kline (1973) accounting for why different nations came up with similar documents in the 1960s, suggests that it was partly imitation, but more to do with the emphasis and direction which mathematicians were favouring. He thus treats them as a single movement characterised by common features and content.

It is useful in a comparison of influences in the two periods to have some understanding of the nature of the curricula and how they are different. The 'New Maths' of the 1960s had its origins in the structure of mathematics itself and was concerned with children learning the laws of mathematics from its axiomatic base. Content was organised around algebraic structure and there was little concern for pedagogical matters (Neyland, 1991). Emphasis was placed on rules and the one way of solving a mathematical problem. The curricula which came out of this were 'teacher proof' and textbook driven (Apple, 1992a).

In contrast to this, changes in mathematics curricula in the 1990s focus on the teaching and learning of mathematics with an emphasis on problem solving and multiple ways of ‘doing mathematics’. The curriculum aims to "help students to develop a variety of approaches to solving problems involving mathematics, and to develop the ability to think and reason logically" (Ministry of Education, 1992, p8). It is stated that "mathematics is best taught by helping students to solve problems drawn from their own experience... real-life problems are not always closed, nor do they necessarily have only one solution" (Ministry of Education, 1992, p11). Students construct new knowledge and refine their existing knowledge and ideas (Ministry of Education, 1992). The use of technology is encouraged as a tool for learning. Mathematics is perceived as a human activity, culturally produced and socially constructed (Walshaw, 1994).

A Curriculum Development Framework

Placing curricula within their complex contexts, there are a significant number of influences which have the potential to shape curriculum development. These can be discussed in terms of specific contributors to the writing of the document, or as broader pressures that influence curriculum change.

Dealing specifically with mathematics curriculum development, Begg (1994) acknowledges not only the crucial role of curriculum writers, but also that their thinking is influenced by other groups, who have in turn been influenced by their own experiences, their professional reading, international trends and concerns, and political views. These groups include teachers, mathematicians and statisticians (academic and professional), mathematics educators, the community (parents and employers), resource producers, politicians and students. These groups could either be seen as influencing curriculum writers, or as potential writers themselves.

Taking a broader view than this, Howson (1979) identifies four types of pressures that can serve to initiate curriculum development. These are societal and political, for example, the demand for mathematicians to enhance national prosperity and economic growth; educational, pressure from, for example, teachers; mathematical, changes in the discipline of mathematics and the influence of university based mathematicians; and change itself, the international factor with nations desiring to 'keep up' with neighbours. Using this as a framework for analysis, curriculum change in mathematics in the 1960s and 1990s will be compared.

'New Maths'

Societal and political influences

In the 1960s, societal and political pressures on curriculum development were strong. Price identifies the importance of three events in the American context. The first of these was World War Two and the role played in it by scientists and mathematicians. The second event was the launching of Sputnik I and Sputnik II by the Russians in October and November 1957, and thirdly there was the race to be first on the moon (Price, 1989).

The assumption made was that for America to 'lead the world' there was a need for scientists and mathematicians and that the curriculum needed to be changed to produce these. The School Mathematics Study Group (SMSG) had its origins in a committee requested by the National Science Foundation to "devise a practical programme which will improve the general level of instruction in mathematics in elementary and secondary schools" (Price, 1989, p391). By 1960 sample texts had been written for each of grades 7-12 and there were also at least seven other curriculum study groups which had written texts or issued reports with recommendations for improved programmes (Price, 1989). This growing political demand for science and mathematics has also been attributed to change in New Zealand; "mathematical achievement and national prosperity were coming to be regarded as directly linked" (Openshaw, 1992b, p142).

Political/societal pressures influence curriculum development not only at this level of larger agendas, but also in terms of the curriculum reform process in place. In New Zealand, curriculum development at this time was centralised through the Department of Education but occurred on a 'piecemeal' basis with different subject areas being reformed at different times. In the case of 'New Maths' initial change came from teacher innovators but state approval was required to disseminate the changes in a wider, nationally organised way.

New mathematics innovators initially required Departmental legitimation to prepare an alternative examination syllabus. It is interesting to note here that in negotiations with the Department, reforms were labelled 'modern' rather than 'new' to facilitate "the most crucial test of successful curriculum innovation in New Zealand ... [which] remains its translation from a limited regional experiment into a national curriculum reform movement" (Openshaw, 1991a, p15). The relative success of the second pilot scheme was certainly dependent on Departmental backing and nominal Departmental leadership; "to the extent that central support was vital for both resource allocation and access to curriculum making on a national basis, the Department can be regarded as a bureaucratic gate-keeper" (Openshaw, 1992a, p206). Although the role of the Department is not seen as being large, through the conferences and inservice courses which enabled dissemination of the changes, they did play a role as selective interpreters of innovation (Openshaw, 1991a).

Educational influences

"Dissatisfaction with the existing secondary mathematics curriculum among some mathematics teachers was of long standing in New Zealand" (Openshaw, 1992b, p140); the impetus for change came from a group of teachers. McCausland (1974) identifies the focus for change as a group of Christchurch secondary teachers who met to consider ways of introducing new ideas into the mathematics syllabus. Meeting to write texts, and trialing material in the classroom, this Christchurch Mathematics Group was given official recognition and set up and marked its own School Certificate paper called Pilot Mathematics (McCausland, 1974). All members of this group were Heads of Department and were given by their schools an afternoon per week to meet.

Thus change stemmed from a localised group of teachers but required Departmental approval and funding to be implemented on a national scale. It is also important that change occurred in assessment. A limited pilot scheme for School Certificate began in September 1961; this became more comprehensive in January 1964 after a teacher refresher course. An alternative School Certificate Paper was introduced in 1965 and by 1969 twenty per cent of School Certificate mathematics candidates from seventy secondary schools sat the new mathematics option (Openshaw, 1992b). Change in assessment meant there was also change in curriculum at junior secondary and primary levels. The move towards general acceptance is illustrated by the Alternative Mathematics School Certificate paper being renamed Mathematics in 1973 (McCausland, 1974). Teacher Refresher Course Committee courses were run, booklets put out and texts written.

Openshaw identifies two significant points from his interviews with those involved with the development of 'New Maths'. These are the importance of small groups of teachers in initiating and in sustaining curriculum reform, and the need for innovators to be in step with the times they are living in and the dominant rhetoric of those times (Openshaw, 1991a). This latter point is significant, as the teacher innovators did conform to the dominant rhetoric of the time. 'New Maths' changes were occurring internationally, and were seen as a way of improving the nation through the development of 'better' mathematicians.

Mathematical influences

Mathematical influences are changes occurring in the body of knowledge which is defined as mathematics, and the influence of mathematicians. There is a sense in which all who engage in 'mathematics' are mathematicians; the emphasis here is on university based academics Internationally, there has been considerable focus on university based pure mathematicians and the part they played in the development of 'New Maths'. In America, they were requested by the state to assist in the programme of mathematical curriculum change. A criticism made by Kline (1973) is that the college professors leading curriculum reform lacked pedagogical skill, and were generally not interested in the connections of mathematics with the real world. Education professors were not utilised as they did not have a mathematical background .

In New Zealand, mathematicians in universities have not been identified as significant leaders in curriculum change. The universities did, however have a role, and changes were occurring there even though they didn't directly influence what was happening at the secondary school level.

> As students at the University of Canterbury, McDowell, Goldsmith, Parr and Strange had first met, each having been '...exposed in our last Degree years to some of the interesting unifying concepts in mathematics' and in consequence coming to ask 'Why [weren't] these in the school programme?' (Openshaw, 1992b, p145).

However, "the absence of any large scale new mathematics curriculum projects closely associated with prestigious secondary schools and controlled by University interests undoubtedly was a factor in contributing to a situation which, in this respect at least, was at variance with developments in Britain and elsewhere" (Openshaw, 1992b, p145).

Change itself

From the 1959 Royaumont Conference in France, which eighteen countries attended under the umbrella of the Organisation for European Economic Cooperation, a number of projects were developed internationally. At this international meeting the abandonment of virtually all the familiar courses in high school mathematics (including Euclidean geometry) had been urged (Kline, 1973). There is a sense in which the change occurring had its own impetus, not directed by any single nation, group or individual. "By the mid 1960s well over one hundred curriculum projects aimed at introducing radical changes to secondary school mathematics and science programmes had been initiated" (Openshaw, 1992b, p140).

Although the teacher innovators do not speak much of the international nature of the change itself and how much this had an effect, they were aware of projects developing in other countries. McDowell remembers that Max Riske came back to New Zealand "waxing eloquent on the new emphasis on mathematics in the United States.... he provided the catalyst and it really set us off" (cited in Openshaw, 1991b: 32).

The 1990s

Societal and political influences

> As the first curriculum statement to be developed as part of the Achievement Initiatives policy under the umbrella of the *New Zealand Curriculum Framework*, the document is embedded in the broader social and political milieu of an era characterised by the most extensive restructuring and policy reformulation in education for more than a century (Walshaw, 1994, pi).

Privatisation has occurred across social systems; in education, school administration and management has been decentralised with Boards of Trustees responsible for the running of schools. However the 1989 Education Act dismantled the Curriculum Development Unit of the Department of Education and curriculum came under Ministerial control with greater centralisation than at any point in the past (Peters, forthcoming). By May 1991 the Minister of Education had published his plans for comprehensive reform of the school curriculum to "bring our schooling system into line with the needs of the 90s and the 21st century and the imperatives of the modern competitive international economy" (Smith cited in Peters, forthcoming, p10). The discussion document for the National Curriculum Framework was launched in a speech at the Post Primary Teachers' Association Curriculum Conference, with frequent reference to labour market needs and a narrow and functionalist core curriculum (Peters, forthcoming). This rationale has seen the development of a National Curriculum with an instrumental emphasis on essential, generic skills.

Within this context, mathematics curriculum change was initiated in 1991. Curriculum change has occurred within

> ... a context increasingly defined by the right. Its agenda includes the marketisation and privatisation of education, the slashing of budgets, the standardisation of pedagogy and content through the development of national testing and national curricula, the defining of the needs of business and industry as the primary goals of schooling, the return to a romanticised 'common culture' based on the 'western tradition', and a schooling system that is much more highly stratified in terms of student outcomes than before (Apple, 1992b, p439)

Educational influences

Political demands and the overall agenda of the state have therefore been dominant in this period of curriculum change. What educational influence was there on the curriculum in this period? The rhetoric of the government has been the involvement of teachers in the writing groups of curriculum documents.

> The national curriculum statements are developed by the Ministry of Education following widespread consultations with teachers, other educators, boards of trustees, and the wider community, including the business community.... Drafts are sent to schools for comment and

> trialing before final statements are published for implementation (Ministry of Education, 1993, p22).

With the closure of the Curriculum Development Unit, curriculum development is contracted by the Ministry of Education. In the case of mathematics, the contract was awarded to the Wellington College of Education, under the name of an individual. There was a team of writers, a policy advisory group, and a review group. The original writing group was disbanded, with the rationale that it was dominated by mathematics educators and that more teachers were needed on it. However, the second group was also dominated by mathematics educators.

Another educational influence, which is crucial, is constructivist theory, which is currently prominent in many areas of education. Briefly, constructivists hold the belief that knowledge is actively constructed by the learner, and that learners are not discovering an independent pre-existing world (von Glaserfield cited in Neyland, 1991). With regard to mathematics, there is an "emphasis on the process of mathematising; problem solving strategies, investigative approaches, communication and modelling" (Neyland, 1991, p41).

Mathematical influences ... mathematics educators?

Mathematical influences, defined earlier as those occurring within mathematics at university level, have not been explicit in the curriculum of the 1990s. Here the limitations of Howson's framework are evident in analysing mathematics curriculum development of the 1990s. Where are mathematics educators?

The first International Congress on Mathematical Education (ICME) was held in France in 1968, followed by another in 1972. Although under the name of mathematics education, these two conferences were attended by mathematicians. It was not until the 1976 conference in Germany that there began to be talk of an academic discipline called 'mathematics education'. With regard to New Zealand involvement, one New Zealander attended in 1976, six in 1980 when the conference was held in the United States, and more in 1984 when it was in Australia. Mathematics education developed in the 1980s as a discipline for those with both educational and mathematical backgrounds, working in the field of the teaching and learning of mathematics. Before this, there had been teachers or mathematicians. Britt, (cited in Openshaw, 1992c, p53) states that

> the mathematics educators I still feel are getting off the ground in New Zealand. They are I think to a very great extent associated with the Teachers' Colleges. There are many teachers' colleges that now see themselves as Mathematics Education Departments and as mathematics educators.

This statement is made a year prior to the announcement of curriculum change in mathematics, which was dominated by mathematics educators. Mathematics educators predominantly work in Colleges of Education (as lecturers in pre-service education or advisors to teachers), and increasingly in universities. The first official mathematics education appointment was in 1992.

Change itself

Howson's recognition of change itself as a pressure on curriculum development is also relevant in analysis of New Zealand in the 1990s. Again there is a strong sense in which what is happening internationally informs the changes in New Zealand. *Mathematics in the New Zealand Curriculum* incorporates many features found in other publications, particularly in the United States and England (Howson, 1994). Constructivism, as an internationally prominent, or even dominant, philosophy of learning could almost be seen as having an impetus of its own, independent of specific agents of change. "The present change in perspective in mathematics education stems from the increasing acceptance of the constructivist view of learning" (Begg, 1993, p212). However, there have been challenges to constructivism and it will be interesting to see how these will inform future periods of curriculum development internationally.

Mathematics Educators: Shaping the Curriculum?

In both periods of curriculum change in mathematics there have been a range of influences. Figure 1 summarises the periods and enables similarities and differences to be identified. Particularly significant is the dominance of political demands as an overall agenda shaping the direction of change. International competitiveness and the role that it is believed education, and particularly mathematics can play in this is highlighted in both periods. This international competitiveness is also played out in the way that both periods of reform have been part of larger international change, with 'change itself' having its own impetus. A key difference, is the shift in 'expert' influence from mathematicians to mathematics educators. In the 1960s, mathematicians in universities were key figures in international change. By the 1990s the growth of mathematics education saw mathematics educators at the fore-front of curriculum writing.

Influences	**1960s**	**1990s**
Societal/political		
state education body	Centralised curriculum development but 'piecemeal'	Ministry of Education Education Act 1989 NZ Curriculum Framework- overall agenda
political demands	Demand for scientists and mathematicians 'Space Race' - international competitiveness	Skilled workforce and international competitiveness 'New Right' reforms
Educational		
teachers	Small group of teacher innovators - localised	Rhetoric of being on writing team and opportunities for all to comment on draft document
maths educators	***Did they exist as a community?***	***Prominent on writing group***
Mathematical		
mathematicians	University led reforms by pure mathematicians	What role did they play?
Change itself		
international	Influence of overseas projects	Constructivism

Figure 1. Summary of influences on curriculum development, 1960s and 1990s

The complex nature of curriculum development makes it difficult to pull from the contextual web key factors in curriculum change. It could be asked to what extent change is under the influence of any of these individual factors? Does change have its own momentum independent of specific actors or decisions? This rather rhetorical question has no simple answer. What it suggests however is that curriculum change cannot be attributed solely to any particular influence. However, although recognising this, it is argued here that mathematics educators have played a key role in shaping the curriculum document of the 1990s. It is not to suggest that other influences have not also been significant.

The growth of mathematics education as a recognised field has provided the setting for research into the teaching and learning of mathematics. Mathematics educators draw from mathematics *and* from educational theory, and interact with other disciplines, for example, psychology, sociology and philosophy.

Research into the teaching and learning of mathematics within constructivist theory has occurred internationally. Constructivism has been put on the agenda of mathematics curriculum writing by mathematics educators. Constructivism with its particular focus on children learning through discovery and investigation suggests particular ways of teaching. This has countered the direction the document might have taken under the influence of the National government, with its concern for privatising, measurable outcomes, testing and competition. The agenda of the government has been in some sense countered by constructivism, the influence of mathematics educators. This is evident in the words of Jim Neyland who led the writing team for the curriculum document. He states that "the task we set ourselves was easy to state: Marry the directives from the Government with the findings of the latest research and scholarship in mathematics education and with the very best of current teaching practice" (Neyland, 1993, p7).

Therefore, mathematics educators have shaped the direction of the document through the development of research and the theory of constructivism. Specific individuals may not have made the difference, but mathematics education as an academic discipline has shaped curriculum change in the 1990s.

Conclusion

This paper has analysed influences on curriculum development, comparing two significant periods of change in mathematics curricula. In these two periods, curriculum change was strongly influenced by societal and political factors, particularly concern about international competitiveness. However, the curricula developed were very different. It has been suggested here that the development of mathematics education and mathematics educators has shaped curriculum change through the placing of constructivism on the curriculum agenda. This has in some sense countered the 'New Right' agenda of the national curriculum framework.

Two possible agendas for research come out of this paper. Firstly, the role of mathematics educators needs to be explored further with regard to the influence they exert on the teaching and learning of mathematics, and more particularly, to curriculum development. Howson, (1979, p136) expresses concern that previously

> a feature of most projects is that initiation has usually been unilateral in the sense that it was made by mathematicians or by educationalists, and, indeed, that it often concerned mathematics alone, mathematics *qua* mathematics, and not mathematics as a component of a general education.

The development of mathematics education as a discipline, and the role that mathematics educators have played in curriculum development has perhaps allayed some of Howson's concern. It may be particularly interesting to analyse the relationship of mathematics educators with the government, teachers and mathematicians.

Secondly, further comparisons of curriculum change can be undertaken. Comparison could occur historically, between and/or within nations, and between subject areas. Influences on curriculum development need to be made visible. Curriculum development is a complex process involving a number of different groups working within structural constraints. Understanding and analysis of this process can inform future curriculum development and enable discussion to begin around the questions of 'what will be identified as influential factors in future periods of curriculum change in mathematics?' and 'what influence will mathematics educators have?'.

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The 'Ups' and 'Downs' of Subtraction: Young Children's Additive and Subtractive Mental Strategies for Solutions of Subtraction Word Problems and Algorithmic Exercises

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This paper reports on a longitudinal study across Years 2 and 3 of 104 children's performance and strategy use for two-digit separation and missing-addend word problems and vertical and horizontal algorithmic exercises with and without regrouping. The study showed that while Year 2 children predominantly used a subtractive *removing* strategy for separation problems and an additive *building-on* strategy for missing-addend problems, Year 4 children were more mixed in their strategy use. It also showed that children used subtractive strategies predominantly for vertical and horizontal algorithmic exercises.

A subtraction example can be solved *subtractively* by removing the minuend from the subtrahend (with materials, by counting down or by subtracting components) or *additively* by building from the minuend to the subtrahend (with materials, counting up or by adding components). Research has indicated that such subtractive and additive strategies or procedures are used spontaneously by young children for word problems (e.g., Carpenter & Moser, 1984), by young children for number facts (e.g., Steinberg, 1985), and by older Years 8 to 12 children for algorithmic exercises (e.g., Perry & Stacey, 1994); and can be taught to children (e.g. Fuson, 1986a, 1986b; Fuson & Willis, 1988; Thornton, 1990). Subtraction word problems can be presented with a subtractive or separation semantic structure (e.g., *There were 6 apples and 2 were removed. How many apples are left?*) and an additive or missing-addend semantic structure (e.g., *There were 2 apples. More apples were added to the bowl. There are now 6 apples in the box. How many were added?*) (Carpenter & Moser, 1984).

Research on relationships between children's strategy use and semantic structure for subtraction problems has been inconclusive and contradictory. Some studies have reported a direct relationship. Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993) found that kindergarten children used subtractive strategies for separation problems and additive strategies for missing-addend problems. Similarly, De Corte and Verschaffel (1987) found that Years 1 to 3 children used procedures which modelled the semantic structure of the word problems. Other studies have indicated that this relationship only holds for young children. Carpenter (1986) reported that older children are not limited to procedures that directly match problem structure. In a 3 year longitudinal study of addition and subtraction strategies, Carpenter and Moser (1984) reported that Years 1 to 3 children solved missing-addend problems additively, but only initially solved separation problems subtractively. Later, the children tended to solve separation problems additively. Carpenter and Moser (1983) postulated that additive strategies are used earlier and with higher frequency than subtractive strategies, and that some children may never use subtractive strategies. However, it should be noted that the number combinations used in this study favoured the additive *count up* strategy.

This finding that young children prefer additive strategies for subtraction problems has been supported by yet other studies. Secada (1982) and Steffe, Spikes, and Hirsten (1976) reported that Year 1 children preferred additive to subtractive strategies for both separation and missing-addend problems. Fuson (1986a, 1986b) and Fuson and Willis (1988) reported that Year 2 children also preferred additive over subtractive strategies for separation and missing-addend problems, and that these children learnt multidigit subtraction earlier than is normal for that Year level by employing an additive method. Boulton-Lewis (1993) argued that children favoured additive strategies because they involved less processing load than the subtractive strategies.

This preference for additive strategies has not been found in studies of basic facts and algorithmic exercises. Steinberg (1985) reported a higher incidence of subtractive strategies in a study of Year 2 children's spontaneous derived facts strategies. Heirdsfield (1996) found that Year 4 children did not appear to prefer additive strategies for subtraction number facts. She reported that the dominant strategy was *immediate fact recall* (50.4%), with additive strategies 16.5% and subtractive strategies 19.5%. She found that subtractive strategies were predominantly used when the example proved too difficult to be solved using *immediate fact recall.* In a longitudinal study of 82 Years 1 to 3 children's understanding of multidigit addition and subtraction operations, Carpenter, Franke, Jacobs and Fennema (1996) found missing-addend problems more difficult for children to solve than separation problems.

This paper reports on a 2 year ARC funded longitudinal study of Years 2 to 4 children's mental strategies for addition and subtraction. Two and three digit separation and missing-addend word problems, and two and three digit algorithmic exercises were presented to children in Years 2 and 3, and beginning of Year 4 (some results of this study are presented in Cooper, Heirdsfield, & Irons, 1995a, 1995b, in press). Mental strategies refers to those strategies children employ to perform "arithmetic calculations without the aid of external devices" (Sowder, 1988, p. 182). As mental strategies were not specifically taught to the children in the study, it was hoped to elicit invented strategies. In order to increase the likelihood of children's employing invented strategies, no materials (concrete and pen and paper) were available. Carpenter and Moser (1984) have reported that children often use less advanced and less efficient strategies than they are capable of if materials are available. The paper compares the children's choices of additive and subtractive mental strategies for 2 digit separation and missing-addend word problems, and describes the children's strategy choice for subtraction algorithmic exercises. Efficiency of strategy use over the two years is also discussed.

The Study

Subjects

The subjects were 104 children of varying mathematical abilities (one third each of above average, average, and below average) in 6 Brisbane primary schools (3 State and 3 Catholic) representing a variety of social backgrounds. Although the teachers selected the children, their mathematical abilities were not revealed to the researchers to avoid bias in the interviews. The children participated in the study from the beginning of year 2 (1991) to the beginning of year 4 (1993).

Interview procedures

The interview method was Piaget's revised clinical interview technique (Ginsburg, Kossan, Schwartz, & Swanson, 1983). The questions (for the purposes of this paper) consisted of 2 digit separation and missing-addend subtraction word problems relating to money, and 2 digit algorithmic exercises (vertical and horizontal), presented in visual and oral form (the interviewer read the questions). The problem types are summarised in Table 1 (based on Carpenter & Moser, 1984).

Table 1
Subtraction word problems

Problem type	Example
separation (subtractive)	If John had 82 cents and spent 54 cents on bananas, how much money does he have left?
missing-addend (additive)	If Nancy had 47 cents and the chocolate costs 75 cents, how much extra money does she need?

Further, examples with and without regrouping were included for both word problems and algorithmic exercises in most interviews. The types of numbers involved in all subtraction questions are summarised in Table 2. As much as possible, numbers for the questions were chosen to elicit a range of spontaneous strategies, in particular, wholistic.

Table 2
Subtraction question types

Question type	Example (of numbers)
Word problems	
separation, 2 digits, no regrouping	48 - 25
separation, 2 digits, regrouping	95 - 47
missing-addend, 2 digits, no regrouping	86 - 30
missing-addend, 2 digits, regrouping	92 - 57
Algorithmic exercises	
horizontal algorithm, 2 digits, no regrouping	68-23
horizontal algorithm, 2 digits, regrouping	55-27
vertical algorithm, 2 digits, no regrouping	89 - 37
vertical algorithm, 2 digits, regrouping	45 - 26

The structure of the interview was to present questions for each problem type in increasing order of difficulty, until the children's responses indicated they were experiencing difficulties. Further, as children progressed in their ability to solve more difficult questions, easier questions (for instance, 2 digit subtraction without regrouping) were no longer presented to them. Thus, by interview 4, fewer children were presented with these easy questions, in order to maintain a maximum length of the interview. Further, vertical algorithms for examples involving no regrouping and horizontal algorithms (both regrouping and no regrouping) were no longer presented.

The children were interviewed 6 times: beginning, middle, and end of year 2 (interviews 1, 2, and 3), beginning and end of year 3 (interviews 4 and 5), and beginning of year 4 (interview 6). However, interview 4 contained no missing-addend problems and is not reported here.

The children were withdrawn from the classroom and interviewed on a one to one basis in a separate room. The interviews, whose length was kept to a maximum of 30 minutes, were videotaped. The questions were presented visually, as the researcher read each question and asked the children to explain their solution strategy. No materials (concrete or pen and paper) were provided for the children as aids in calculations. However, they were permitted to use their fingers.

Analysis

The videotapes were transcribed into protocols and behaviours analysed for strategy choice. The coded responses for each interview were tabulated and analysed for each question type. The strategies were categorised in terms of whether both numbers were split into place values, giving rise to strategies called *right to left separated* and *left to right separated* (depending on whether the ones or tens were computed first), *aggregation* (depending on whether only one number was split into place values), or *wholistic* (numbers were treated as a whole and not split into place values (reported in Cooper, Heirdsfield, & Irons, 1995a, 1995b, in press). For this paper, analysis was aimed at identifying additive and subtractive strategies for each question type over the 5 interviews. Examples of *additive*, *subtractive*, and *combined* strategies are presented in Table 3. *Combined* refers to both additive and subtractive processes being used in the same example or example type, for instance, using an additive process to calculate the tens, and subtracting to calculate the ones.

Table 3
Mental computation strategies

Strategies	Example (for 95-47)
R→L Separated	
subtractive	15-7=8, 80-40=40, 48
additive	7+8=15, 50+40=90, 48
combined	7+8=15, 80-40=40, 48
L→R Separated	
subtractive	90-40=50, 50-10=40, 15-7=8, 48
additive	40+50=90, 7+8=15, one less 10, 48
combined	90-40=50, 50-10=40, 7+8=15, 48
Aggregation	
subtractive	95-40=55, 55-7=48; or 95-7=88, 88-40=48
additive	47+40=87, 87+8=95, 48; or 47+8=55, 55+40=95, 48
combined	47+40=87, 95-7=88, 40+8=48
Wholistic	
subtractive	95-50+3=48
additive	45+50=95, 50-2=48

Results

The results for each question type for each interview are presented in Table 4. The data reported include the percentage of the population who were able to attempt the question type at each interview, the percentage of correct responses for those who were able to attempt the question type; then considering again the children who attempted the question type, the percentage who employed additive, subtractive or combined strategies, and finally, the percentage of those using each type of strategy who were correct. Thus, efficiency of strategies may be compared, but small numbers employing strategies must be kept in mind when comparing some choices.

General trends for question types indicate that an increasing percentage of students were able to attempt questions over the interviews, and accuracy levels tended to improve. Further, more children were able to attempt separation word problems than missing-addend problems, although accuracy levels were about the same (taking into account the number of students attempting each question type). For examples that did not require regrouping, more students were able to attempt missing-addend problems than algorithmic exercises; however, the trend was reversed for regrouping examples where more children attempted algorithmic exercises than missing-addend problems. Further, there is little difference in accuracy levels for missing-addend problems and algorithmic exercises, although it may be hazardous to draw such conclusions with small numbers .

Subtractive strategies were predominantly employed for separation word problems and algorithmic exercises, and additive strategies were predominantly used for missing-addend word problems. Overall, there is little difference in the accuracy levels of strategy choice, that is, whether additive, subtractive or a combination strategies were chosen had little effect on accuracy. However, very small numbers for some strategy use for some question types made it difficult to compare accuracy levels. Although separation word problems and algorithmic exercises were predominantly solved using subtractive strategies, a higher percentage employed subtractive for algorithmic exercises. In the first few interviews, most students solved word problems using the strategy that reflected the semantic structure of the problem. However, by the beginning of year 4, a greater variety of strategies was evident, although most solution strategies still reflected the question structure. Thus, as the children progressed through their schooling, they were not necessarily limited to procedures that directly matched the semantic structure of the problem. Responses to algorithmic exercises showed less variation. It appears that students consistently considered algorithmic exercises as subtractive.

Table 4

Children's performance and strategy use for subtraction problems and exercises

Measure	Strategies	Interviews				
		1 (*n*=104)	2 (*n*=104)	3 (*n*=90)	5 (*n*=104)	6 (*n*=104)
A. Separation, 2dig. probs no rgp						
% attempting question (of *n*)		24.3	62.5	72.2	73.1	64.4
% correct (of those attempting ques.)		38.6	52.3	60	77.6	86.6
% who used denoted strategy (of those attempting question)	additive	13.6	7.7	10.8	19.7	23.9
	subtractive	81.8	82.4	80	61.8	64.2
	combined	4.5	10.8	9.2	11.8	9
% correct (of those using denoted strategy)	additive	33.3	40	85.7	80	70.6
	subtractive	41.7	50.9	57.7	80.9	93
	combined	0	71.4	50	88.9	100
B. Miss-add. 2dig. probs no rgp						
% attempting question (of *n*)		34.6	51	55.6	57.7	44.2
% correct (of those attempting ques.)		38.9	60.4	52	81.7	71.7
% who used denoted strategy (of those attempting question)	additive	86.1	92.5	90	83.3	62.2
	subtractive	13.9	7.5	10	15	32.6
	combined	0	0	0	1.7	2.2
% correct (of those using denoted strategy)	additive	38.7	59.2	46.7	82	63.3
	subtractive	40	75	100	88.9	86.7
	combined	NA	NA	NA	0	100
C. Alg Ex, 2dig. no rgp vertical						
% attempting question (of *n*)		25	34.6	26.7	NA	NA
% correct (of those attempting ques.)		26.9	47.2	45.8		
% who used denoted strategy (of those attempting question)	additive	0	2.8	8.3		
	subtractive	100	97.2	91.7		
	combined	0	0	0		
% correct (of those using denoted strategy)	additive	NA	0	0		
	subtractive	26.9	48.6	50		
	combined	NA	NA	NA		
D. Alg Ex, 2dig. no rgp hor						
% attempting question (of *n*)		19.2	38.5	44.4	NA	NA
% correct (of those attempting ques.)		15	42.5	50		
% who used denoted strategy (of those attempting question)	additive	0	0	5		
	subtractive	100	100	92.5		
	combined	0	0	2.5		
% correct (of those using denoted strategy)	additive	NA	NA	0		
	subtractive	15	42.5	54.1		
	combined	NA	NA	0		
E. Separation, 2dig. probs rgp						
% attempting question (of *n*)		4.8	14.4	18.9	68.3	77.9
% correct (of those attempting ques.)		40	26.7	35.3	28.2	51.9
% who used denoted strategy (of those attempting question)	additive	0	20	17.6	9.9	3.7
	subtractive	100	60	82.4	70.4	75.3
	combined	0	20	0	19.7	21
% correct (of those using denoted strategy)	additive	NA	33.3	66.7	57.1	0
	subtractive	40	33.3	28.6	26	45.9
	combined	NA	0	NA	21.4	82.4
F. Miss -add. 2dig. probs rgp						
% attempting question (of *n*)		1.9	3.8	5.6	21.2	37.5
% correct (of those attempting ques.)		50	25	0	50	61.5
% who used denoted strategy (of those attempting question)	additive	100	75	100	72.7	59
	subtractive	0	0	0	18.2	35.9
	combined	0	25	0	9.1	5.1
% correct (of those using denoted strategy)	additive	50	33.3	0	62.5	47.8
	subtractive	NA	NA	NA	25	78.6
	combined	NA	0	NA	0	100

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Table 4 (cont.)

Children's performance and strategy use for subtraction problems and exercises

Measure	Strategies	Interviews				
		1 (*n*=104)	2 (*n*=104)	3 (*n*=90)	5 (*n*=104)	6 (*n*=104)
G. Alg Ex, 2dig. rgp vertical						
% attempting question (of *n*)		7.7	9.6	10	63.5	62.5
% correct (of those attempting ques.)		50	40	44.4	25.8	46.2
% who used denoted strategy (of those attempting question)	additive	0	0	0	1.5	4.6
	subtractive	100	90	88.9	98.5	95.4
	combined	0	10	11.1	0	0
% correct (of those using denoted strategy)	additive	NA	NA	NA	0	0
	subtractive	50	44.4	50	26.2	48.4
	combined	NA	0	0	NA	NA
H. Alg Ex, 2dig. rgp hor						
% attempting question (of *n*)		2.9	4.8	13.5	NA	NA
% correct (of those attempting ques.)		0	60	28.6		
% who used denoted strategy (of those attempting question)	additive	0	0	0		
	subtractive	100	100	100		
	combined	0	0	0		
% correct (of those using denoted strategy)	additive	NA	NA	NA		
	subtractive	0	60	28.6		
	combined	NA	NA	NA		

An increasing percentage of children attempted separation, 2 digit without regrouping word problems from interviews 1 through 5 (Table 4, A). The decrease in interview 6 resulted from a growing number of students being able to attempt the more difficult regrouping problems, and thus, not being presented with the easier problems. Although the less able students were presented with the separation no regrouping problem types in Years 3 and 4, the level of accuracy continued to rise (to 86.6%). Subtractive strategies were predominantly used in the first 3 interviews, but by interview 6, additive strategies were also used. However, subtractive strategies were more accurate than additive strategies.

As with the separation problems, missing-addend problems were also attempted by fewer children in interview 6, as more students were attempting regrouping problems (Table 4, B). However, the accuracy level fell in this interview. Although additive strategies were dominant throughout, subtractive had become significant by interview 6, and were consistently more accurate.

Presentation of vertical and horizontal algorithms without regrouping ceased in interview 4, to allow for more difficult question types to be presented (reported in Cooper, Heirdsfield, & Irons, 1995a, 1995b, in press). Data shown in Table 4, C and D, indicate the preference and higher accuracy levels for subtractive strategies. However, additive strategies were present by interview 2 for vertical algorithms and by interview 3 for horizontal algorithms.

For separation word problems with regrouping, subtractive strategies were most popular; however, additive strategies were significant in interviews 2 and 3, and combination strategies were also quite popular in interviews 5 and 6 (Table 4, E). Additive strategies proved to be the most accurate of the strategies in interviews 2 (equal with subtractive), 3, and 5 (although the numbers were quite small). Combined strategies were most accurate in interview 6. However, none were particularly accurate, but this is consistent with the overall accuracy levels for subtraction with regrouping. Although percentages attempting the regrouping missing-addend question type increased over the interviews, few children (37.5%) were capable of attempting it even in interview 6 (Table 4, F). The dominant strategies throughout were additive; however, by interview 6, subtractive had also become significant. Subtractive strategies were more accurate in this interview, although the small numbers makes it hazardous to make such assumptions.

The percentage of students attempting the vertical algorithm increased over the interviews, although accuracy levels did not improve (Table 4, G). This fact could be accounted for by the dramatic increase in number, but not ability. Subtractive strategies remained dominant; however, a few children also chose additive strategies in interviews 5 and 6, although, with no success. The low accuracy levels for subtractive are consistent with the low accuracy levels for subtraction in general. The horizontal algorithm for regroup exercises was not presented beyond interview 3 (Table 4, H). As mentioned above, this was to allow for presentation of more difficult problems and exercises. Subtractive were the sole strategies employed for this question type.

Discussion and Conclusions

The children in this study exhibited both subtractive and additive strategies for all separation and missing-addend word problems with and without regrouping. This is consistent with the findings of Carpenter and Moser (1984). Further, the children used both subtractive and additive strategies for algorithmic exercises, with and without regrouping. This was also reported by Perry and Stacey (1994) in their study of much older students (years 8 to 12). However, in this present study, subtractive strategies were predominantly used by the younger children.

For word problems, the children initially used strategies that related to the semantic structure of the problem. Similar findings were reported by Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993) and De Corte and Verschaffel (1985). However, in line with the findings of Carpenter (1984), this relationship was not as strong by Year 4 (i.e., subtractive for separation and additive for missing-addend had reduced). Unlike Carpenter and Moser (1984), this weakening of relationship moved both ways. Both the use of additive strategies for separation problems and the use of subtractive strategies for missing-addend problems increased, although subtractive for missing-addend increased more. There was not the emphasis on additive strategies that was present in the studies by Carpenter and Moser (1984), Fuson (1986a, 1986b), Fuson and Willis (1988), Secada (1982), and Steffe, Spikes, and Hirsten (1976). In fact, for problems with regrouping, children's strategy use showed a tendency to move towards subtractive. By Year 4, the use of subtractive strategies for missing-addend problems with regrouping was higher than for problems without regrouping while the use of additive strategies for separation problems with regroup was much lower.

For algorithmic exercises, both vertical and horizontal, subtractive strategies strongly predominated (91.7 to 100%). This was in line with the findings for young children and basic facts (Steinberg, 1985) and for older children (Perry & Stacey, 1994).

Unlike much of the literature, the children in this study did not prefer the additive strategies. Initially, they followed the semantic structure of the word problems, and preferred subtractive strategies for algorithmic exercises. There was some evidence that the subtractive strategies began to predominate as the children became older.

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Moments In The Process Of Coming To Know

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This paper reports on research which attempts to conceptualise, operationalise and document instances of meaningful learning in the secondary mathematics and science classroom, and the conditions under which this learning occurred. A "Linkage Criterion" for meaningful learning is proposed and applied to the data from 67 interviews of students reflecting on their classroom learning experiences. Instances of meaningful learning were few and comparatively impoverished where they occurred. The data suggests that the role of out-of-class experiences as classroom focussing mechanisms is a significant one. If teachers plan their instruction with the goal that students should acquire significant new knowledge in every lesson, our data suggests that this goal is achieved sufficiently rarely as to call it into question.

Classrooms are meant to be places in which learning is facilitated. It is reasonable to expect that this learning has some meaning for the student. Our goal is the creation of a learning environment in which students are not simply accumulating isolated facts and procedures, but are involved in integrating, or linking, new concepts and skills into an already existing conceptual framework. This form of interconnected knowing we refer to as *meaningful learning*. In this view, meaning is measured by the richness and complexity of these links. Using this image of linkages, the phrase "attaching meaning" takes on a very specific metaphorical character.

This paper reports on research which attempts to conceptualise, operationalise and document instances of meaningful learning in the secondary mathematics and science classroom, and the conditions under which this learning occurred. Such information could inform attempts to facilitate meaningful learning, and evaluate the effectiveness of teaching strategies designed to enhance or promote this form of learning. This paper is concerned only with the occurrence of meaningful learning and some of the characteristics of the classroom episodes with which it was associated.

Learning has been studied in many ways and in many settings. Recognition of the situated nature of learning has led to studies of workplace learning (for example, Lave, 1988). By contrast, learning has been studied through Teaching Experiments in clinical settings involving small groups of students (for example, Steffe, 1991). Classrooms are distinguished by being environments with the specific purpose of serving as sites for learning. Learning in classroom settings is no less situated than learning in other settings. This may seem an obvious assertion, but one consequence of the current enthusiasm for workplace learning could be an eroding of the status of the classroom as a setting for legitimate, meaningful mathematical activity. If the community is to continue to employ schools as the principal means by which young people are educated and enculturated into academic disciplines, then educational research must attend to the characteristics of the learning environments in such settings and to the processes by which learning occurs in classrooms. Many researchers have made the classroom the focus of their efforts (for example, Yackel & Cobb, 1993). There are certain characteristics of structure and purpose shared by all classrooms. Equally, each classroom presents a unique social setting, which facilitates and constrains the actions of all participants and the form of the consequent learning. Classroom research typically attempts to capture the characteristics of learning in one classroom in a form that will bear translation to other classrooms. This paper documents one attempt to characterise the form taken by this classroom learning.

Research Methodology

The research procedure we adopted was developed in an attempt to study learning in legitimate classroom sttings, while minimizing the need for researcher inference regarding student thought processes. To infer student thought processes and the significance of classroom events on the basis only of videotape data seems an unjustified extrapolation. Clarifying and corroborative data can be provided by the students themselves in interview situations, where their accounts of the significance of classroom events and their associated thought processes can be reconstructed with the assisting prompt of the classroom video record. More detail regarding the rationale for this research technique can be found elsewhere (Clarke, in press; Clarke & Kessel, 1995).

A total of thirty-three secondary maths and science classes were videotaped using two cameras. One camera was directed at the teacher, while the other camera was focussed on a group of about four students. The teacher's utterances were recorded through a radio microphone and a single microphone was used to record the conversations of all four students. The two video images were mixed to produce a composite picture in whch the students occupy most of the screen with the teacher image superimposed in a corner of the screen. This combined image was recorded onto video-8 tape using a compact videorecorder attached to a laptop computer. The researcher, seated at the rear of the classrom, was able to listen simultaneously to both student conversations and teacher utterances and recorded field notes onto a word processing document on the computer using *CVideo* software (Rochelle, 1992). The field notes can be "time-tagged" to corresponding events in the video record. The field notes enable the researcher to document impressions of significant classroom episodes and learning events as well as provide reference markers for the subsequent interviewing of student subjects.

Relating the interview to the video record

Students from the groups which had been videotaped were interviewed immediately after the lesson, a total of sixty-seven student interviews. The videotape record was used in the interview to stimulate student recall of classroom events. The use of the *CVideo* software enabled the researcher to locate within the field notes refernce to actions of the student which seemed to be of significance either to the researcher or to the student. Having found a particular item in the word document, the software could be used to very quickly find the corresponding moment on the video record. This was then played back and discussed. The audio record of the interview provided a third source of data, which in fact was the primary source of data for this paper. A more detailed account of the research methodology can be found in Clarke (in press) and Clarke and Kessell (1995).

Operationalising Meaningful Learning

In order to operationalise meaningful learning, we have generated a list of characteristic student actions, which we have employed as indicators that "meaningful learning" has occurred as a consequence of classroom experiences. These characteristics are set out below.

A Claim to New Knowledge: In order to substantiate a claim to have learned something meaningful, a student must, first of all, claim to have learned something new.

The Communication of New Knowledge: Having claimed to have learned something, a student should be able to articulate or demonstrate what it is they think they have learned, with some degree of clarity and accuracy. This elaborates their claim to have learned, but it does not justify classifying the learning as "meaningful"

The Demonstration of Interconnected Knowing: For the learning to be regarded as meaningful, there should be demonstrable links with an existing framework that the student already possesses. If there are no links (that is, the learning has the form of an isolated fact or procedure) we would argue that the student has not attached meaning to whatever it is that they have learned. Moreover, it is our suggestion that the greater the number of linkages which can be formed with the new knowledge, the greater chance there is that the new knowledge has meaning for the student. To meet the Linkage

Criterion for meaningful learning, the student must therefore articulate some form of relationship between the lesson's academic content and at least one of the following:

(i) the content of previous lessons in the same school subject
(ii) the content of other school subjects
(iii) future studies (in the same school subject or other subjects)
(iv) previous personal experiences
(v) subsequent uses to which the lesson's content might be put outside the classroom.

Interview Questions

A number of issues were discussed with the students at the conclusion of the lesson. This paper forms part of a broader study, so only interview questions designed to elicit information specific to the theme of this paper are documented below.

1. Student claim to new knowledge

Students were asked to summarise the content of the lesson. Sample questions included: *What was that lesson about? If someone were to ask what you learned in that lesson today what would you say? Was this something you knew before? Was there something which you did not know before this lesson, but which you would now claim to know?*

This question was designed to establish whether or not the student could articulate the primary content of the lesson; to establish a claim to have learned something new.

2. *Communication of new knowledge*

Sudents were asked to elaborate in more detail what it was they learned in the lesson. Sample questions include: *What was it that you learnt from this lesson? So could you complete this sentence "In this lesson I learned that..."*

3. *The Demonstration of Interconnected Knowing*

Further questions were asked to tap in to the connections the student made with prior learning and possible future use. Examples of these are: *Was this related to work you did last year? Can you imagine using anything that you learned in that lesson? Why would you want to know that?*

The interviews were transcribed and analysed for reference to these areas in the discussion between student and interviewer.

Levels Of Learning

The extent to which students made connections can provide insight into the meaningfulness of learning, which could be imagined along a continuum from a situation where the student learned an isolated fact or procedure to a situation where a rich and complex linkage with existing conceptual frameworks has occurred.

Meaningful learning could involve the elaboration or consolidation of an existing construct, or relinquishing a transitional construct in favour of a more correct or elaborate one. It could also involve the synthesis of constructs to form a new construct at higher level. For example, a student moves from knowing how to find the area of a square, a rectangle or a triangle as a procedural exercise to an understanding of area as a general concept.

Results And Discussion

Sixty-seven interviews with year seven, eight and nine students at an independent school in an outer eastern suburb of Melbourne were examined for the evidence of meaningful learning, as operationally defined above. The prime focus of this study was to document the presence and extent of meaningful learning. The following discussion provides examples from the data of situations along the continuum from isolated, non-meaningful learning to examples of learning where rich connections were articulated.

There were many instances in which students neither claimed that learning had taken place, or if they did, they were unable to elaborate what they had learned with enough clarity to have that claim substantiated. When a student did substantiate a claim to new knowledge, there were a very small number of instances in which the students made connections between their new learning and existing frameworks, in terms of the categories set out above.

The examples below are taken from interviews in which the student claimed to have learned something new. In selecting examples, we have attempted to illustrate instances of learning categorised according to "meaningfulness". Some students articulated isolated pieces of knowledge. Others demonstrated that they were struggling to find meaning but were not able to achieve it. Very few actually met our criterion for meaningful learning.

Non-meaningful learning

In this interview, Shona described how she tends to simply apply the most recent thing she has learned, regardless of its relevance to the task at hand:

S: If you try to do a fraction and you're trying to put it into lowest common denominator um, then I'll end up doing something different like the newest type of fraction that we've done and I work a different sum for it completely and I'll just go on to a different track like I'll be doing timesing the two fractions together instead of changing it to the lowest common deminator, so I'll do the wrong thing . . . Well I look at it and I think "Oh, well, if it's not right, I'll just do it anyway, just to, at least I've got answer down there and that might be right." So I just kind of hope. 'Cause I can remember how to do the newest ones, so that's the one that I usually do.

Shona's attitude to mathematics echoes the experience of students referred to by Schoenfeld (1992) who react to the arbitrariness of methods imposed on them by teachers and texts.

> Students may simply give up trying to make sense of the mathematics...they may come to believe that mathematics is not something they can make sense of, but rather something almost completely arbitrary (or at least whose meaningfulness is inaccessible to them) and which must thus be memorized without looking for meaning - if they can cope with it at all
>
> (Schoenfeld, 1992, p. 343).

Students' inability to envisage any potential use for new knowledge or a procedure was a feature of several interviews relating to both mathematics and science classes. For example, Harry had just completed an experiment which served two purposes: to calculate his own lung capacity and learn a technique (displacement) for finding the volume of an irregular object. Harry was able to articulate what he learned, but unable to link this to previous lessons or to any other school subject and when asked if he could imagine using anything he learned in that lesson, the only possibility he could think of was if he were to become a teacher himself.

Struggling for meaningful learning

There was a sense with some students that they were trying to find connections, but were unable to do so. Shona, in a year 8 science class, had just done an experiment designed to measure the energy content of food. Students heated various items of food using a bunsen burner and measured the temperature increase of water in a test tube. She was not yet comfortable with the connection between the the amount of energy in the food and the rise in temperature of the water.

I: OK, unpack that one for me, if you had to explain it to somebody else, I mean, I don't know if you've got a younger brother or sister, a year 7 kid, somebody who peers in the window and they say "You guys are burning peanuts, that's really weird", how would you explain?

S: Um, well I'm burning the peanut to see, 'cause then all the steam and everything comes off the peanut, and comes off the water in the test tube and it makes the temperature of the test tube um, product, higher, and that way you find out how much energy's in it.

I: Did you understand?

S: I did understand how it was supposed to work out how much how much energy was in it, 'cause like you're just burning it and seeing how, what the temperature of the water is going to change to, I didn't see how that was going to be, that was going to find out what the energy was going to be in the food, I didn't know the connection kind of.

Later in the interview she returns to this issue:

S: You find out how much energy is in the food like I believe if you do that experiment and then you find out how much energy's in the food, I just don't understand how it actually all comes together and what you find out, like I know that (it) works, like if you do it you'll find out what the energy is, I just don't know why it works.

I: OK, this is really, really, really interesting because I mean you're grappling with a really important problem here, OK, let me get this straight, you understand everything that happens in the experiment, you know all the things that you do.

S: Yes. And like I believe that it works, like if Mr. Smythe told me that if you find out the energy by doing this, I'd say OK, like I'd say how you do the experiment that's how you find out what the energy is, but I don't see what the connection is between the energy and the temperature of the water.

I: Right.

S: Like I don't find out how you actually put them together and then why it works.

I: How you connect the temperature change in the water to the energy. That's the bit that you've still got to work out.

S: Yes, like why it happens.

I: But you believe it works because-

S: Mr Smythe told us.

Shona concludes the interview with a further statement of her dilemma:

S: I know how to do the experiment and everything, I just don't understand how it's all linked together and it tells you what the energy is.

Connecting with prior school mathematics experience

Hamish and Colin both participated in the same lesson on probability. The activity they did in the class involved a simulated horse race, with horses numbered from two to twelve (the possible outcomes of rolling two dice and taking the sum). Each race involved ten throws of the pair of dice and the students predicted which horse would be the winner. For Hamish, the learning was meaningful in that he made links with what he had done in grade 6, and showed that what was intuitive knowledge had become more sophisticated. He was able to articulate what he had understood the previous year and describe how his knowledge had developed. He was now able to explain why, when two dice are thrown, the chances of a seven are higher than for any other combination:

H: Well, I thought it would be easier to get more sevens because um, seven was in the middle and there would be more combinations of getting seven, like five and two, four and three, and six and one, but for numbers like twelve there'd only be one combination to get it, so I…

I: What did you understand before this year? If I'd asked you when you were in grade six?

H: I understood that seven or numbers in the middle of there like seven, six, eight, and five and nine are easier and even more likely to roll than the two dice than the twelve and the eleven, ten, one, two.
I: Could you do me the why? Could you explain that one to me?
H: Well, there's like these numbers like seven, eight and six and they're right in the middle and there's more combinations with the dice to get it and with ten there's only two combinations four and six, five and five, but with um, six, there's four and two, five and one, and three and three.

On the other hand, for Colin, who had participated in the same lesson, there appeared to be no development from grade 6 to year 7. He had a sense that it was the numbers "around the middle"that come up more often but had not yet grasped the notion of combinations, or *why* seven was the winner.

I: What about the bit where you backed the six and the seven and the eight ... and seven came out the winner?
C: Yeah, we did it last year in grade 6. Where it's usually around the middle that wins.
I: OK. What's educated about your guesses?
C: Well, that it's all around the middle.
I: Well, that's right, I mean you knew that, didn't you? Now what made you think that was true?
C: Because I did it last year.
I: And you did it once last year and it came out so you figure it's going to happen again?
C: Yeah, 'cause like the odds and all that. 'Cause there's more chance of getting things around the middle instead of up higher.

The sense of progression in understanding that is conveyed in Hamish's responses is absent from Colin's. The linkage which Hamish formed allows us to attribute a meaningful character to his learning, that does not apply in Colin's case. However, the restriction of the student's conceptual linkages to "within class" experiences poses serious questions with regard to the applicability of the student's learning to non-classroom contexts. While the linking of current content to previous studies in the same subject can be seen as an essential component of school learning, if this is the only form of linkage promoted or occurring, it is our feeling that the evolving understanding of concept or procedure is a fairly impoverished one.

Examples of meaningful learning

Using the criteria listed earlier, we were unable to find many instances of meaningful learning. In mathematics classes, where links with previous experience were made, none were with experiences outside the mathematics class. The example described above, of Hamish learning about probability, where he built on the knowledge he already had from the previous year is typical. Sarah, a year 8 mathematics student, provides another example in the context of learning about the difference between rate and ratio:

I: Do you feel like you learned anything new in this lesson?
S: Yeah ... Well, I know now what rate meant.
I: OK, so when did you decide you understood it?
S: Um, well first she related it to things we already sort of knew about, like about distance and time and all that kind of stuff we'd just done last year.

It was only in science classes that we found examples of students who made links with experiences outside the classroom. Sometimes these links were initiated by the teacher, and sometimes students responded by describing their own experiences. An illustrative example from a year 8 science class comes from Kevin, who, when dissecting a sheep's heart, drew on what he had learned from seeing his mother's ultrasound:

I: So when you talk about the four chambers, is that, are you picturing what you saw or are you picturing the diagram?
K: Well, I'm sort of picturing um, both, and um, well, Mum's going to have a baby soon.
I: All right.
K: And we, we saw an ultrasound.
I: Oh, OK.
K: And we saw the four chambers of the heart in the baby, um, all at once, and linking them in with the others in, so I sort of pictured that as well.

And further on in the interview, he was asked where he learned this:

I: So, in as much as you know something, whatever you got out of it afterwards, where did you learn that stuff from? Do you remember?
K: From the ultrasound probably, and well, well, we got videoed, think we took the video off back home so we could watch it on TV, and we, Mum sort of told a few things about it while we were watching...
I: What about [the fact that] the heart has four chambers? Where did you learn that? First time.
K: Ultrasound.
I: At the ultrasound and you really didn't know before then, that hadn't come up before then.
K: No.

Kevin's account illuminates two aspects of meaningful learning: the occurrence of purposeful and significant learning outside the classroom; and the significance of the prompted recall of prior out-of-class experiences as mechanisms to focus student attention on subject content.

In the following excerpt, a student in a mathematics class described the process of confirming her predictions:

I: So could you complete this sentence "in this lesson I learned that."
P: Um, I just really learned, well I kind of already knew but it just sort of put it into perspective how I realized that I already knew it but I didn't realize 'cause I sort of, you half, I half knew it but, but when you have the aspects when you fold it, will sort of change, but I already half-knew because it, but, well I already knew everything to go into it but I didn't know the combination, that they were in, do you know what I mean?
I: I think I, yeah, I think so.
P: The pieces, it was like a puzzle, that you sort of, I knew all the pieces I just had to put it together.
I: Yeah.
P: And that's where I didn't know, but I knew everything that went into it. So that was. 'Cause I knew how to do the aspect ratio and I had a feeling that when you fold it in half it would be equal because well, I had a feeling that this would happen, because when you fold it in half in, in the four it's just going to be a scale version of the bigger one, but smaller so it's going have the same aspect ratio, but it just sort of, I already had a feeling that I knew half of it, I just learned that it was really.

It appears that this student had experienced the empirical nature of mathematics, for it involved for her the observation of patterns, testing of conjectures, and estimation of results. Because of the links the student made with her own conjectures, we feel that this instance should also be regarded as an example of meaningful learning.

Conclusions

It appears that the Linkage Criterion (for Meaningful Learning) can be applied with success to data of the type generated in this study. It is noteworthy that of the 67 interviews analysed for this paper, less than 10 survived the stringent application of our criterion for meaningful learning. Within these examples, we have mentioned our concerns regarding the limitations of subject-specific linkages that make no reference to objects or events outside the classroom. Our data suggests that the role of out-of-class experiences as focussing mechanisms is a significant one. It is our intention to explore this particular outcome in a subsequent experimental study. Other issues to emerge included the recognition of intermediate states in the process of coming to know involving "transitional concepts" (Moschkovich, 1994). Student difficulties in articulating new knowledge raise the question of whether a claim to know can be made legitimately if the claimant is unable to express or demonstrate the knowledge claimed. Certainly, students in this study made claims to learning, but found the articulation of the new knowledge extremely difficult.

Is it the expectation of educators that students should acquire significant new knowledge in every lesson? It is our impression that teachers plan instruction with this goal in mind. Our data suggests that this goal is achieved sufficiently rarely as to call it into question. How frequently should we expect a student to engage in "Meaningful Learning" as defined by our Linkage Criterion? How important is it to distinguish the meaningful learning of a single fact or procedure, with whatever richness of association, from the more global form of meaningful conceptual learning, which the community has dignified with the term "understanding"? This distinction goes beyond Skemp's "instrumental" and "relational" categories for understanding (Skemp, 1976), since neither of these require reference to any objects outside the local system of mathematics (or science), which might have formed the basis for a lesson's content. It may be that the scale of the most meaningful forms of conceptual learning precludes their identification with any single lesson or activity, but rather are a consequence of a long-term synthetic process to which classroom events contribute incrementally. If this is the case, and if this form of learning is our ultimate goal, then classroom settings may not be either necessary or sufficient sites for our research.

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"I've never had such a long conversation with him" Video as a Means of Gaining Access to Elementary Children's Engagement With Mathematical Processes.

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Te Whanau o Ako Pai ki Te Upoko o Te Ika

This paper discusses the potential of video as a means of discovering young children's explanations of mathematical tasks. This is illustrated with an observation made as part of a larger study into promoting mathematical processes in New Zealand junior classrooms. It highlights the potential power of video as a means of providing the teacher with an insight into the thought processes of young children doing mathematical activities independently of the teacher.

The value of talk in mathematics classrooms has been well documented by various commentators (Cobb, Wood, Yackel & McNeal 1992) and has been highlighted in current curriculum documents. The New Zealand mathematics curriculum document (1992 p.11) states

> Learning to communicate about and through mathematics is part of learning to become a mathematical problem solver and learning to think mathematically. Critical reflection may be developed by encouraging students to share ideas, to use their own words to explain their ideas, and to record their thinking in a variety of ways, for example, through words, symbols, diagrams, and models.

The purpose of talking about mathematical ideas for the student as a means of explaining their thinking to others, as well as challenging and questioning the thinking of others is suggested in the above quote.

This paper will focus on the potential of student talk from the teacher's point of view as an indication of students' progress. There has been an affirming of school based assessment of children's learning by informal rather than formal means in recent curricula and policy documents in New Zealand (Ministry of Education 1992, 1994a). In a guide to the *Beginning School Mathematics* resource (Ministry of Education 1994b p.20) provided at a national level in new Zealand to junior classrooms, it is noted that, "By observing and interacting, teachers can gain a great deal of information on each child's progress". It is suggested that this information can be used to plan further learning. *Assessment Policy to Practice* (Ministry of Education, 1994a p.17) states that, "Observation can range from the informal (for example, casual monitoring of an aspect of student behaviour) to the more formal (for example, using a standardised observation schedule)."

However, many junior teachers have reported difficulties in accommodating observation in their mathematics teaching and learning programmes typically based on a Piagetian model of child-centred classroom practice (Bennie, Henry & Ratcliff 1990). A usual classroom mathematics session observed, as part of the wider study of which the excerpt in this paper is a part, consisted of children organised into small groups (most often on the basis of stage of learning determined by time at school). After a brief introduction the teacher worked with one of the groups while the other groups worked on self selected exploration from a range of equipment or a mathematical game or activity chosen from a teacher selected range, or a group task set by the teacher of either a collaborative or individual nature. For most sessions the children could choose whether they would work alone, or in a pair or as part of a small group of typically three, four or five children from within their teacher designated group. The teacher's role in such typical sessions is constrained by the classroom social arrangements of three to four group; the teacher working intensively with a small group, albeit two groups in total over a single mathematics session, giving limited opportunity to observe children's talk as they work on the self-directed tasks.

The following excerpt from Ruby School illustrates how a teacher inadvertently finds out the processes that Campbell, whom she regarded as reticent, was using when engaged in a mathematical task - building a structure. The children observed were 5 to 6 year olds in their first year of school. Each mathematics session was observed and these observations were recorded using field notes and audio-tape. In addition a video camera was positioned on a tripod and left to run unattended. The excerpt that follows traces Campbell's progress as he works on a self-directed activity while his teacher is working with another group. It then presents Campbell's and the teacher's response to a videoed classroom sequence, played back a week later, of Campbell making the model.

Observation Notes Taken at the Time

Campbell, Stan, Alister and Amber are sitting at a table. All of Campbell's group have chosen to work by themselves on different tasks using mathematical equipment. On other occasions they chose to work together on group tasks, such as a game, or the construction of a model. They are each working with different pieces of equipment - Stan with the interlocking cubes, Amber with the seriated tubs and Alister with the counters and balance. Campbell is using flat interlocking triangular blocks to build a large structure. He has to work out ways of balancing the pieces as he extends the model upwards and outwards.

Campbell: *I'm having a bit of trouble doing this.*
(He is trying to fit the interlocking triangles together.)
Campbell: *Look at that.*
(Campbell has now connected several of the triangles.)
A few minutes pass.
Campbell: *Look at this.*
Stan, look at this. Arrows.
(He has made arrow shapes with the triangles.)

(Campbell continues to comment to Stan as he builds.)
Campbell: *I made even longer than this [sic]. Longer than this room. Over to the windows and under the tables.*
(After a few more minutes.)
Campbell: *Look at that. I'm going to make it even bigger.*
(A few more minutes pass.)
Campbell: *I'm going to make it even bigger.*
(Stan puts away the interlocking cubes.)
Campbell: *(to himself) Guns. Facing out.*

Alister: *I want that one to win and that one to lose.*
See that one losed [sic].
(Alister is loading up the different sides of the balance. He is trying to make one side touch the table while the other side, with very little on it, is up in the air.)

Campbell: *Put that in.*
Campbell: *No, put that one there in.*
(Campbell holds up the model so Alister can attach one of the interlocking triangles underneath.)

Alister: *I'll make that one win next time. (He places another piece on the balance.)*

(Campbell keeps putting the blocks together.)
Campbell: *(to Amber) Be very careful [of the model].*
Amber: *I'm not going to knock it down.*
(Alister puts away the counters and balance.)
(Amber and I have a conversation.)
Campbell: *Stan look at this. I think it's going to take up the whole table.*
(After a few minutes)
Campbell: *Can I just move it over there a little bit more?*
(Campbell lifts the whole model and repositions it on the table.)

Amber: *Look. I did it. I did it.*
(She holds up the stacked seriated tubs.)

For the next few minutes the boys go on working with the equipment in silence while Amber is more interested in continuing to interview me about my life. After some time of working in silence, Campbell comments to Alister about what he has built.

Campbell: *This is the back of it.*

The initial impression is of procedural comments in the main such as Campbell's opening comment, *I'm having a bit of trouble doing this,* comments to the other children of *Look at this* and comments relating to the positioning of the blocks, such as *No, put that one there in.* Other comments, relating to themes of "bigness" and "winning", as the children worked at their different activities reflect the peer cultural context in which they were working. (Higgins 1995). For instance, the focus on "bigness" is evident in Campbell's comments throughout this excerpt such as *I made even longer than this [sic]. Longer than this room. Over to the windows and under the tables.* Alister picked up the theme of "winning", this time applied in relation to the balance, when he talked about, *I'll make that one win next time.*

A fascination with balance is reflected in the way in which the boys, in particular, approach these activities. Campbell's model was finely balanced particularly as new pieces were added. On other occasions the boys were again observed to be experimenting with balance. For instance, when building towers with the rods Alister and Tom discussed which rod at the bottom of the tower they would need to pull out to make it fall down. On another occasion, Campbell shifted an interlocking triangle from the bottom of the structure to the side without threatening the balance. He commented to Tom, *Look at that, Tom. Look at that. I even took out the bottom one.*

The Response to the Video

Amber reminded me on numerous occasions throughout the series of observations that I promised I would come back and show the class one of the videos of their maths session. The day before the last day of the school year I went back for the screening. Despite Amber having insisted that I come back, throughout the screening she appeared to be more interested in playing with the hair of the girl who was sitting in front of her.

However, what was unexpected to both the teacher and to me was the way other children in the class responded to the video. In contrast to Amber's response Campbell, who was usually very reticent, became animated in his response to the sequence of his model building. The teacher commented that she had never seen him like this before - he was quite different to what the teacher had thought. Overall the teacher was surprised by those who were engaging with the video and those who were distracted and restless. Several of the children participated in the videoed discussion as if they were having the discussion again.

The teacher commented that she was amazed that the children could remember so much about the maths session.

The following is an excerpt of the running commentary that the children gave in response to their actions on the video.

> *Alister: (to me) I moved over there cos I wanted you to look at me too.*
>
> *Campbell:* *I didn't really know what it was. I made a stand and tried to balance as many triangles as I could on it (He pauses) Soon I had to start standing up cos you can't see my face now.*
>
> *Teacher:* *(to me) I've never had such a long conversation with him.*
>
> *Campbell:* *I'm standing up [he had to do this to reach the top of the construction].*

Campbell continued to describe the process of making his model in a way which explained what he was trying to do and the reasons for his actions.

The teacher later commented in conversation with me.

> *Teacher:* *I would never have expected that from Campbell. I wouldn't have thought he would come up with that at all. If it had been a triangle [group] I wouldn't have been so surprised. I feel blown away by the experience. He remembered all the details of what he was doing whereas I would have only looked at the "zen" of the experience. I wouldn't have considered him at this level. Look at the number of decisions he was making.*

Discussion

This commentary on the video revealed Campbell's thought processes in a way that was not evident in the transcript recorded as he completed this task where the nature of his statements were mostly procedural. This sequence, while not part of the planned research methodology, raised many questions for me as a classroom researcher, particularly in relation to statements in recent policy documents highlighting teachers' interaction with students. I was left pondering about the teacher's surprise at the quality and nature of this child's explanation. The rest of this paper will discuss possible reasons as to why this video provided this teacher with an insight into Campbell's thought processes and will identify key factors in realising the potential of student talk as an indicator of progress.

In the policy documents cited at the beginning of this paper varying forms of student talk appear to be absorbed under the broad category of communication or interaction. This is not surprising given the purpose of such documents, but does not provide guidance for teachers on the significance of the type of student talk, various ways of hearing this talk and how they might use talk to judge student progress. Thomas (1995) is critical of the too ready acceptance by mathematics educators of the value of talk between children. In an investigation of the nature of the talk that occurs between children working in groups independently of the teacher, she found that the talk depended on a complex interrelationship of the nature of the child, the demands of the task and the features of the classroom context. She suggests that there is a critical balance between social and cognitive demands of the task.

So what type of talk does provide an insight into the thought processes of children completing mathematical tasks and under what circumstances might the teacher observe this? Random observation of short duration, such as teachers can realistically manage alongside their direct teaching responsibilities, may give the teacher a limited idea of the nature of the talk between children, even though the teacher's presence may have a moderating effect on its nature and substance. In Campbell's case, it would appear that the teacher would probably have observed procedural talk in a short random observation. The fact that she thought of Campbell as reticent may relate to the settings in which she observed him, in which he typically said little. Clarke, Clarke and Lovitt (1990) refer to such students as "invisible" and for this and other reasons suggest a systematic approach to informal assessment.

There has been much debate about the links between critical learning processes and private talk. In two recent studies of children's classroom learning Alton-Lee, Nuthall & Patrick (1992) gathered data using remote microphones worn by individual pupils, to investigate the lived culture of the classroom (Apple & Weiss 1983) in relation to children's processes of schema generation. Whilst they caution against over-interpreting these private utterances, they suggest that the utterances may provide the teacher with valuable insights on internal cognitive processing. They suggest three distinct categories of private talk identified according to intended audience as unofficial/no audience, unofficial/ restricted target audience and unofficial/general peer audience. They see the identification of audience as informing an analysis of children's individual negotiations of the lived culture of the classroom. According to these categories the private talk I reported on in this paper would fit `verbalisations accessible to peers but not a public statement' to an unofficial general peer audience.

Palincsar and Brown (1989) working from the Vygotskian tradition see private speech as an aspect of self-regulatory behaviour in the way in which it becomes a verbal directing or self-questioning of the child's activity. They argue that dialogue as an important means of acquiring private speech. In this sense they describe this private speech as being of a dialogical nature, even if the comments are clearly directed at no one. However, Thomas (1995) found few instances of "thinking aloud" in her investigation of talk in junior mathematics. She defined this as a sub-category of reflective talk where the child articulated their reasons for taking a particular action. She notes that this is consistent with Bennett and Dunne's (1992) finding of a complete absence of abstract talk generated by mathematics tasks. In the example in this paper, some of Campbell's comments, such as "Guns. Facing out" would appear to fit her category of "thinking aloud" and be explanations for his placement of the blocks on the model.

What teachers need to inform their planning of classroom programmes is an insight into children's thinking. In another study by Nuthall and Alton-Lee (1995) children were interviewed about rote recall questions immediately and up to 12 months after the administration of an achievement test. Their finding that students had strong recollections of the original learning experiences, led them to suggest that the process of recall is "controlled by the students' metacognitive awareness and understanding of their learning experiences, and it is conditioned by the multilayered and fragmentary nature of their memory for those experiences" (p.219).

It would seem that the video enabled Campbell to relive his original learning experience building a model with the triangular blocks. The commentary on his own actions appeared qualitatively different from his commentary to his peers during the activity itself. Although the teacher had found Campbell to be reticent, in my observation he actually said a lot about the task. Yackel (1995) suggests that children change the ways in which they talk about their activity according to the different situations. From a symbolic interactionist framework her argument is that a child's interpretation of a social event rather than the classroom social arrangements constrain the nature of a child's explanations. She sees this constraint as being reflexively linked to the obligations and expectations that a child associates with a particular social setting. Campbell's response to the video of explaining his actions may well be a result of the expectation and obligation to explain himself that he associated with this setting. In doing so he revealed some of the thinking and decision making that in the other settings was not evident to an observer, be it teacher or classroom researcher. It would seem that the use of video may be one of a number of strategies that can be used to gain insight into children's thought processes as they complete mathematical tasks.

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A methodology for using classroom teachers to gather information about using mathematics from the work place.

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This paper describes an AAMT project which is endeavouring to develop better understandings of what it means for a person to use mathematical ideas and techniques when completing practical tasks. To help us we have collected some snapshots of young people at work. This was done by using teachers, from around Australia, to go into the work place, shadow a worker for up to a day, followed by an interview and then to write up their findings. This paper looks at the processes used, the issues that have arisen as a consequence of doing it, and at the usefulness of the experience.

Overview of the project

The Commonwealth Department of Employment, Education and Training (DEET) have provided a grant, to the Australian Association of Mathematics Teachers (AAMT), to support a project which aims to promote an informed view of the competency *Using Mathematical Ideas and Techniques* (UMIT) to educators in all relevant areas and provide strategies for the effective development of this competency in young people.

UMIT is one of a set of Key Competencies (KCs) developed by a national committee, chaired by Eric Mayer, in 1992 *(Mayer, 1992, p,5)*. The key competencies were seen as "essential for effective participation in the emerging patterns of work and work organisation...also...for effective participation in further education and in adult life more generally"*(Ibid, p, ix)*. A detailed description of the UMIT competency can be found on pages 33-35 of that report.

The AAMT project entitled Rich Interpretation of Using Mathematical Ideas and Techniques (RIUMIT), commenced in October, 1995 and is due to finish at the end of October, 1996. It is one of a number of projects, funded by DEET, with the purpose of exploring the Key Competencies.

RIUMIT has a wide range of broad outcomes which include -

- a description of the competency which is clear and defensible and where its links to other key competencies, school and training mathematics, and numeracy have been made;
- a clear understanding of the relevance of the competency in educational, training and vocational settings and the applicability of the competency to work and life generally;
- documentation of exciting and innovative examples of how the competency is (and could be) articulated in school, TAFE and training curriculums, and of how, in these same settings, the competency is (and could be) learned, assessed and reported;
- a good understanding of the issues surrounding the learning (and teaching), assessing and reporting of this competency making particular reference to the implications for social justice;
- guidelines for the future development of curriculum support materials, school and Vocational Education and Training (VET) learning environments, and for professional development programs;
- shared understandings and a supportive environment for further development work in this area, and a wider group of interested and involved people around Australia.

As an AAMT project the intention is that, wherever possible, the state and territory affiliated mathematics teachers associations will get involved in the project. Hence each affiliated association has a contact person to assist in keeping the local association membership informed and to help with getting the work done in that location.

RIUMIT is endeavouring to ensure that its research is well grounded in the action of the work place and of education and training situations. This paper focuses on the methodology of the work place research, the initial findings from the trial and outlines some of the possibilities that could be achieved from the research data.

The work place research

The main purposes for the work place research are -

- to develop a set of rich descriptions of a range of people operating in the work place in order to increase our understanding of how mathematical ideas and techniques are used in practical situations and support our pursuit of the project outcomes.
- to develop a methodology that would involve as many people from affiliated associations as possible. It was hoped that the methodology itself could be used by other associations when faced with doing research.

The intention was to -

- develop a description of what was happening in work places without imposing a preset view of what mathematics is and hence corrupting the analysis of what mathematics was being used and how it was being used;
- minimise peoples 'fears' of mathematics to reduce the interference with the process;
- be open to new ideas and themes which may arise when viewing the data; and
- establish a research process which could be a learning experience for all involved.

The scope of the work and size of the sample needed to take account of the cost, and broad geographic coverage required. It was recognised that this would also impact on the training of the researchers and, hence, the quality of the descriptions.

The UMIT competency attempts to describe what a person does when using mathematics to assist in doing a practical task. A premise of the RIUMIT project is that if descriptions of the UMIT competency could be developed which exemplify how it manifests itself in practice, then educators, employers and workers will be better able to identify and develop it. Thus one focus of the research needed to be the work place, where the intent is to describe the actual work as it is being undertaken, and then conduct an analysis from the perspective of how mathematics is used. A similar methodology was used by Baturo et al (1990) who investigated the mathematical demands of a number of unskilled occupations undertaken by people with intellectual disabilities. The research team developed rich descriptions of each worker's job by interview and observation within the work place. These descriptions were then analysed for evidence of the mathematics associated with the work place, the informal and formal mathematics identified in the work place and the metacognition associated with the mathematics. Their descriptions and ensuing analysis supported the power of such a methodology.

In our research workers, from a sample of work places around the country, are being shadowed for a half day by a researcher who is looking to describe the work situation. This information will be analysed by a range of people - the worker, the researcher and others associated with the project. The worker will be interviewed to explore issues that had arisen during the observation and to get a more detailed description of the work. The aim of the interview is to clarify what were routine problems, where new and unfamiliar situations had occurred. Where necessary the worker will talk through the processes they use to make decisions about what to do in the routine and non-routine situations identified. The researcher can negotiate with the worker and employer to return for clarification purposes.

Following the visit the researcher will prepare a descriptive story with vignettes of how the worker dealt with one or two 'interesting' situations. The description and vignettes will be accompanied by an initial analysis by the researcher and, where possible, the worker. The meaning of a set of events are subject to interpretation in a variety of ways so multiple perspective's will be portrayed. The worker will be able to read and comment on what was written by the researcher. They will be encouraged to write an analysis themselves. This will be used to develop a deeper understanding of the work. Other people involved in the analysis will be asked to write their perspective of each story.

The information gathered in each case will be analysed to identify the mathematical ideas and techniques that people used, what is involved in using these mathematical ideas and techniques, what meta-cognitive aspects are evident and what attitudinal aspects were important to the story.

The next phase will involve applying the following questions to all of the descriptions. What generalisations can be made about these stories? What are the commonalities? The differences? How are the mathematical ideas and techniques, using them and the other meta skills related? Can they be separated? What education and training will foster this capacity? In what ways might these stories be used for professional development activities? How might these stories impact on our thinking about curriculum, teaching and learning, assessing and reporting?

The researcher is also being asked to reflect and write on the experience of doing the research. This should provide some insights into the extent to which the methodology provided opportunities for learning for the researcher.

At the time of writing it is imagined that there could be four types of product- the stories themselves, a description of how the UMIT competency seems to manifest itself in the contexts described, a report which addresses the key questions described above and the final project report which will bring together this part of the research with data collected from other sources.

A research methodology was trialed and refined in WA in December of 1995 and January, 1996. Affiliated association contact officers were then invited to find five members who were willing to participate as work place researchers. The aim was to generate about 40 stories covering examples from small business, large business, industry, home, community, self employed, volunteers and the like and including types of work such as hospitality, engineering, retail, agriculture, office and clerical, tourism, building and construction and so on. Workers should come from a range of backgrounds. The term 'work' is being used in its broadest sense. It includes both paid and unpaid work in, amongst other contexts, both domestic and community settings. Researcher received a detailed information package detailing the procedures and examples of stories written in the trial.

The process, permission for the public use of the example in project writings and work shops, and for the description to be used in future publications are all negotiated with the workers and employers prior to their agreement to participate. The workers have the opportunity to critique the descriptions produced and their responses will be noted and described. It is anticipated that any differing interpretations by the workers and researchers will be an important additional information for the analysis. In some settings access to what people are doing will limited by safety constraints. This raises the issue of insurance for the researcher while in the workplace. Teaching staff will be covered by insurance provided the work is seen to be part of discharging their duties on behalf of the school. Each of the contracts, if being completed by a teacher, then need to be endorsed by the School Principal.

Issues from the trial

Four secondary mathematics teachers, from an outer metropolitan senior high school in Western Australia, were involved in a trial of the process late last year. After a short briefing they each spent a day in the work place, wrote up their findings and then spent a half day debriefing the experience. Two of these teachers, along with two new to the experience, trialed the process again in February of this year. As well as helping to refine and improve the research methodology the trial also ensured that we take seriously the following interrelated issues.

1. The placement of researchers : In the first stage of the trial it became apparent that locating the researchers and workers needed to be handled carefully. In this stage there were no restrictions as to where the researchers were situated and it was decided that the easiest way to set the exercise up would be for the school industry link teacher to find each of them placements in local industry. Despite the fact that it had been stated that the workers could be doing any sort of work this exercise reinforced how much people's understanding and feelings regarding mathematics intruded on the type of job to be shadowed. All placements were in large industries and all the jobs were middle management professional positions. One personnel officer, from a large mining and processing company, commented that she had looked all over the site and found that the accountant was the only person, who she could think of, who did mathematics in his job. In the second stage of the trial the researchers were asked to locate their own worker to shadow using their own network of relationships. This enabled a much easier entree into the work place. It removed the need for 'middle persons', such as school industry link teachers and personnel officers to be involved in negotiating the placement. It also assisted with the relationship between the worker and the researcher (see below). It was decided to use this method for the work across the country. The researchers found their own placements which were checked, in consultation with the project director, to ensure that a range of types of work and areas of work were achieved.

2. The relationship between the researcher and the worker : Not surprisingly some workers were nervous about being watched and often devalued their own work. One worker said "I'll just be doing the same thing over and over so you might as well go home now". Some seemed to be somewhat intimidated by the presence of a mathematics teacher. One researcher, in the first stage of the trial, said "People seemed frightened of having someone look over the shoulder. They seemed intimidated". Clearly the process needed to ensure that the researchers take the time to build the relationship and that they needed to value the work being done by the worker.

Having the researcher find their own work site, through someone they knew (see above), was the first step in helping the relationship become 'easier'. In order to get some balance in the relationships the researchers were asked to be up front and say that they (the researcher) were there to learn from them (the worker). Both of these aspects were included and emphasised in the second stage of the trial with, it seemed, good effect. One of the researchers involved in both trials wrote "This time around was more relaxed, and more enjoyable".

It was also felt that there may be opportunities for the researcher to help with some of the work and get the worker to teach them about what they do and that this reversal of roles would assist develop equality in the relationships, as well as improve the quality of the description of the work (see below). To this point opportunities for this have not occurred.

The information package given to all the researchers included detailed notes on these issues.

3. Selecting the researchers : It was important to get a range of teachers from different contexts involved. It seemed quite easy to get teachers to volunteer for the trial. There was a danger that they could all end up as male secondary mathematics teachers. Contact officers, of the affiliated associations, were instructed to ensure that - there was a balance between male and female researchers, the researchers came from a range of locations and to get some who were not secondary mathematics teachers. In fact the recruitment of volunteers took somewhat longer than anticipated due mainly to the considerable industrial turmoil in many states and territories.

4. The focus of the shadowing : In the trial one work place set up a 'visit' and invited the researcher to talk to a variety of people throughout the day. Given the project's focus on detailed descriptions of people's work it was decided that the researchers needed to stay focussed on the one person.

5. Collecting the data : It was hard for the researcher to "get inside people's heads". One suggestion was to, if possible, become a participant observer. That is, to assist the worker do the work and to ask them to explain to the researcher how they do particular tasks. This had also been identified as a good idea in helping with the relationship between the researcher and worker. It was reaffirmed that it was more important to document some things in depth than get a sketchy overview of everything. Another suggestion was that the researcher should, at the completion of the observation session, take some time to clarify those elements about which more information was needed from the interview. These suggestions were included in the information package.

6. Training the researchers : It was clear from the trial that ideally all researchers should have a briefing and a debriefing however due to the considerable constraints of time, money, and geography this was not possible. In the place of a training program an information package was supplied to all researchers which included a description of the task, notes on collecting the information, two written examples from the trial, and the contracts that they would be required to have completed. The researcher contract includes a clause which requires them to debrief with a project officer and allows for the possibility of returning to the workplace to gather more information.

Beginning the analysis

At the time of writing detailed analysis of the work place stories was not possible. A snippet from one of the stories collected so far is included here to demonstrate the scope and potential of this work. The full description which includes background information, a summary of observation notes, some vignettes, and the researcher analysis runs to some 10 pages. This story was written by Lisa Thomson and describes the work being done by two women in a health food store.

The Health Food Shop Workers by Lisa Thomson

Vignette

Weighing the dry goods - the weighing scales (which are electronic ones, calculating the cost of the items after a code is keyed in) are situated on the other side of the shop to the bulk goods, so 'D' partially filled a bag with nuts and brought it over to be weighed. Her guess was accurate to within 50 grams. She was consistently close to the required amount with each different item, which indicates that she has become very familiar with the weights of these goods. When asked whether some items were more difficult to estimate than others (for example, was the weight of small grains harder to estimate than that of large nuts?), she said that it wasn't how full the bag was that mattered, but how **heavy** it felt. She added that she had gotten to know that, for example, one scoop of sunflower seeds weighed about 150g, whilst one scoop of bran flakes was only about 50g, and that it was easy to calculate larger amount knowing these basic measurements.

Researcher analysis

Another impressive display of these estimation skills was seen in the weighing of bulk nuts/cereals/grains. To be able to select enough to be within 50g of the required amount, and to be able to do this consistently, regardless of the size or weight of the product, is surely a valuable tool in this industry. 'D' brushed this comment aside, saying that over time, she had just gotten to know how much one scoop of most things weighed, and that she just multiplied accordingly for larger amounts - but what is a **full** scoop? what is a **half** scoop? The scoops were the common, sloped, round edged ones used by most bulk goods outlets, and I found myself wondering about all the different combinations of 'fullness' that could be found amongst the workers just in this one store. When watching each of them, it became obvious that each had a different idea of 'full'. Repetition must play a large part in learning the weights etc of products, but each person had developed her own standard against which to judge and compare all others.

Figure 1

Even in this small slice of work in a health food shop we begin to get some insight into how this person uses mathematical ideas and techniques to assist in the completion of practical tasks. The final KC description of UMIT (Mayer, 1992a, p, 35) summarised the competency as involving the

"• clarification of the purposes and objectives of the activity;
- selection of mathematical ideas and techniques;
- application of mathematical procedures and techniques;
- judgment of level of precision and accuracy needed; and
- interpretation and evaluation of solutions." (Mayer, 1992a, p,34)

Elements of these criteria exist in the story described in figure 1. The worker is clearly competent, in this context, in using particular mathematical ideas and techniques successfully to complete the task of giving the customer the required amount of goods. She knew what was required of her to complete the task. She had to call on her understandings of number and weight, the idea of comparison, and the technique of using a standard measure to calculate larger quantities. She then applied the ideas and techniques reliably and efficiently to a required level of precision and accuracy. She was able to check that the answer made sense in this context.

However these criteria do not seem to capture the full richness of the performance of the worker. In particular the way that the worker had developed her own idea of a standard measure. This is an essential skill - whether or not the worker actually used 'heaviness' of a scoop as the standard measure (as she claims) or whether she had just learnt how much a full scoop of each type of food contained and remembered it. Other Key Competencies (eg working with others), and attitudinal aspects (eg the motivation to get it right for the customer) are also evident in this example.

The story also hints at the differences between experiences of learning in the work place and in educational settings. Resnick (1987) refers to these differences as working with others when you need to rather than on your own, doing rather than just thinking, contextualised reasoning rather than symbol manipulation and situation-specific learning rather than generalised learning. How did the worker in this story learn to be competent? Did she learn, in the first place, from a colleague? What role did 'doing' play?

It is hoped that the full range of workplace examples will enable us to explore

- the meaning of the UMIT competency,
- the relationships between the UMIT competency and other elements such as the other KCs, metacognitive and generic skills, and attitudinal aspects which could be involved in being competent, and
- how learning in the workplace occurs.

Some reflections of the teachers involved : Given that it was an intention that this work would involve teachers and that the experience be something of value to them the researchers have been asked to write on how doing the research impacted on them. Some of the comments made by the teachers involved in the trial have been included here. There is no sense in which these particular reflections are being supported or endorsed by the writer.

The teachers involved have talked of the improved understandings they had of the work place. They were interested in the way problems were tackled and solved. The larger industries, it seemed, were putting a real emphasis on teams to make suggestions about what needed doing and that the team had to discuss problems and make decisions together.

Some of the researchers were struck by the peculiarities of some work places where workers used their own way of doing things 'because they work'. It seemed that these workers don't really understand why their methods work (nor care). They observed that some workers had built up their own systems for doing things over a long time by trial and error.

The researchers were all confirmed by the amount of mathematical ideas and techniques that were being used in the work place (being a mathematics teacher might be important after all!). There was considerable discussion, at the trial debrief, about their observations of the level of accuracy in calculations (or not!) which occurred in the different contexts. With one work place having very liberal tolerances in their calculations and truncated everything while another put cost efficiencies as vital and calculated everything to the nth decimal point. One researcher talked of the "insight gained to technology in the work place, and the use people made of this technology".

Some have written of the misconception they had of what they expected in the work place. "My preconceived idea was that you would have to know a lot about money to work in Real Estate but Jenny said you could refer to tables and charts and then refer people with more expertise in these fields when conveyancing was necessary".

Some have begun to ponder how their observations in a work place might impact on their own classroom practice and the experiences that they provide for their students. "It seemed to me that the workers were engaged for about one third of their time in getting on with others, one third of their time in solving new problems and one third on interpreting information. It seemed to me that I should break up my lessons in class in the same way." Another researcher noticed the importance of "the ability to follow a set, standard procedure" but also that "the ability to modify the procedure to suit individual preferences, while ensuring that the job is done properly, is also important." He went on to say "This job would be pretty boring to me but didn't seem to be for S and the others that worked there....Teaching should perhaps be a blend of 'boring' and 'challenging/thinking' work. Maybe I should do less challenging and more steady, routine type stuff. A bit of hard thinking followed by practise." A third researcher went on to talk of the need for workers to learn how to manage change and, hence, that learning should focus on the process of learning.

Clearly such comment depends on the workplace visited by the researcher as it does on the world view of the researcher. The comments of the researchers in their writing up of the story could also be open to analysis and conjecture. The researcher's observation of the work, self reflections about the tasks being observed, and comments made in their analysis could prove illuminating about their relationships with the curriculum, their students and the workers.

Conclusions

The majority of stories will be completed by the end of June, this year. As the stories are collected they will be distributed to the project steering committee for review and comment. They will then be analysed based on the key questions set out in this paper. This data will be put together with the information collected from school and VET settings in order to attend to our project outcomes.

It is expected that the research will be able to provide insights into how people use mathematical ideas and techniques to do practical tasks. This information should be of great interest to those who are currently looking into the numeracy requirements of school and training systems and to people involved in research into situated cognition and mathematics education.

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The views expressed here are those of the writer. They do not necessarily reflect the views of the AAMT or the affiliated associations and do not necessarily represent the views of the Commonwealth Department of Employment, Education and Training.

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Change Trajectories in Teacher Professional Growth

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Professional growth is a form of learning. As such, it is not surprising that the response of a teacher to a professional development program is a very individual one. This paper reports the professional growth of two teachers involved in the same professional development activity, and with many common elements in their personal histories and work situations. However, the teachers' responses to the professional development program were significantly different with respect to: the level of classroom experimentation (both reported and observed); the degree to which each reflected on their teaching and the consequences of the program; the changes they reported in their practices and beliefs; and their engagement in collaborative activity with their colleagues. The challenge for the researcher investigating teacher professional growth is to find a suitable mechanism to describe the growth process. This paper employs "change trajectories" as a suitable descriptive device. The metaphor of a trajectory offers an image of movement over time, and draws attention to issues like: the dimension along which change occurred; the rapidity of the change; and the factors facilitating and inhibiting the change. In the case of the two teachers reported here, the change trajectories are sufficiently different to provide a useful insight into the individuality of a teacher's response to a professional development experience.

Historically, academics and policy-makers have sought strategies by which to effectively reform the curriculum. Such strategies include: the development of innovative curriculum materials; new assessment initiatives; and, new professional development activities and programs for teachers. The calls for reform of the teaching and learning of mathematics over the past ten years have led to professional development activities assuming an increasingly important part of mathematics teachers' lives.

Whilst much research has led to common agreement about the complexity of the change process, and growing agreement about features of effective professional development, there is more that needs to be known about teachers, professional development and change (Sparks & Loucks-Horsley, 1990). The intricacies of the change process are not clear, and so recommendations for improving professional development to assist teacher change in mathematics are difficult to make (Brown, Cooney & Jones, 1990). As Clarke (1993) reports, "A question of interest now is the order in which the change process occurs, and the nature of the interactions between the variables involved. A greater understanding of these aspects is essential if staff development programs are to make a major contribution to teacher professional growth" (p.46).

With these considerations in mind the study reported in this paper sought to unfold more about the process of change for teacher participants of a mathematics professional development program. In particular, the study was designed to examine teacher growth through the investigation of changed practice, both observed by the researcher and reported by the teachers, and the identification of factors that influenced the process of change for the teachers.

In establishing a descriptive framework for the study, the researcher made use of the Interconnected Model of Teacher Professional Growth (Teacher Professional Growth Consortium (TPGC), 1994) as a model for examining the process of change for the teachers involved in the professional development program. The Interconnected Model, presented in Figure 1, is a model of Professional Growth. This may be influenced by Personal Activity or Beliefs or it may, in turn, be an influence on Personal Activity or Beliefs. In this study the Interconnected Model is used as a means for describing change in the teacher's Curricular Activity. Personal Activity and Personal Beliefs are both facilitating and inhibiting factors which influence change in the teacher's Curricular Activity, and may also be changed in turn as a consequence of teacher growth in such aspects as an increased valuing of reflective activity and an increased professional self-esteem. Curricular Activity in this study is described in terms of three components of teaching: Teaching Strategies, Use of Resources, and Assessment Strategies.

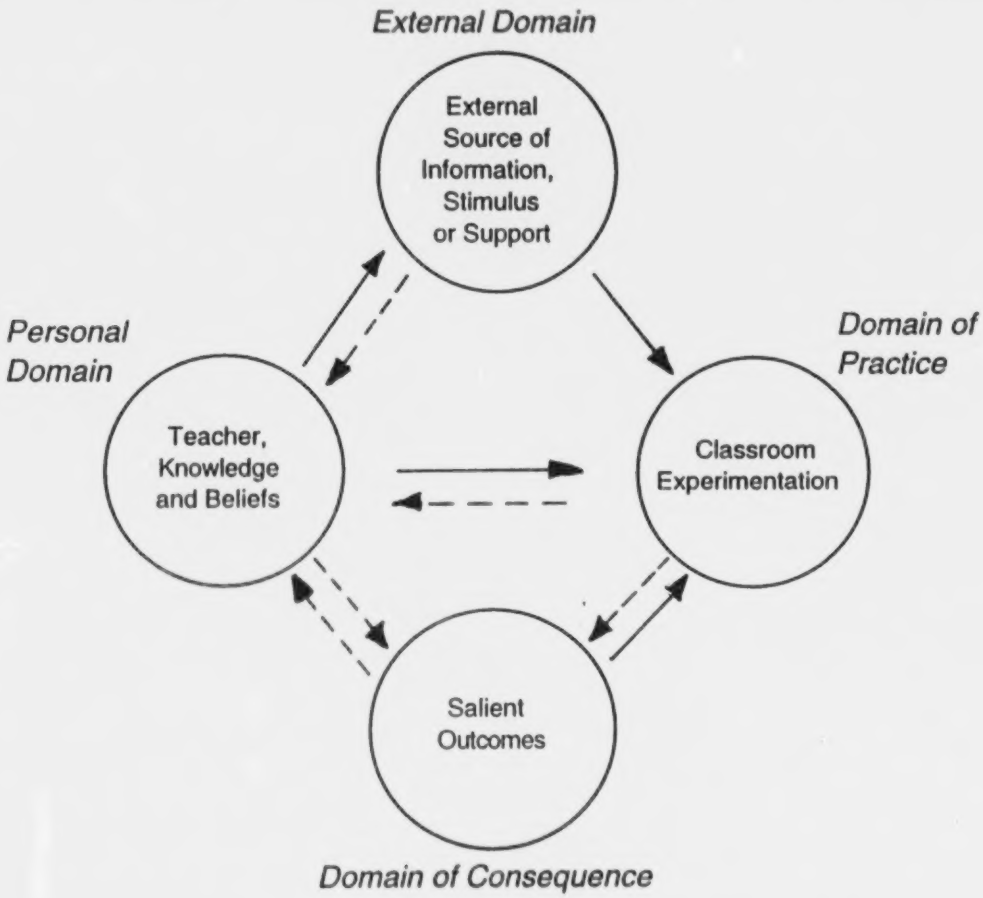
External Domain
External Source of Information, Stimulus or Support
Personal Domain
Teacher, Knowledge and Beliefs
Domain of Practice
Classroom Experimentation
Salient Outcomes
Domain of Consequence

Figure 1. An interconnected model of teacher professional growth (solid line = *enactive* mediating process; broken line = *reflective* mediating process). (TPGC, 1994)

This paper reports some aspects of the professional growth of two teachers involved in the study and introduces the use of "change trajectories" as a suitable mechanism for describing the growth process.

Methodology

The study involved a detailed investigation of the process of change for teacher participants of the Victorian mathematics professional development program, Exploring Mathematics In Classrooms (EMIC). The EMIC program was designed to affect the practice of teachers by encouraging them to reflect on their current practice, read appropriate professional literature, and trial effective teaching strategies in their classrooms. It was the aim of the EMIC program not only to inform teachers of current trends in mathematics education, but also to encourage them to develop and expand their repertoires of effective strategies and practices (Ministry of Education, 1990).

Data was collected from six teacher participants of a ten week EMIC program. The teachers were interviewed and observed, both during their EMIC participation and in their classrooms, over a period of 18 months, beginning at the commencement of their involvement in the program. Documents and teaching records of each participant were also examined. The study traced the perceptions and practice of the teachers in relation to their Curricular Activity from their initial involvement in the program through to a period of consolidation and refinement 12 months after completing the program.

Classroom observation record charts were completed for each observed lesson. Interviews were transcribed and teacher statements were categorised using a coding system developed to classify components of the teachers' Curricular Activity. Field notes were also generated in relation to a selection of the EMIC workshops, but this data is not reported in this paper. In addition, field notes were maintained relating to informal conversations with the teacher-subjects. Teaching documents, such as work programs and assessment records, were also collected.

The data were rigorously analysed from a variety of perspectives derived from the key components of Curricular Activity. The intention was to identify structure and relationships within the data. It was in this fashion that the potential significance of certain scales, such as Range of Resources or Diversity of Teaching Strategies, was recognised. The notion of change across a variety of dimensions suggested the use of "trajectories" as a suitable metaphor for describing the process of change, and as a powerful method of communicating the data arising from this study.

Results

Consistent with the descriptive framework employed in this study, Curricular Activity data were collected with specific respect to Teaching Strategies, Resources, and Assessment Strategies. The results reported in this paper focus on the professional growth of two teachers, Geoff and Paula. While both teachers experienced the same professional development activity, and had common elements in their personal histories and work situations, their responses to the professional development program were significantly different with respect to: the level of classroom experimentation (both reported and observed); the degree to which each reflected on their teaching and the consequences of the program; the changes they reported in their practices and beliefs; and, their engagement in collaborative activity with their colleagues. Some of the key differences between the responses of the two teachers with respect to each of these are summarised below. Two forms of activity are distinguished and different representational forms are employed to report the analysis of each activity type. Practical Curricular Activity is the outward evidence of professional growth, and, as such can be expected to change incrementally. Classroom experimentation, teacher reflection, changing teacher practices and beliefs, and teacher collaborative activity can be identified with particular elements in the Interconnected Model. Whereas Curricular Activity is clearly an outcome of professional growth, these other elements can be seen as characteristics of the growth process. it is these characteristics that are discussed first.

Level of classroom experimentation

Both teachers engaged in some classroom experimentation throughout the study. In interviews, Geoff was much more likely to discuss his experimentation than was Paula. It was certainly true that Paula had a more extensive repertoire of teaching strategies at the commencement of the EMIC program, than did Geoff. Analysis of classroom observations and teaching records suggest that over the period of the study, Paula and Geoff engaged in a similar range of teaching strategies and with a similar level of frequency. This is an important point for researchers investigating professional growth. Whose professional practice is more praiseworthy: the teacher who continues to maintain a reasonably varied teaching repertoire, or the teacher who moves from a comparatively impoverished range to one that is more extensive? It may be that the limited nature of Geoff's initial teaching repertoire made him more inclined to discuss this aspect of his classroom experimentation. Certainly the researcher gained the impression of a greater willingness to experiment from Geoff, than from Paula. Both teachers explored the use of a similar range of teaching resources. With regard to the range of assessment strategies employed, Geoff was observed (and reported) using a wider variety of assessment techniques than did Paula. The visibility of Geoff's experimentation was, in part, due to his willingness to discuss, in interviews and informal conversations, his use of new techniques.

Level of reflection

From the commencement of data collection, it appeared that Geoff was much more likely than Paula to reflect on his teaching practice or on his experience of the professional development program. Paula did not make statements that suggested she was reflecting on either the program's content or her classroom practice. These topics were only addressed in response to questions from the researcher, whereas Geoff would volunteer observations about his teaching and his use of ideas encountered through the program. An example of this occurred during Classroom Observation 2 as Geoff's students worked with concrete materials on a number of tasks related to fractions. He commented about the "great things" that they did the day before, and how he felt that they were really "looking at the sizes and comparing them" (Field Notes - Observation 2). He noted that one student who usually had difficulty staying on task when more rigid guidelines were set [as he inferred was the case with his previous mode of teaching], could actually "work on this activity all day". It appeared that these positive salient outcomes were impacting on Geoff's view of his mathematics teaching as he was involved in experimentation with strategies recommended in the EMIC program.

Changing practices and beliefs

Geoff reported change in practices and beliefs with regard to all three areas of Curricular Activity: Teaching Strategies, Resources and Assessment.

> I use groups and pairs and things like that rather than just this is how you do it, putting 30 problems up on the board and saying open your books and go for it. So a lot of it's different... just a whole lot of things to make maths more interesting. (Interview 3)
>
> The kids are using more concrete materials in maths now as a result of me doing that [EMIC]. Things like fraction kits and cuisenaire, and just anything that's going to be of benefit for what we're doing rather than having everything in the abstract... my starting point is different. Before I was the type of person that if I had a maths course or a couple of maths books, I'd tend to start at page one and work through. Now if I'm doing like that fractions unit, I'm more likely to go in search of maths books and text books and look through to see is there anything in these books that's relevant to what I'm doing? So now I'm looking for anything I can find that might be of help to what I'm actually doing. So I'm working in the opposite direction I think, coming from a different place. (Interview 2)
>
> I always thought that the way to assess children's performance in mathematics was to give them a test and whatever they got, like if it was worth 20 and they got 19 they had done really well, and if they got 10 they had done rotten, which is a very old style. Now I've realised that important things are observation, and there are so many more things to evaluation than just a score on a test. So when my kids are working in pairs or groups I spend as much time watching the children, the way they interact with one another and I tend to have a list and make comments about things that they do just for a personal record. I think that's as important as anything else that they can do. (Interview 3)

He reported that his whole approach to teaching mathematics had changed considerably since his involvement in the EMIC program.

> Well when I first spoke to you about this I was always extremely formal. My maths teaching was always really formal. Obviously I've learnt that there are better ways and more interesting ways to teach maths.
> (Interview 3)

These reports appeared to be consistent with observations made in Geoff's classroom over the period of the study. He demonstrated changes in his practice in each of the three areas of Curricular Activity.

Paula, on the other hand, whilst demonstrating some experimentation in each of the three areas of Curricular Activity in classes observed, reported change only with regard to her use of Resources.

> Using things around the room, and things around the school, more of that...
> I think I'm more aware of what's around me, and I make use of those. (Interview 3)

She also, however, reported an increase in confidence in relation to Assessment.

> I'm certainly more confident in that area [Assessment]. (Interview 3)

Level of collaborative activity

It was noteworthy that Geoff would make repeated reference in interviews and in informal conversation to his increased level of collaborative involvement with other staff.

> Well, Patricia and I, we do all our planning together, if not at school, then out of school. We're doing a totally integrated program now with whatever we're doing. We go through each particular unit of work that we're going to do and then we select the maths that's relevant to that unit. Then by discussing it and also using Cathy [EMIC Tutor] as well, then we come up with different ideas that we can use. (Interview 3)

> There's a lot of sharing here. It happens all the time. (Interview 3)

This change in Geoff's behaviour was significantly obvious for it to be reported by the Program Tutor.

> I worked very closely with Geoff... I look at him now and he's right in teaching with Patricia... and he's really holding his own... he now makes a very sound contribution school wise... (Tutor Interview)

Paula mentioned the availability of assistance and support from other staff, however there are no data to suggest that she took advantage of this directly, during the course of the study.

Change Trajectories

The challenge for the researcher investigating teacher professional growth is to find a suitable mechanism to describe the growth process. If we consider Curricular Activity with respect to each of the three identified components of teaching practice, we can locate a teacher's Curricular Activity on a notional scale with respect to each of Teaching Strategies, Resources and Assessment. One attempt to do this is set out below. The Curricular Activity of both Geoff (G) and Paula (P) are located on each of the scales shown. Pairs of scales are employed in order to make clear the extent of change over the course of this study.

Teaching Strategies

Narrow Repertoire Diverse Repertoire
During Program
G P
10-12 months later
P G

Use of Resources

Narrow Range Diverse Range
During Program
G P
10-12 months later
GP

Assessment Strategies

Narrow Repertoire Diverse Repertoire
During Program
GP
10-12 months later
P G

While the scales above give some idea of the nature of professional growth for each of the teachers, the approach is one of snapshots in time, rather than conveying a sense of a change process. Since data were collected at four points over the period of the study, it is actually possible to give a more continuous picture of this process of professional growth. The metaphor of a trajectory offers an image of movement over time, and draws attention to issues like: the dimension along which change occurred; the rapidity of the change; and, the factors facilitating and inhibiting the change.

Figures 2, 3 and 4 extrapolate from the scales employed above to show graphically how each component of the teacher's Curricular Activity changed with time during the course of the study. In each Figure the solid line represents changes to Paula's Curricular Activity and the broken line represents changes to Geoff's Curricular Activity.

Discussion

As can be seen in the change trajectories presented in Figures 2, 3 and 4, Geoff appeared to experience the most significant change in each area of Curricular Activity investigated. His professional growth with regards to Teaching Strategies and Assessment was particularly rapid, whilst Paula's most significant area of change was in her Use of Resources.

A range of supporting and inhibiting factors may have impacted on the process of change for Geoff and Paula. Whilst as noted earlier they had many similarities with regards to their personal histories and work situations, each one brought to the professional development program their own needs and interests, and as would be expected each responded to the program ideas in their own way.

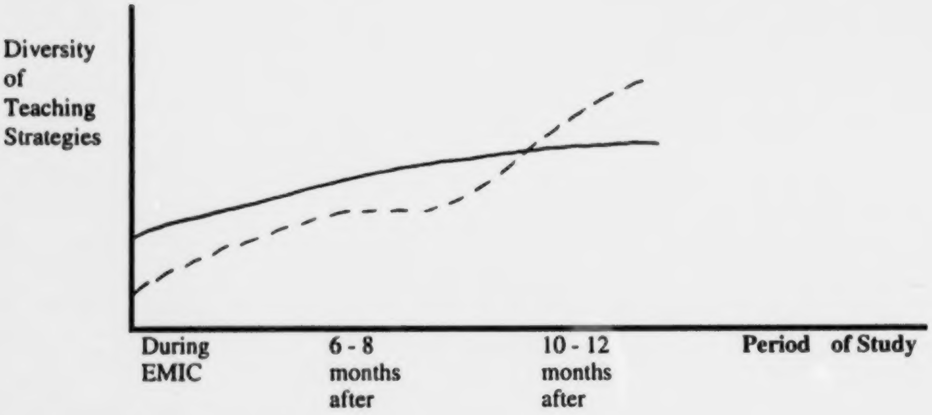

Figure 2. Changing Teaching Strategies

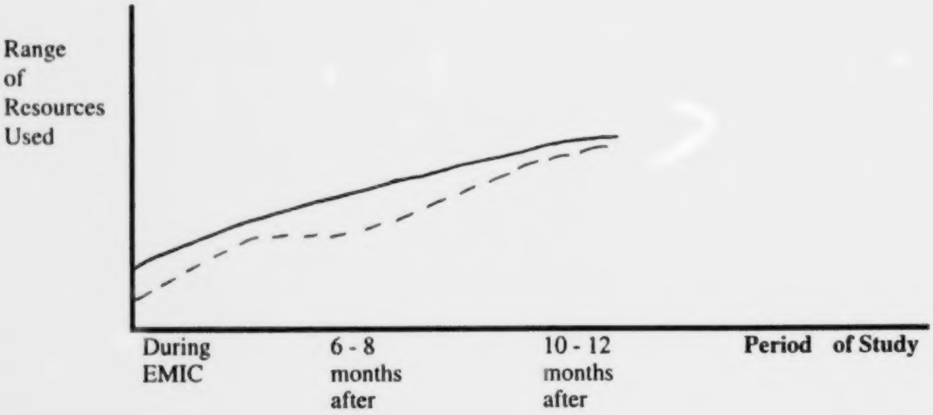

Figure 3. Changing Use of Resources

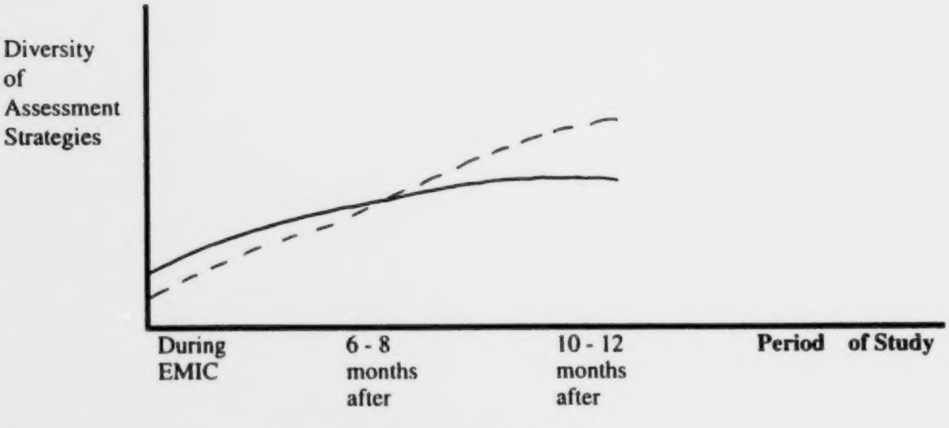

Figure 4. Changing Assessment Strategies

Factors associated with the teaching context and the professional development program context were very similar for each teacher, so it could be assumed that differences with regards to the changes in their Curricular Activity were the result of each teacher's own Personal Activity and Beliefs as they responded to the EMIC program.

For Geoff, two associated factors appeared particularly significant in supporting his professional growth throughout the study. The first was related to his motivation for involvement in the professional development activity. Whilst he was initially reluctant to participate in the program due to personal constraints such as the need to commit time to attend the program sessions, he regarded his mathematics teaching as problematic, and the EMIC program provided him with a stimulus to reconsider his mathematics teaching methods and beliefs. A further associated factor that appeared to support his professional growth in each area was his "inclination to reflect" on both the outcomes of his classroom experimentation and the content of the professional development program. As he explored new territory in his mathematics teaching, and reflected on the impact of that exploration, he became excited about the outcomes for both his students and himself, and he sought further ideas, strategies and resources to explore.

Whilst Paula was also coerced into participating in the EMIC program, she did not regard her mathematics teaching to be as problematic as Geoff did his. Paula explored many of the ideas presented in the program and reported that she considered the program useful and beneficial. However, she did not appear to reflect on her classroom explorations with the same level of detail as Geoff, or report the outcomes of her explorations with the same level of satisfaction or enthusiasm as he did. Paula appeared to regard the ideas presented in EMIC to be a useful addition to her usual mathematics teaching and learning program.

Conclusions

The teachers described here differed in key aspects of the growth process and in the specific outcomes arising from that process. Those of us who are interested in promoting teacher professional growth, need new ways to think about the process of growth, new models, and new metaphors. In this paper, I have explored the potential of the metaphor of a "change trajectory" as a tool for helping me to understand and compare the changes I observed in the Curricular Activity of two teachers. I feel that, in the case of the two teachers reported here, the change trajectories are sufficiently different to provide a useful insight into the individuality of a teacher's response to a professional development experience. Certainly I will continue to apply this method in seeking to understand differences between individual teachers' approaches to professional development. It will be interesting to see whether other researchers find value in this approach.

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Aboriginal Educators' Views Concerning the Learning and Teaching of Mathematics

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There has been no reported investigation in mathematics education involving the views of Aboriginal educators about the learning and teaching of mathematics for primary school Aboriginal students. This paper reports on such an investigation. It concentrates on two interviews with Aboriginal Education Assistants teaching in a geographically remote primary school in New South Wales. These two Aboriginal educators identify the importance of personal relationships, relevance and humour in the learning and teaching of mathematics to Aboriginal students. They set a pedagogical agenda for teachers in their interaction with Aboriginal students and communities as they facilitate the learning of mathematics.

Introduction

Across Australia, colonisation has destroyed much of the Aboriginal and Torres Strait Islander culture. Until recent times, anthropological studies had inferred that Aboriginal people had little if no mathematics in their culture (Crawfurd, 1863; Dixon, 1980; Blake, 1981). Over the last 208 years the Aboriginal peoples of Australia have experienced conquest, integration and assimilation. The learning of school mathematics by Aboriginal and Torres Strait Islander students has taken place in this context. Until recent times, little attention has been given in Australia to the contexts in which Aboriginal students learn mathematics. Social and cultural factors influence the development of people's beliefs, attitudes and values about mathematics and it is in the environment of the mathematics classroom that these factors impact upon the learning and teaching of mathematics (Lerman, 1994). Teachers need to be sensitised to the fact that Indigenous people and people from varying cultural backgrounds may have a view of mathematics and mathematical concepts quite different to those contained in the accepted mathematics curriculum (Rinni & Rinni, 1992).

Mathematics and Cultural Influence

Many mathematics educators hold the view that students bring *"with them different competencies in, and conceptions and beliefs about mathematics"* (Mtetwa & Jaji 1992, p.7). The view that people develop their own perceptions about the nature of mathematics is certainly different from a universal notion of mathematics. All cultural groups have developed their view of mathematics and its relationship to the events that happen within their lives (Bishop, 1988; Lave, 1988; Mtetwa & Jaji, 1992).

It is necessary to investigate Aboriginal peoples' views of mathematics, in order to be able to build effective bridges from the *"indigenous mathematics to the new mathematics to be introduced in schools"* (D'Ambrosio 1986, p.6). In undertaking an investigation of Aboriginal people's views about mathematics one needs to include an appreciation of:

* the social, cultural and historical diversity of Aboriginal communities,
* respecting and valuing Aboriginal people's views of learning,
* listening to Indigenous people's views towards learning,
* the importance of recognising Aboriginal people's views of mathematics,
* how Aboriginal people relate to mathematics, and
* how Aboriginal people use mathematics.

As displayed in most curricula, there appears to be only one mathematics. There are several mathematics. However, there are social and political implications for cultures who do not own the one mathematics as prescribed in a set curriculum. Each culture has developed its own view and concepts of mathematics. Their world view of the place of mathematics in the environment about them has influenced their thinking about mathematics. Such thinking leads to a set of mathematical beliefs, attitudes and values. As cultures become more aware of each others' views they further develop their own view of mathematics. As individuals learn, evaluate and discard notions their view of mathematics develops, modifies and changes. A cultural view, as an individual's, will evolve with time.

In some communities there may well be several views of mathematics held by differing groups. Aboriginal people may hold many differing view of mathematics than other groups within the community. The diversity of Aboriginal groups within Australia may have resulted in many views of mathematics being developed. The views held by coastal communities may differ from inland groups. Those views held by rural people may differ from urban and geographically isolated communities. Many of these views may differ from non-Aboriginal groups within Australia.

Much of the present teaching of school mathematics, particularly in the primary years, has Aboriginal students doing mathematics that is not related to their world and their everyday experiences (Graham, 1988). By the time that many of these students have reached the latter years of primary school they have been alienated from mathematics. The students can do the sums, addition, subtraction, multiplication and perhaps division but they do not feel in control of the mathematics and particularly the language of mathematics that is being used. School mathematics is taught and learnt through a language that has been developed by the dominant culture. Often student success is dependent upon and measured by how well the meanings of this language of mathematics are grasped.

Conflicts between groups who have differing world views of mathematics may be identified by one group and not accepted or even seen by the other (Barton, 1992). Such conflicts can give rise to one world view of mathematics being seen as superior to the other. This latter view was often the approach to mathematics taught in colonised countries. The mathematics achievement levels of people in such countries has been acknowledged as a major concern.

In not appreciating Aborigines' views concerning the nature and conceptual notions of mathematics and a lack of consideration of Aboriginal learning styles in the varying contexts in which Aborigines live little value, in the past, has been given by curriculum developers and educators to their world view of mathematics. By not allowing Aboriginal or culturally based education into the mathematics curriculum, Aboriginal students have been disadvantaged both in their schooling and in the development of their own ideas and patterns of mathematical thinking (Mtetwa & Jaji, 1992). More respect and regard needs to be given to the mathematics of Indigenous peoples and the issues related to their learning of mathematics in the various education sectors.

Through not :

- appreciating the nature and conceptual notions of mathematics of Aborigines,
- valuing their learning styles and the context in which they lived, and
- allowing culturally based education into the mathematics curriculum

Aboriginal students have been disadvantaged in their schooling through teachers not appreciating their understanding of the nature of mathematics and in the lack of development of relevant mathematics curricula. Aboriginal people have called for appropriateness and relevance within the curriculum as they acknowledged the role of the dominant society in curriculum development and control of the curriculum content.

> *The content element of curriculum development is given much attention by sensitive non-Aboriginal teachers as well as Aboriginal educators. There is a constant need to evaluate the quality, accuracy and appropriateness of the content of material used to teach about Aboriginal society and to teach Aboriginal children. A critical analysis must be maintained at all costs. It needs to be recognised that curriculum is shaped by the dominant society, which in this country is now Anglo-Saxon .*
>
> (National Aboriginal Education Committee 1985, p. 8)

The introduction of Aboriginal learning styles into schools and the mathematics classroom needs to be done by and in co-operation with Aboriginal people. Such co-operation between the community, students and educators can help bridge the difficult social and learning experiences that many Aboriginal students face in learning mathematics. Through being involved in the curriculum decision making processes of the school Aborigines bring to the school their perspective and their identity.

For many Aboriginal and Torres Strait Islander children the knowledge they bring with them to school and the language they use is not part of the culture of a Western based mathematics class. Yet, it is too simplistic to suggest language and cultural factors are the only reasons that Indigenous students are not achieving their potential in mathematics. *'Aboriginal children are being taught mathematics in our schools, but they are not learning the things that matter. Such knowledge is not just to do with getting sums right, though that is part of it. Rather, it is to do with the way people talk and think about what they know'* (Graham 1988, p. 132). Time needs to be provided for children to develop mathematical meaning through action in real life experiences, in discourse about those experiences and in reflection of what they have learnt needs to be provided.

Teachers and curriculum developers are becoming more aware of the cross-cultural situations that exist in our classrooms from pre-school to tertiary level. Harris (1991) has indicated that there are difficult challenges for the teacher to meet when teaching mathematics in cross-cultural situations but they have to be met and well answered for the continued empowerment of Aboriginal people.

There needs to be *'in class'* and *'out of class'* investigations of the sociocultural contexts in which learning mathematics takes place. One suggested area of research has been *'to obtain a more elaborate picture of teachers' knowledge, beliefs, judgements, and decisions as they apply to their diverse student populations and as related to their notions of equity'* (Secada 1992, p. 22). This comment has been extended, in this study, to investigate the views held by Aboriginal students, educators, parents and their teachers towards the learning and teaching of mathematics and the reporting of those views in the terms used by the people interviewed.

Methodology

In New South Wales primary schools there are a number of Aboriginal people employed as Aboriginal Education Assistants (AEAs). AEAs are appointed to schools to assist with the teaching of Aboriginal students and the implementation of Aboriginal perspectives across the school curriculum. The number of AEAs in any one school depends on the number of Aboriginal students enrolled.

This paper draws on data collected during 1994 which was part of an nine month ethnographic study involving Aboriginal students, parents educators and teachers living in a remote country community in New South Wales. Over a period of 6 years the author had come to know the community and be known by the Aboriginal educators working within the schools (Cutmore & Howard, 1995). Conversational interviews were conducted with 11 participants. This paper reports on the interview conducted with two Aboriginal Education Assistants, Sandy and Phil [pseudonyms]. For this interview, a set time was agreed upon and the interviewees selected the location. Sandy and Phil preferred to be interviewed together and agreed for the interview to be audiotaped.

Analysis of Aboriginal Educator's Interviews

Interviews were transcribed and reviewed using a constant comparative method (Glaser & Strauss, 1967). From the transcript reviews, seventeen categories were identified and used to analyse all the interviews. Not all seventeen categories were identified in each interview. There were six categories identified in Sandy and Phil's interview.

Relevance: identifies comments related to how meaningful participants find mathematics to them.

Emotions: relates to me and mathematics. This category identifies how an individual feels about mathematics, thinks about mathematics and how it relates to them.
Language: identifies those comments related to role of language and mathematics.
Contexts: identifies issues related to the social, cultural, economic, historical and political contexts in which the learning of mathematics takes place.
Teaching: identifies comments related to the organisation of teaching and learning activities.
Family identifies concerns and values that are held by family members Concerns: towards learning and mathematics.

Relevance

Sandy and Phil believed that mathematics was very important for Aboriginal children. Phil spoke about the fact that mathematics was all about you, yet often the students did not know that the things they see about them involve mathematics. It was obvious that both Phil and Sandy could identify many instances of the presence of mathematics in the lives of the students.

*Phil: We do maths everyday. We don't know that we do. When you look around a room. What shape is that there? Window obviously a square. What shape is a car? We say like a box. Things like that. But mate they don't know that the things they see around is involved with maths.
*Sandy: Just walking to school.
*Phil: Navigating their way home.
*Sandy: Then at home like cooking, you know.
*Phil: Putting two eggs into the cake or something like that.
*Sandy: They don't realise .....
*Phil: ... that's maths.

Sandy and Phil thought it would be good for children to go into the community to identify the mathematics being used. This would be very different to what happened in the mathematics at school. .

*Sandy: It would be good for the kids to go out into the community where they could see things and maybe to explore.
*Interviewer: Do they do much now?
*Sandy: No not really. It's mainly just in the class. All you see is stencils.

Emotions

Phil and Sandy did not know if the students worried about mathematics but they did believe that the mood that a child was in when they came to school influenced their learning of mathematics. Phil said that students would use strategies to avoid doing mathematics and identified some that were used.

*Interviewer: Do kids here get worried about maths?
*Sandy: I don't know really.
*Phil: It depends on how they come to school really isn't it? If they come to school in a good mood I don't think they'll worry about it. But if they come in a bad mood they'll chuck a sickie or go to the toilet, ask can I go and have a drink during the lesson and then they miss out on an important part.
*Sandy: Yeah it probably depends on their mood.

Language

Sandy believed that Aboriginal children were good at mathematics but that it was the language in the form of long words, students' lack of understanding of the words that are used when they are writing that makes mathematics hard for them.

*Interviewer: The long words are a problem?

*Sandy: Yeah.

*Interviewer: In what ways do you think?

*Sandy: Probably because they don't understand them.

*Interviewer: That's a big issue. The kids are good at maths.

*Sandy: Yeah.

*Interviewer: But they find it hard when it's writing?

*Sandy: Yeah, you know the words that are used and that's where they fail because they don't understand.

Sandy believed that the language of the test was a major reason for Aboriginal children failing in mathematics tests.

*Sandy: ...a lot of them find it hard to do especially with the maths cause of the language, you know they fail. A lot of kids are good at maths but when they see it in writing what they got to do they find it hard.

Context

The contextual issue of identity, of being Aboriginal, was seen as an important pedagogical issue when teaching mathematics to Aboriginal children. Sandy and Phil were firmly of the opinion that Aboriginal children learn better if concrete materials are available. They believed that it made mathematics clearer for the students helping to overcome the language difficulties of writing, teacher expressions and long words.

*Interviewer: How important is the Aboriginal culture issue ?

*Phil: Real important I reckon. Different cultures have different ways of saying that kids have to learn this way and that way and with Aboriginal culture I think the most important thing is hands on stuff. Kids learn better I reckon that way.

*Sandy: Yeah.

*Phil: It just seems clearer to them than all this writing on the board and you know teacher expressions and big words.

Teaching

Phil also talked about the importance of fun and humour in learning. He believed that the students should focus on their work and be rewarded for the work that they did. Phil believed that as the students realised that they could do the work it would make them want to do more.

*Phil: You got to have fun in doing the subjects and that goes right across the curriculum. If all the kids have fun doing their stuff when they come to class, get down, knuckle down and do a bit of work and then have free time as a bit of a reward. ... If the kids know yeah I finish this, yeah that's good, yeah I can do this then. It sort of makes them knuckle down a bit more. If you just go into the classroom and have a dull, old stencil then they'll just sit there and take their time. You've got to reward them in the end.

Phil was not supportive of the use of stencils and Sandy believed that the children get bored and do not want to listen. Phil really believed that learning should be fun and that in mathematics there was none that he saw.

*Sandy: Yeah they don't want to listen. Get bored. They turn off.

*Phil: Some of the kids they sort of want to have fun when they are doing their learning activities and if there were a funny part about maths they'd learn good.

*Interviewer: Do you see much fun going on in mathematics?

*Phil: None mate, no. (Laughter)

There needed to be thought given to the structure of a mathematics lesson for Aboriginal children. Phil had already spoken about stencils and the need for more hands on material. The issue of students misusing the manipulatives was raised and Sandy believed that the teacher just had to be consistent and that the relationships between the teacher and the students had to be positive. The role of the teacher and their relationships with the students was seen as a crucial influence on Aboriginal students' learning.

*Sandy: The students have to get on good with the teacher and know that teacher.

*Phil: Yeah. A lot of kids don't like the teacher cause the way they teach it, or the way they are or the way they look.

*Sandy: A lot of kids if they hear a teacher growling they say I don't like that teacher. You say why and they say because they growl.

*Phil: If they are consistent with the students they would see down the road that if you play up you're going to get roused on. It's hard being consistent with kids every time. Sometimes I know that some of the kids want you and they don't want you to be with other kids. You can pick them and sort of dodge them and say I have to look at these kids first.

Phil was strong in his belief that relationships was the most important variable in children's learning yet he was honest in saying that it is not an easy thing to change.

*Phil: I think the relationship is number one. Then you break down the barriers and you slide it into their system that you can't do this at school. You don't go off or nothing like this .... You can't change things that have been around for ten, twenty years in one little short term.

Family concerns

There does not seem to be the support from home for the school. For Sandy and Phil there needed to be more community involvement to support the children's learning.

*Interviewer: I've yet to meet a parent who doesn't want their children to learn and I'm yet to meet kids who don't want to learn. How do we get them learning at school?

*Phil: More community involvement in the school like in the classrooms. Teachers and parents coming and sitting in the class and helping them with their work and saying you did good today with maths and English. Being positive always.

Though, Phil was committed to the belief that community involvement would help the children at school he identified the need for the school to create a more relaxed atmosphere for the parents, for most of them had set views on how they were supposed to behave in school.

*Phil: I think for the parents, too, when they come up to school they think oh we're in school now we've got to behave like this (Said in a whisper). If kids go into the community and get the parents involved it's in a more relaxing atmosphere, isn't it?

Discussion

There was a stated need to make mathematics more relevant to the students. The students needed to be able to recognise mathematics in their activities outside of school and in those that they saw about the community. The Aboriginal Educators believed that mathematics had to not only be made more relevant but structured activities had to be set up for the students to identify mathematics in what they did themselves and what they could see in their community.

Phil and Sandy raised the emotional issue of one's mood and how it influenced a student's learning. Coming to school to do mathematics in a poor mood could lead to the students using avoidance strategies when it came to doing mathematics.

Though much has been written about the language difficulties experienced by Aboriginal students (Graham, 1988) these Aboriginal Educators still identified teacher expressions, writing and the students understanding of long words in mathematics as constraints to students learning mathematics. The language of and in the mathematics classroom continues to be of major concern for these Aboriginal educators for, they believe, it does lead to student failure.

For Sandy and Phil, teachers needed to acknowledge that the learning of mathematics by Aboriginal children is fostered through the students using concrete materials. They believed that the materials helped to make mathematics clearer, overcoming some of the language difficulties of writing, teacher expressions and long words.

Teaching was given special focus by Phil and Sandy. They identified that consistency, positive relationships, teaching style, teacher's body language and open communication as important factors in fostering effective learning overall and in mathematics in particular. It was important to be consistent and the children needed to be told what was happening in their class. The prime factor for Phil was positive teacher/student relationships. Both Sandy and Phil were united in the view that learning mathematics should be fun and that currently they did not see much of that happening.

These two Aboriginal Educators recognised the need for more community involvement to support children's learning. Though, many of the parents are anxious to become involved in schools from past experiences and that a relaxed atmosphere has to be fostered within schools and that children working in the community may indeed be one way to have the community become more involved.

Conclusion

For mathematics educators a knowledge and sensitivity to the historical, psychological and social contexts in which Aboriginal children learn mathematics could only but benefit the Aboriginal student's learning. The classroom for many Aboriginal children in the senior years of primary school becomes an alien place comprising tension and conflicts about relationships and the value of what they are being taught. Relationship between the teacher and the Aboriginal student has been identified by these two Aboriginal educators as a crucial element for the effective learning of mathematics to take place.

The history of and the effect of the imposition of a Western education and its mathematics on Aboriginal children is an issue that needs to be raised in teacher education courses, induction programs for teachers and their continuing professional development. Such an inclusion would better prepare mathematics educators to better understand the complex nature and culture of the mathematics classroom for many Aboriginal students.

Aboriginal students want to learn mathematics, they want to do well and they want to maintain their identity. This has implications for the overall mathematics program of any school with Aboriginal students. Teachers need to develop teaching strategies that address the Aboriginal child's learning of mathematics as well as their identity. The evaluation of the quality, accuracy and appropriateness of the mathematics curriculum for Indigenous students continues to require attention.

In future reports the views espoused by these two Aboriginal educators towards the learning and teaching of mathematics will be supported and expanded through the comments of other Aboriginal educators as well as Aboriginal students, parents and their teachers. Together, they present a holistic view of learning and teaching.

Their views were wide ranging. They felt free to express them and it was important to them that they did so. There is no easy solution to the difficult issues facing Aboriginal students learning school mathematics and Aboriginal educators in teaching school mathematics. Already the issues of language, relationships and one's identity in the learning of mathematics have emerged together with specific comment on pedagogical aspects including fun in mathematics, the influence of one's mood on learning and the need for teacher consistency. Through open conversations both the pedagogical issues and possible solutions as perceived by those involved in the learning process can be voiced.

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Assessing the Impact of Graphics Calculators on Mathematics Examinations

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Graphics calculators will be allowed in VCE mathematics examinations from 1997, but questions must be set so that candidates who only have access to a normal scientific calculator are not disadvantaged. A sample of mathematics teachers, and two VCE examiners, were asked to assess the impact of the graphics calculator on thirteen multiple-choice questions used in a 1995 VCE mathematics examination. The teachers generally agreed with the assessments of the examiners, but where clear differences existed, tended to attribute less potential advantage to a graphics calculator user than did the examiners.

Introduction

In February 1995, the Victorian Board of Studies approved the use of graphics calculators in some year 12 Victorian Certificate of Education (VCE) mathematics examinations to take effect from 1997 (Board of Studies, 1995a). More recently, it has decided that graphics calculators will be allowed in *all* VCE mathematics examinations from 1997. In July 1995, the Secondary Education Authority of Western Australia announced that graphics calculators could be used in the mathematics Tertiary Entrance Examinations from 1998 onwards (Bradley, 1995). In Victoria only graphics calculators without symbolic processing capabilities have been approved, but Western Australia will allow the use of graphics calculators with limited symbolic processing capability (such as the HP 38G). Examination boards in other Australian states are also expected to eventually allow graphics calculators in their examinations but have yet to set timelines.

While the decision to allow graphics calculators in public mathematics examinations has been seen as a relatively radical step in Australia, it is merely reflecting a similar move in the UK, where all examination boards have allowed the use of graphics calculators in 'A' level examinations since 1994, and the US, where graphics calculators have been allowed in the Advanced Placement Calculus examination since 1993 and have been required since 1995. However, there does not seem to be a consensus as to the role that graphics calculators should play in examinations. In the UK, some examination boards have set examinations that are 'graphics calculator active'; that is, students are *expected* to have a graphics calculator in the examination and questions are set accordingly. However, other boards *allow* students to have a graphics calculator but set examinations that aim to be 'graphics calculator neutral'; that is, questions are set in a way that does not unduly advantage a student who has a graphics calculator compared to a student who only has access to a normal scientific calculator. In contrast, for the US Advanced Calculus examination which now assumes students have access to a graphics calculator, part of the examination is designed to be calculator active while the rest of the examination forbids the use of *any* calculator. In instructing examiners to set questions 'such that students are not advantaged by having a graphics calculator' (Board of Studies, 1995a, p. 14), the Victorian Board of Studies requires that, at least for the time being, VCE mathematics examinations will be graphics calculator neutral. Furthermore, this has to be done within the constraints of the existing syllabus. But how is this best achieved?

Unfortunately, in this regard, the literature on the use of graphics calculators in examining has mainly been concerned to date with identifying the associated problems (for example: Harvey, 1990; Boers & Jones, 1994; Kissane, Bradley & Kemp, 1994; Jones, 1995; Kemp, Kissane & Bradley, 1995; McCrae, 1995; Taylor, 1995; Tobin, 1995) and, in particular, offers little guidance as to how graphics calculator neutral examinations might be constructed. One possible and reasonably obvious approach, which has been adopted by the authors, is to start with the existing examination papers and see what difference having access to a graphics calculator might make. In analysing the 1994 VCE Mathematical Methods 3/4[1] Common Assessment Tasks 2 and 3 (Board of Studies, 1994a & 1994b), Jones (1995) found that the questions on these examinations ('CATs') appear to fall into three reasonably distinct categories. Firstly, there are those questions on which the graphics calculator could be classified as having *no impact* because it either contributes nothing to the completion of the task or no more than a scientific calculator. The second category of questions are those on which the graphics calculator *impacts*, by providing the user with an alternative, but still mathematically valid, method of solution. Finally, there are a category of questions which the graphics calculator effectively *trivialises* because it provides an alternative method of solution that requires little or no mathematical input from the user. In Jones' analysis, the focus was on determining the potential of the graphics calculator to assist in the *generation* of answers to questions. It ignored the potential of the graphics calculator as a device for *checking* answers generated by non-graphics calculator methods (see, for example, Kissane, Bradley & Kemp, 1994).

Using the same categorisation scheme, McCrae (1995) extended Jones' analysis to a second VCE mathematics subject, Specialist Mathematics[2], and produced table 1 which shows in summary form the percentage of marks that would have been affected if graphics calculators had been allowed in the 1994 VCE mathematics examinations[3].

Table 1. *Percentage of marks affected by graphics calculator use* (McCrae, 1995)

Examination	No impact	Impacts	Trivialises
Mathematical Methods CAT 2 (Multiple-choice)	69.7	12.1	18.2
Mathematical Methods CAT 2 (Short-answer)	35.3	64.7	0
Mathematical Methods CAT 2 (overall)	58.0	30.0	12.0
Mathematical Methods CAT 3	61.7	38.3	0
Specialist Mathematics CAT 2 (Multiple-choice)	90.9	3.0	6.1
Specialist Mathematics CAT 2 (Short-answer)	100	0	0
Specialist Mathematics CAT 2 (overall)	94.0	2.0	4.0
Specialist Mathematics CAT 3	83.3	10.0	6.7

On the basis of these results, it would appear that, while graphics calculators do have the potential to impact on a significant proportion of the questions currently used to assess Mathematical Methods 3/4 and Specialist Mathematics, the effect has not been to been to trivialise the majority of questions, but rather to broaden the methods available to answer many questions. As long as the potential impact on content validity is taken into account, such questions would appear to have a place in an examination which permits the use of graphics calculators. There are, though, a small number of question types that would be trivialised and which would no longer be useable in their present form in such an examination. But do VCE mathematics teachers agree with this assessment? And are there other types of questions that would need to be modified or replaced if the examination is to be graphics calculator *neutral*? These are the concerns of the present paper.

Method

The thirteen calculus multiple-choice questions from the 1995 Mathematical Methods 3/4 CAT 2 (Board of Studies, 1995b) were independently analysed by the authors, both of whom are VCE mathematics examiners[4], and twenty-seven VCE mathematics teachers. The calculus Mathematical Methods multiple-choice questions were chosen for analysis because in the previous year the corresponding questions had shown themselves to be the questions most likely to be trivialised by the introduction of the graphics calculator (see table 1). The teachers in the study were not chosen at random but were volunteers from schools where graphics calculators were known to be in regular use. The authors and twenty-four of the teachers were TI-82 users, two teachers were Casio fx-9700 users and one was a Sharp EL-9300 user. Although these three graphics calculators have similar capabilities, it was decided to exclude the three non-TI users from the sample so that there was no variation in calculator useage. Of the twenty-four teachers who were TI-82 users, two rated themselves as barely familiar with the non-statistics capabilities of the TI-82, five as somewhat familiar, thirteen as reasonably familiar and six as very familiar. The authors both rated themselves as being very familiar with the TI-82. The TI-82 has what can be regarded at present as standard features for a graphics calculator that will be allowed to be used in VCE mathematics examinations. The features relevant to the current investigation include the ability to draw and analyse graphs, evaluate derivatives and definite integrals numerically, and solve equations numerically. It does not have symbolic processing capability.

The questionnaire given to the teachers asked them to make an assessment of the impact of the graphics calculator on the thirteen multiple-choice questions using the following three categories:

A. The availability of a graphics calculator would have no impact on the question
B. The availability of a graphics calculator would have an impact, but the question could remain unchanged
C. The availability of a graphics calculator would have an impact and the question would need to be modified or replaced

These categories broadly follow those used by the authors to classify questions on the 1994 examination papers, but avoid giving a reason why the respondent might put the question in a particular category. The reasons for the teachers' classifications were solicited separately on the questionnaire and the preamble made it clear that 'impact on the question' meant whether a graphical calculator user would have an advantage over a student who only had access to a non-graphics, scientific calculator.

Results

For most questions, the classification assigned varied from respondent to respondent. A summary of the teachers' responses is given in table 2, together with those of the authors (the 'examiners').

Table 2. *Summary of teachers' and examiners' classifications of each question*

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13
Category A	1	0	24	1	0	22	0	0	0	8	11	19	15
B	9	13	0	14	6	0	2	1	11	12	11	2	3
C	14	11	0	7	18	0	21	21	13	4	2	1	4
Examiner 1	C	B	A	C	C	A	C	C	C	B	B	B	A
2	C	C	A	C	C	A	C	C	C	C	C	B	C

To assist with data interpretation, the alphabetic categories A, B and C were numerically rated as A =1, B = 2 and C = 3. Thus, the higher the numerical rating given to a question, the greater the potential advantage attributed to the graphics calculator user on that question. An 'average' rating was then obtained for each question, for both the teachers and the examiners, and these were plotted against question number as parallel line graphs: one line for the teachers' ratings and one for the examiners' ratings; see figure 1.

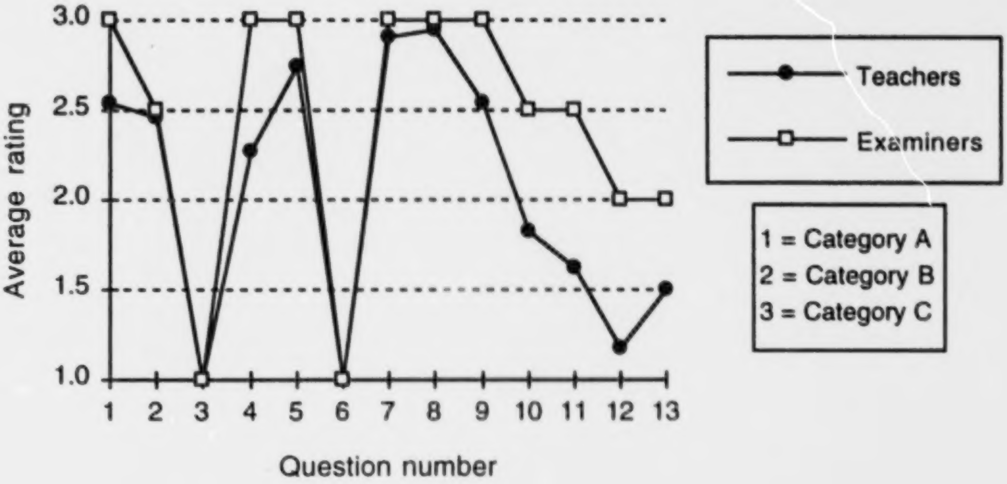

Figure 1. Parallel line graphs displaying average rating given to each question by the examiners and the teachers.

Analysis of Results

The overall pattern of results evident in figure 1 shows that, in general, the teachers and the examiners rated questions in a similar manner, but where differences existed, teachers tended to attribute less potential advantage to the graphics calculator user than did the examiners. According to the teachers' average ratings, at least three and up to six of the calculus multiple-choice questions on the 1995 Mathematical Methods CAT 2 could not be included on a graphics calculator neutral examination. By comparison, the examiners would be more conservative and exclude between six and nine of the questions[5].

Rating Agreement

Two of the questions (3 & 6) were rated A (graphics calculator no advantage) by all respondents, while three of the questions (5, 7 & 8) were rated C (graphics calculator a clear advantage) by most respondents. Typical of the latter group of questions is question 8 which asks candidates to select, from five alternative graphs, the one that best represents the graph with equation $y = 3\sin(2x) - 1$. Having a graphics calculator would clearly be an advantage in answering such a question, since little or no mathematical knowledge is required by the user to generate the desired graph .

Rating Disagreement

Clear differences (more than half a category) in the *average* ratings of the teachers and the examiners occurred for four of the remaining eight questions (namely questions 4, 10, 11 & 12). However, close inspection of table 2 shows that there are significant differences in the ratings of the individual teachers, either between themselves or with the examiners, for *each* of these eight questions. An examination of the reasons given by teachers for their ratings indicates that these questions can be classified into three groups for the purpose of our analysis.

The first group comprises questions 1, 2 and 4. Question 2 is representative of this group and is reproduced in figure 2. This question was classified as B by about half (54%) of the teachers and as C by the remaining teachers (46%), with a typical reason for choosing category B being that 'graphing each alternative would take much longer'. The teachers who chose category C also seemed to assume that a 'trial and error' approach would be used with a graphics calculator, but evidently weren't concerned about the time factor. Surprisingly, only one teacher suggested that an advantage of a graphics calculator would be that it could be used to verify an answer obtained in another way.

If the graph of $f: R \rightarrow R$ crosses the x-axis exactly three times, which one of the following rules could *not* be the rule for f?

A. $f(x) = x(x^2 - 4)$

B. $f(x) = x(x - 2)(x + 4)(x^2 + 1)$

C. $f(x) = (3 - x)(x^4 - 16)$

D. $f(x) = (x^2 - x - 6)(x - 4)$

E. $f(x) = (x^2 - x - 6)(x^2 - x - 12)$

Figure 2. Question 2, Mathematical Methods 3/4 Part I (Board of Studies, 1995b)

Question 4 was one of the questions in which there was a clear difference in teachers' and examiners' ratings, with the teachers favouring category B (graphics calculator some advantage, but no need to alter the question) whereas both examiners felt a graphics calculator would be of such an advantage as to require a change to the question (category C). This question is like questions 1 and 2 in that it shows a graphical relationship between quantities x and y and asks candidates to select the equation relating x and y from five alternatives. However, it is distinctive in that each alternative gives the *form* of an equation, for example $y = a \cos(bx)$ where a and b are positive constants, rather than a specific equation.

Questions 9, 10 and 11 constitute the second group of questions. Question 10, see figure 3, was the only question (out of the whole thirteen) for which the teachers' ratings were fairly evenly spread across all three categories, with 33% choosing A, 50% B and 17% C. Clearly, a graphics calculator could be used to generate the graph of the function, but a knowledge of terminology and some interpretive skills are still needed to obtain the answer. The variability in the teachers' ratings reflects the different weightings they gave to these factors. A typical reason given for choosing category B was that 'they could graph it but students would still need to know the terminology'.

A trigonometric function is given by $f: R \rightarrow R$ where $f(x) = 3\cos(2(x-\pi)) + 1$.
The amplitude, period and range, respectively, of the function are

	amplitude	period	range
A.	3	π	R
B.	2	$\frac{2\pi}{3}$	$[-4, 4]$
C.	2	$\frac{2\pi}{3}$	R
D.	π	3	$[-2, 4]$
E.	3	π	$[-2, 4]$

Figure 3. Question 10, Mathematical Methods 3/4 Part I (Board of Studies, 1995b)

Similar considerations were evident in the rating of question 11, reproduced in figure 4. Here, a couple of teachers (8%) chose category C with the remainder equally split (46%) between categories A and B. A typical comment was that 'the calculator is an aid but interpretive skills needed'. No teacher suggested that the ability to graphically check one's answer would give a graphics calculator user an advantage. There was a clear difference in the average ratings of teachers and examiners on both questions 10 and 11, but once again in each case the teachers were more inclined to allow the question to remain unchanged than were the examiners.

If $f(x) = a \cos x + c$, where c is a positive real number, then $f(x) < 0$ for all real values of x if

A. $c > a$ B. $c < -a$ C. $c = 0$ D. $-a < c < a$ E. $c > -a$

Figure 4. Question 11, Mathematical Methods 3/4 Part I (Board of Studies, 1995b)

The last group of questions for analysis consists of the two remaining questions, numbers 12 and 13, both of which involve differentiation. Question 12, which is reproduced in figure 5, was rated as B by both examiners but as A by most (87%) of the responding teachers. This question could be answered with the support of a graphics calculator because the *graph* of the answer can be generated. Alternatively, the answer is identified if the numerical derivative is calculated at $x = 1$, since each option clearly has a different value at that point. It is possible that some teachers recognised these possibilities but considered them either too time-consuming or too sophisticated to constitute an advantage to graphics calculator users. However, there was no evidence of this in the teachers' comments and it is more likely that the graphics calculator's (numerical) differentiation capabilities were unknown to most of the teachers—something that will obviously change with increased useage.

The derivative of $2\sqrt{x}$ is equal to

A. $\frac{1}{2\sqrt{x}}$ B. $\frac{1}{4\sqrt{x}}$ C. $\frac{1}{\sqrt{x}}$ D. $\frac{4x^{3/2}}{3}$ E. 2

Figure 5. Question 12, Mathematical Methods 3/4 Part I (Board of Studies, 1995b)

Question 13 shows a graph of an 'unknown' function f and asks candidates to select the graph of the derived function f' from five alternatives. However, the graph is easily recognisable as that of sin x (on a restricted domain) and so a graphics calculator could be used to generate the answer and this is why one examiner rated it as category C. The other examiner rated it as an A, agreeing with the teacher who argued that if a student recognises that f is the sine function, then the answer is essentially given on the formula sheet anyway.

Discussion and Conclusion

From the analysis of the results, it would appear that the sample of teachers generally agreed with the way in which the examiners rated the impact of graphics calculators on the thirteen multiple-choice questions used in this study. Where clear differences in ratings occurred, the teachers tended to attribute less potential advantage to a graphics calculator user than did the examiners who tended to take a more conservative stance. In particular, if a graphics calculator could be of assistance in answering a question but considerable (mathematical) interpretive skills were still needed to obtain the answer, then the teachers did *not* regard the question as being unsuitable for inclusion on a graphics calculator neutral examination.

Although there is often the potential for a graphics calculator to be used for checking answers obtained by other means in the sort of questions analysed here, explicit mention of this aspect of the potential impact of graphics calculators was deliberately omitted from the explanation of the categorisation scheme given to the teachers. However, the opportunity existed for them to recognise this potential when they commented on their reasons for rating each question, but few did so. This may be because they generally failed to recognise this potential—in contrast to one of the more experienced graphics calculator users in the sample who added an overall comment that 'much of the advantage to the students is that they confirm part or all of their work'.

Alternatively, it may be that the teachers tended to discount the checking potential of the graphics calculator because of the critical factor that time plays in multiple-choice testing in VCE mathematics (there is an average of less than two minutes available to answer each question) and the mathematical sophistication needed to implement some checking methods.

Further research in the area is clearly needed, particularly as teachers become more experienced and skilful in the use of graphics calculators and relevant utility programs become readily-available for the various models. This research also needs to include an investigation of the impact of graphics calculators on *extended-answer* questions. You are often required to solve an equation or evaluate an integral as part of such questions and these procedures can be performed graphically and/or numerically on graphics calculators. There may also be more opportunity to check answers in these longer questions, with time not being as critical a factor, and more incentive since (unlike with multiple-choice questions) the correct answer is not generally given.

In conclusion, from the analysis presented in this paper, it would appear that the scheme used by the authors to categorise the potential impact of the graphics calculator on the sort of multiple-choice questions used in current VCE mathematics examinations yields similar results for both teachers and examiners, and that where ratings of questions do clearly differ the examiners tend to be more conservative. Examiners regard fewer of the current questions as being suitable, in their present form, for a graphics calculator neutral examination than do teachers. We must not be too distracted, however, by the pursuit of graphics calculator *neutral* examinations. To ensure that the great potential of graphics calculators is fully realised in the teaching and learning of mathematics, it is important that mathematics examinations become graphics calculator *active* as soon as possible.

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[1] Mathematical Methods 3/4 is a calculus-based subject, with some statistics and probability content. It is a prerequisite for almost all mathematics-related university courses in Victoria.

[2] Specialist Mathematics is a second calculus-based mathematics subject which is taken, in addition to Mathematical Methods 3/4, by students wishing to specialise in mathematics. It is a prerequisite for some tertiary engineering courses.

[3] There is a third VCE mathematics subject, Further Mathematics, which is a non-calculus based subject for students not needing mathematics in future years. No analysis is included for this subject because graphics calculators were found to have only minimal impact on the CATs.

[4] Jones is chief examiner for Further Mathematics and McCrae is chief examiner for Specialist Mathematics.

[5] Note that, for comparison with the first row of table 1, in both 1994 and 1995 there were thirty-three multiple-choice questions on the Mathematical Methods CAT 2, each worth one mark. Hence 3, 6 and 9 questions represent 9%, 18% and 27% respectively of the available marks.

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Investigating The Use Of Language And Mathematics In A Workplace Context

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This study investigates how the linguistic devices of metaphor and metonym produce and regulate mathematical knowledge in the performance of work related tasks. Data are drawn from the front desk operation of three metropolitan district motels in various parts of Australia, and analysed making use of 'critical incident' methodology. Findings indicate that within the workplace, language use and mathematical knowledge are deeply entwined

Background and aims of study

Edward Thorndike, in his General Introduction to the *Psychology of Arithmetic* (1922) made the following assertions:

> The aims of elementary education, when fully defined, will be found to be the production of changes in human nature represented by an almost countless list of connections or bonds whereby the pupil thinks or feels or acts in certain ways in response to the situations the school has organized and is influenced to think and feel and act similarly to similar situations when life outside of school confronts him [sic] with them. (p xi)

In some ways, perhaps, this statement represents a high water mark in terms of its confidence in the prerogatives of 'the school', the power of this institution to realise its mission to influence totally the development of the learner, and the rightness of the school in so doing. Since this time, of course, each of these aspects of Thorndike's notion of elementary education has been debated vigorously. Whilst arguments have ranged over domains including psychology, sociology, anthropology, philosophy, ethics, cultural studies, they have usually not accorded a high degree of attention to learning experiences outside the school, such as in the home, the market place, the workplace, and so on. It would thus seem appropriate that in renewing consideration of Thorndike's general assertions, attention be focused on pertinent data obtained within workplace sites. Accordingly, in this study concern centred on the extent to which "already made" knowledge is merely transferred to a given task, as distinct from being produced in response to site specific needs. The approach taken was to examine certain linguistic processes involved in the production and regulation of mathematical meanings within specific workplace contexts. In so doing, the intention was to investigate how these processes were found to shape and so construct the mathematical ideas formed in these sites. If established, these points will enhance claims that out-of-school learning experiences need to be more systematically studied than is presently the case.

Theoretical background

One way to think about the way mathematics is used in a workplace context is to make use of the literature relating to the phenomenon of knowledge transfer. Researches into what transfer is, how it works, and what conditions promote its function currently range over behavioural, cognitive and socio-cultural domains. Given the limited scope of this paper, however, it will be possible to investigate only one salient aspect of this literature base, namely, the nature of epistemological assumptions underlying current research in this area of investigation. It is to this purpose that the following paragraphs are addressed.

A key concern within mathematics education research over the last two decades has surely been the close reconsideration and revision of ideas relating to the epistemological status of mathematical knowledge itself (Ernest, 1991). At one time essentialism (the doctrine that mathematical knowledge is factual in character and thus independent of the knowing subject) occupied a near universal position of dominance within the education field, yet by this time a considerable body of both empirical and theoretical material has now been adduced for a number of alternative epistemological stances; these might generally be grouped under the broad heading of social/individual constructionism. In these emerging alternatives to essentialism, mathematical knowledge is thought to be a construction involving social organisations (Tymoczko, 1986) and/or individual people (von Glasersfeld, 1995), rather than mainly or totally an objective fact of nature. Whilst the nature of these alternatives cannot be discussed here in greater detail, of interest to my immediate purpose, would be to highlight the kinds of research which have lead to the rise of constructionist alternatives and relate these to the issue of usage of mathematical knowledge under circumstances of transfer. Four major lines of development are discernible, and these will be briefly considered in turn.

Firstly, mathematics curriculum theory has for some time drawn attention to 'school mathematics' and 'real mathematics' as consisting of different kinds of mathematical knowledge. For example, Newton (1983) showed that school age children are frequently able to perform elementary numerical operation when they are embedded in naturalistic contexts, but are unable to perform structurally identical tasks within the formal context of the every day school setting. Lave (1988) found similar differences in her study of how adults used mathematical procedures in shopping contexts. Likewise, Walkerdine (1988), in her study of pre-school aged children, found that the discourse between mother and child relating to the preparation of food and mealtimes organisation presented a range of mathematical concepts unique to these contexts and frequently at variance with those later encountered within the formal schooling context. In considering salient contexts for the use of mathematical knowledge, Dowling (1989) offered a rich theoretical map for considering the components of the semantic field framing the operationalisation of numerical knowledge within school contexts. Schoenfeld (1991) also considered what he termed "the unfortunate separation" between formal and informal mathematics. Both, he argued, are required for sense to be made of either. Whilst in each of the studies referred to above, authors conceptualised their data within substantially different theoretical domains, a common set of questions can be seen to arise. These may be stated as: Can mathematical knowledge can be said to occupy an a-contextual space? Does the process of realising meaning require a defining context as a preconditional requirement? To what extent does this context act to shape the mathematical knowledge itself? The general tendency, in the research literature at any rate, has been to respond to the first of these in the negative and the second in the positive, and this has fuelled the shift from belief in essentialism to social constructionism. It is to be noted that the nature of mathematical knowledge transfer/usage on this view is more concerned with the adoption of salient contexts for the operation of knowledge, and is less interested in how knowledge is actually deployed from a cognitive view.

Secondly, developments in the cultural anthropology of mathematics has also provided a challenge to essentialist thinking in mathematics. The work of D'Ambrosio (1985), Carraher, Carraher and Schlieman (1985), Gerdes (1985), Bishop (1988), Saxe (1991), Nunes, Schlieman and Carraher (1993), and others can be cited as making a major contribution to our knowledge of the mathematical ideas and processes made use of by indigenous people and persons generally outside of the hitherto confined circles of academic western dominated culture. Within these broader contexts, essentialism as a doctrine for mathematical knowledge, has appeared increasingly inadequate. Mathematical behaviours have been observed as too diverse and differentially integrated into the cultural and productive life of societies studied for such a doctrine to maintain credibility. In contrast, mathematical knowledge is seen as arising from various interactions among the social, physical, and cultural contexts of the mathematics knowledge user. On this view mathematics knowledge use/transfer and knowledge creation are seen as part of the same process.

Thirdly, recent developments in the psychological literature have generated theories of learning which make little call on essentialist epistemologies for mathematics. For instance, whilst essentialism is not a logical requirement of the associationistic style of learning theory as represented in the works of theorists from Edward Thorndike to John Anderson, this epistemological doctrine is nevertheless both highly compatible and often implicitly assumed within the context of theory formation. In contrast, Piagetian theory posits learning to be the process of active and purposeful reorganisation of thinking and meaning in response to a problem situation, and this so-called constructivist thesis does not depend on the correspondence among accommodated schemata and objectively given facts and conditions. In Vygotsky's theory, also, no such correspondence is either required or entertained. Learning in this theory is the process of internalising inter-psychological involvements as mediated by language and the other artefacts belonging to the learner's context and culture. Arising from these influences a complex body of literature has evolved delineating a theory of 'situated cognition' according to which mental operations are not determined exclusively by factors relating solely to the stage of mental development or other individual characteristics of the person involved, but are also characteristic of the physical, social, cultural context in which they are performed (Pea, 1987; Lave, 1988; Brown, Collins & Duguid, 1989; Collins, Brown & Newman, 1989; Greeno, 1989; Rogoff, 1990, 1995). In summary, situated cognition regards the question of mathematics knowledge transfer/usage as involving two kinds of issues: the first relates to the nature of site specific rendition of knowledge and social practice (Bruner, 1990; Lave and Wenger, 1991), and the second considers the development of task competency in the light of constructs derived from cognitive theory (Pea, 1987; Greeno, 1989, 1992).

In part drawing from the above, a fourth general line challenging essentialist epistemology in the mathematics education literature can be discerned. In this approach researchers have focused on the micro details of interactions between students and their teachers as they go about tasks within normal classroom protocols (Bauersfeld, 1991; Cobb, Yackel & Wood, 1992; Voigt, 1993). Emerging from this research has been a far ranging revision of the Piagetian constructivist paradigm for learning. Cobb et al, for instance, have proposed a theory of social constructivism according to which learners, reacting to given problematic situations, engage in two knowledge building processes simultaneously. In the first they individually construct their own meanings for the mathematical concepts and processes they encounter; in the second, individual meanings are massaged ("negotiated") with reference to the multiple levels of constraint and facilitation occasioned by the many forms of social process that usually accompany learning in actual situations - for instance, interaction with peers, teachers, instructional materials, and so on. Social constructivism is thus posited on a social constructionist epistemology and mathematical knowledge is to be identified with its application within site specific contexts.

In summary, the various perspectives considered above suggest that the use of mathematics can rarely be thought of as a purely technical operation in which certain "ready made" knowledge is merely accessed in order to perform a given task. Instead, knowledge is (re)formulated on each occasion a problem situation arises or a task competency is required. In general, however, the literature surveyed above does not examine what tools mathematics users need to utilise in order to produce and regulate the knowledge required within the operation of specific tasks in naturalistic settings, nor does it fully explore how these tools function. Following in the tradition of Vygotsky (Cole and Scribner, 1974; Wertsch, 1985), language can be thought of as just such a tool; it is for this reason, therefore, that the remainder of this paper will mainly focus on the use of certain linguistic devices in the performance of workplace tasks.

Examining a workplace context - methods

Data for the study were drawn from a significantly larger data base assembled for a study into the contextual nature of workplace competence (Stevenson, 1995). This larger study was conducted by the current researcher as part of a team of Griffith University based researchers and funded by the Australian National Training Authority in 1995 . Motel front desk operations provided the focal point for data collection and analysis; sites were chosen from around Australia and involved motels in both metropolitan and country districts. Data consisted of the video record of transactions which took place in each site over a continuous period of 4 to 5 hours (a) among staff engaged in workplace tasks, (b) between staff and guests relating to check-in and check-out procedures, and (c) between staff and researchers relating to salient aspects of workplace operations. In the study reported here, however, the scope of this larger study was significantly reduced. More particularly, data were drawn from three Metropolitan sites only, reflecting the need felt to reduce the observed variability of the data set whilst ensuring that coverage achieved an appropriate level of depth.

In the current study, data analyses were conducted in the following way. First, transcripts of the interactions were prepared and "critical incidents" involving mathematical knowledge were identified. Criteria used for identifying these episodes were drawn from the documentation provided by the Australian Education Council, *Mathematics - a curriculum profile for Australian schools* (1994). Second, the language used in the critical incidents identified was scrutinised in order to identify uses of particular linguistic devices in the use and application of mathematical ideas and procedures in the workplace operations observed. Linguistic devices chosen for examination were selected with reference to the appropriate research literature. Third, instances of devices utilised were analysed in order to investigate how these were being used to produce and regulate mathematical knowledge within the tasks identified.

Examining the data: Use of metaphor and metonym in the production and regulation of numerical meaning within a workplace site

Linguistic theory (Jakobson & Halle, 1971, Kristeva, 1989) recognises a number of ways meaning for words and procedures can be produced and regulated. Two of these are the use of metaphor and metonym in oral and written communication, and these will be studied in this investigation. Metaphor, relies on vertical relationships between the domain of the words and symbols for which meaning is to be ascribed, and corresponding elements within the domain for which the meaning maker already possesses meaning. Possibly less familiar, however, is metonym, which relies, in contrast, on horizontal relationships established by juxtaposition among words, symbols and other possible referents eg objects (both concrete and abstract), processes etc. In the following, an illustration of findings obtained by analysis of the data will be presented, each of these tropes (as they are known in linguistics) being considered in turn.

The use of metaphorical relationships

Metaphor, as an explanatory device is quite familiar as a commonly used tool for mathematics teaching. For instance, elementary number properties are often first 'illustrated' within a concrete domain consisting of bundling sticks within which operations of grouping can be performed. Later, the bundling sticks are made to correspond to icons eg written strokes and dashes, and grouping is spoken of as 'adding'. Finally, more abstract symbols are produced in order to render the relationships first encountered in the concrete, meaning-rich domain. Notice that on this fairly classical view of mathematics instruction, the key processes revolve around efforts to build vertical relationships which preserve salient features of the meaning-rich domain (often concrete) in the hope that these may then be transferred to the meaning-poor domain (often abstract) between familiar and unfamiliar domains. This is the representational approach to teaching employed systematically by theorists such as Dienes, for instance. Likewise, Resnick (1983) considered teaching strategies effective when, considered as representations of the mathematical knowledge they are designed to teach, they are 'transparent' to the mind of the learner - that using them, the learner will grasp the inherently metaphorical character of the knowledge encountered.

Evidence that indeed respondents made use of metaphorical devices in order to construct meaning and guide respondents in the performance of their workplace tasks was obtained. One example may suffice to illustrate use of this technique. Respondents were observed to have made structure preserving assignments between elements belonging to the price domain (numbers indicated in dollar values) and the domain of goods and services as traded by the motel. In the following episode, in which the respondent is rationalising a customer's account, the reasoning used by the respondent appears to depend critically on the establishment of a one-to-one correspondence between the number of nights the guest in question stayed in the motel and the total cost of this stay. Phone bills were counted as extra. (Note, the following conventions will be used in transcripts: RS - a respondent, G - a motel guest, R - a researcher.)

> *RS:* (Adding machine noise over conversation.) *Yes it's $421.45. Thank you.*
> *RS:* *Hello reception.*
> Inaudible conversation.
> *G:* *How much did you say it was?*
> *RS:* *Yes that's including last night's accommodation, then one week which is $350.00 then of course your phone calls plus last night, which is already there. So that's $421.45.*
> More conversation but inaudible due to adding machine and radio.
> *So you still need to include those at $71.45.*
> Adding machine.

In this transcript, the respondent was able to justify the bill by demonstrating how the process of tabulation preserved the logic of the guest's pattern of consumption. Of significance here is the way in which the consumption pattern was grouped into one block of "one week" followed by "phone calls plus last night". Use of the term "plus" here illustrates the strength of the implicit assignment operating between the numerical domain on one hand, and on the other hand, a domain in which items of consumption are routinely grouped. According to the metaphorical structure implicitly working here, "plus" in the former domain is assigned to the grouping operation which takes place in the latter. On one interpretation, this example illustrates the way metaphor is used to access mathematical concepts in order to better regulate workplace tasks. On another view, it illustrates just the opposite, namely, how metaphor can be used in workplace tasks to produce and regulate mathematical meanings (in this case for "plus") applicable over an extended range of highly contextualised variables. Choice between these views revolves around epistemological issues. In the first, the presumption is that mathematical knowledge exists as a separate entity and is transferred to the task, whereas in the second, it is believed that this knowledge is distributed across elements constituting the task in its particular context.

The use of metonymic relationships

Metonym is a less familiar, though frequently used way of producing and regulating meaning in communicational interactions. In this approach, the meaning to be ascribed to one term derives from a known meaning already ascribed to a second term in circumstances where both terms are juxtaposed, perhaps physically, either by accident, custom or design. In contrast to metaphor, in which is posited a structure preserving map between two domains, metonyms are based on juxtapositional, and hence horizontal relations between terms belonging to the same or similar domains. For example, reference to 'the crown' could be taken as a reference to the monarchy and/or its institutions. In this instance, the 'crown' as an object is juxtaposed with the person of the monarch and thus with the system of monarchical government. An horizontal relation of terms 'crown-monarch-institution of government' is thus created and, by virtue of this chain, the semantic scope of each included term is greatly extended.

Analysis of the data revealed extensive use was made by the respondents of this type of meaning production. Two broad schemes for the use of metonyms were identified. The first involved constructing a nomenclature for motel guests which exploited the numbers of the rooms in which they stayed. In this case the scheme of juxtaposed relationships was the following: 'Motel guests - motel rooms in which they stayed - the numbers of the motel rooms'. Metonyms made possible by these relations were as follows. The first allowed respondents to keep a record of guests using motel room numbers as appropriate labels (*"No charge for Room 37 for a late checkout."*); the second allowed respondents to label rooms with their corresponding room numbers (*"You're in 326 aren't you no 329?"*); and the third allowed respondents to label guests with a numerical identification (*"... what are your numbers Margaret? 23's gone 24's gone 26 has gone, what else? 28's gone, ..., 22's still there. Thanks bye."*).

Situations arose when guests were misidentified or expenses arising from office, bar or restaurant transactions were not properly ascribed to the correct accounts. In these instances, respondents were observed to permute assigned numbers in order to make the appropriate designations. As one respondent noted, in such cases

> RS: You can either try the number again in case you put it in backwards, or you can just ask it to print and it will print out exactly what you put into it and you can check why you don't agree with it.

In another instance, the respondent reported a more complex engagement with the numerical aspect of the assigned nomenclature.

> **R: So how did you find out that was a problem?**
>
> RS: When the charges come down, you put them in and as you're putting them in. The only way you find out its a problem id if you put the room no. in and it doesn't come , there's no guest in the room. So there's two alternatives, either a) the guest is already gone, or there was no guest in the room and they've written the wrong number, or the guests get mixed up and they give the wrong number, sometimes they transpose the number or whatever. So you sort of fiddle around with it. Try the numbers the other way.

In the cases discussed above, respondents were seen to use numbers to form a system of labels. Use of certain numerical properties of numbers constituting labels (eg listing by numerical order, using a place value coding system which allows for error correction) was also observed. In these cases the metonymic relationships established were exploited in order to regulate better progress through workplace tasks.

A second scheme observed for the production and use of metonyms was also observed. In this scheme use was made of the juxtaposition of particular numbers, usually expressed in units of currency, the prices of particular goods and services offered by the motel, and the merchandise offered for sale itself. According to these juxtapositions, dollar amounts were taken to refer to merchandise, as in the following example in which a respondent is explaining to a researcher the method adopted to make a cash balance of transactions conducted in the day.

> RS: I always take my cash out and I balance what I've got in my till. And this should be $250. [float] If its not then I know there's a mistake somewhere. If I've got too much cash it means I've either cash for something and forgotten to post it or I've perhaps made a change error. The scary thing is if you've got money and you've forgotten to post it. because you've got no way of really knowing- you've got to actually - so much happens you've really got to sit and think - `what did I do this morning?' You've got to look at each individual amount and get them to jog your memory and you'll remember

certain things about that person and then you've got to think OK fine - I know I'm up $13 too much - what could $13 be . $13 is probably minibar - who was s'posed to pay minibar and didn't? That's the only way of doing it.

As in the previous example, metonymic structures involving numbers were not only used to construct a convenient system of nomenclature, but they also assisted in actually performing work related tasks. In the following transcript the respondent is seen to make use of the chain of juxtaposed relationships 'merchandise-price-number' in order to produce a richly worked set of highly situated task oriented procedures.

R: **Do you do much in your head or you use that a lot?**
RS: Oh, well to tell you the truth I've got a lot better. Because on check-out when people are telling you, when I say "what did you have for minibar?" "Oh I had a packet of chips, two cans of coke and a beer" - see you haven't got time to get the calculator out you've got to go, $3.00 + $2.80 + $1.40 so you can do it in your head.
R: **So you don't enter each of those items into**
RS: Oh no - you just enter minibar total amount - you've got to do it that way - it would take too long if you entered - I mean some of them drink like fishes so you could end up with 15 or 20 entries for the mornings minibar - it's not worth it. So you do get better - you don't even mean to - it just happens that way. And I find actually, if its quiet like this I work best under pressure - I mean I work fine now, but when things are all happening I find that you don't have time to think, your brain just does it - like when they say that you just know
R: **But you are thinking because you've still got to prioritise that task**
RS: Oh yes - but I mean - like with maths, if I started adding up those numbers I'd be sitting here thinking, thinking, thinking. Whereas, if he said to me "I just had two beers a drambuie and a packet of chips - you just kinda know what it is"...
OK and now we'll try the balance again.

In this episode, number operations are performed on elements within the site specific context (beer drambuie chips) rather than merely on elements within an essentialised framework for mathematical knowledge (numbers). It is concluded that the mathematical knowledge deployed by the respondent in this case was produced entirely within the confines of the domain belonging to the specific task. Moreover, this production was facilitated by the metonym afforded by the following chain of juxtaposed elements sited within the workplace: merchandise-price-number. Put another way, number relations are seen to apply both to numbers and to an augmented set of salient signifiers as designated within the particular context of the workplace. Thus, in this instance metonymic relations are seen to have produced a highly contextual conception of numeration in which tasks of addition are regulated by their association with concrete elements within the domain of the workplace task in which the respondents operated.

Conclusion

This paper has reported on a study into the use of language and mathematics in a workplace context. Two concerns have permeated the work: the extent to which workplace tasks make use of site generated rather than "already made" knowledge; the power of language to produce and regulate of mathematical meanings appropriate to the requirements of a given task. The first of these was investigated both in review of the research literature and in an examination of data obtained for this study. Evidence was provided in support of non-essentialist epistemologies for mathematics, and this finding is consistent with the literature reviewed above. The second concern investigated the linguistic devices of metaphor and metonym and showed how these were used in the sites observed both to produce and regulate mathematical knowledge as respondents performed tasks required of them within a specific workplace setting. Starting from a constructionist epistemology for mathematical knowledge, questions arising from this study are: What limits the range of meanings produced by the linguistic devices observed? How is this limitation expressed within the task competencies observed? It is suggested that answers to these questions will enable us to better understand the nature of effective relationships among the processes of task performance, knowledge assembly, and mathematics curriculum.

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Trigonometry: Comparing Ratio and Unit Circle Methods

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Before the 1960s, introductory trigonometry was taught in Victorian schools using the ratio method, where trigonometric functions are defined as ratios of sides of right angled triangles. With the advent of "new maths", the unit circle method was introduced. This study explored differences between the two methods for teaching introductory trigonometry. Eight classes of students were randomly allocated to either teaching method. The ratio method was found to be much more effective, resulting in better performance and retention in trigonometry and algebra, and more favourable attitudes.

Two methods of introducing trigonometry

Since the advent of the "new mathematics", two methods of teaching introductory trigonometry have been used in Victorian schools. For decades, introductory trigonometry had been taught to Year 9 and 10 students (average age 14 and 15) by the ratio method. In this method, the trigonometric functions are defined as the ratios of pairs of sides in a right angled triangle. From the early 1960s, an alternative "modern" method was advocated by some educationalists (Trende, 1962; Willis, 1967) as a more desirable way for students to learn and understand the topic. This approach, known as the unit circle method, defined cosine and sine as the x and y co-ordinates of a point on a unit circle. Both methods are outlined in more detail below. Since 1982, the unit circle method has been the officially preferred method for the Year 9 and 10 mathematics curriculum in Victoria (Secondary Mathematics Committee, 1982; *Geometry Everywhere*, 1991). Currently there are text books promoting each way of teaching trigonometry and a few try to blend the two approaches.

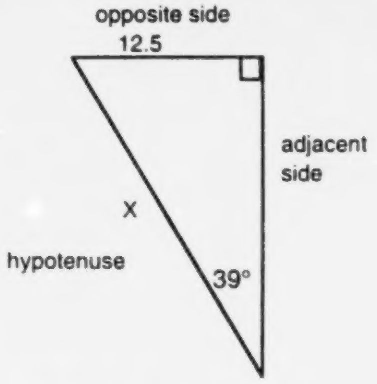
The main purpose of this research was to compare the two methods of teaching basic trigonometry to see which promotes better understanding of the underlying concepts and mastery of skills. In the introductory phase which we are concerned with here, the main skills are to calculate the sides and angles of right angled triangles from known sides and angles. A second goal of the research was to investigate the relationship between success in trigonometry and algebraic understanding. Thirdly, the strategies used by low ability students to achieve success in trigonometry were analysed. Space does not permit the second and third aspects to be reported here, but full details are given by Kendal (1992).

No formal research has been reported which compares the two ways of introducing trigonometry although there has been continuing presentation of anecdotal evidence and opinion. Contributions to the debate on which method should be used and the difficulties experienced by students learning trigonometry include Butler & Lynwood Wren (1970), Ellis (1990), Lloyd (1976), Markel (1982) and McNaughton (1965) discussing the ratio method, Collins (1973), Dooley (1968), Ince and Johnston (1988), Shear (1985), Stokie (1966), Trende (1964) and Willis (1966, 1967) who advocate the unit circle approach and Ellery (1980) and Satty (1976) who favour a combination of the two methods.

When first introduced as part of the "new maths", the unit circle method emphasised the nature of the trigonometry functions as functions taking real numbers to real numbers. One of its virtues was seen to be that no reference to angles or triangles was required, although "the solution of triangles was noted as an interesting and useful outcome" (Dooley, 1968). The trigonometry functions were defined as *functions* of a *real variable*, the length along the circumference from (1,0) to P (see Figure 2). This avoided the undefined notion of angle. Whereas the ratio definitions arise naturally from applications to mensuration and surveying, the unit circle definitions lean more naturally towards applications to periodic phenomena, such as simple harmonic motion. Both of these sources of applications are important. Our question in this study is not which definition should be used - we believe students as they learn more advanced mathematics need both. Our question is which method should be used to introduce trigonometry, given that the principal goal to be achieved at that stage is the solution of right angled triangles.

Description of the teaching methods.

Ratio method. For the ratio method, the trigonometry functions are defined as the ratios of lengths of the sides in right angled triangles. For example, the sine of an angle is defined as the ratio of the length of the "opposite side" to the length of the hypotenuse. Students are often taught to remember the definitions of the ratios using a mnemonic such as SOHCAHTOA (Sine =Opposite + Hypotenuse etc). The named triangle sides are shown in Figure 1, along with one of the hardest calculations that students are expected to do at this early stage. The difficulty of this example arises from the difficulty of solving the algebraic equation.



Find the length of the hypotenuse, x.

$$\sin 39° = \frac{\text{length of opposite}}{\text{length of hypotenuse}}$$

$$\sin 39° = \frac{1.25}{x}$$

$$x = 12.5 + \sin 39°$$

$$x = 12.5 + 0.629$$

$$x = 19.873$$

(OR more accurately, x = 19.863 if the value of sin 39° is not truncated before the final step)

Figure 1. Naming the sides of a right-angled triangle and a hard calculation.

Unit circle method. In the unit circle method, the trigonometry functions derive their meaning from the unit circle, as illustrated in Figure 2. Even though this is the definition which applies to all quadrants of the circle, current textbooks only use the first quadrant. To find lengths and angles in right angled triangles, the given triangle is compared with the standard reference triangles (which students are to learn) and then properties of similar triangles are used for calculation. In order to avoid the difficulties of solving algebraic equations, especially with the unknown in the denominator, recent texts adopt a variation of the unit circle method which uses a scale factor comparing the given triangle with the reference triangle. Figure 3 illustrates the procedures involved in a case where it is difficult to visualise the scale factor.

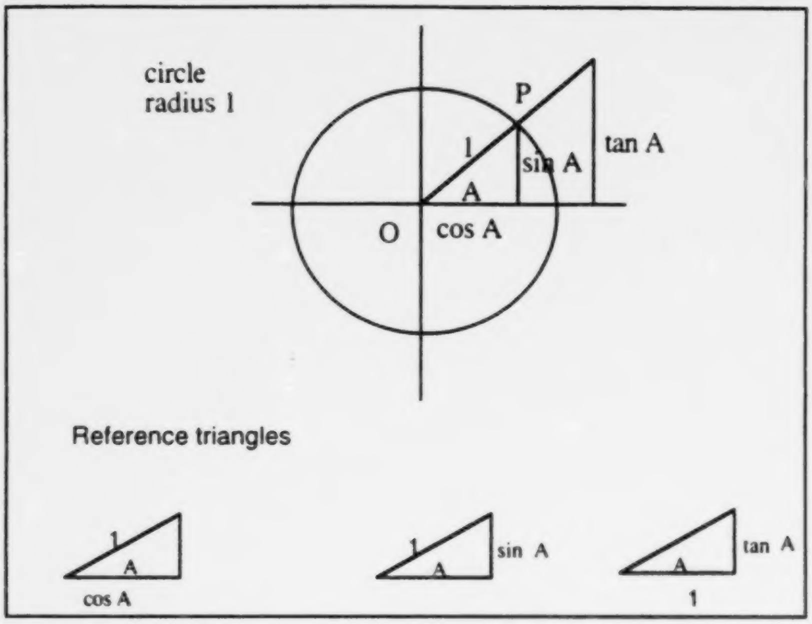


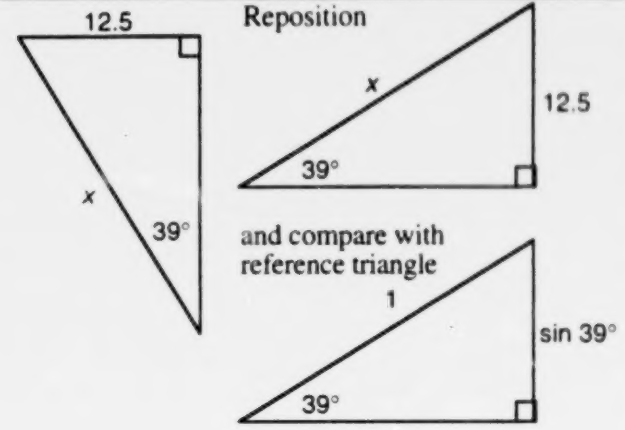
Figure 2. Unit circle definitions of trigonometric functions based on the angle A and reference triangles.

Procedure

An empirical causal-comparative study was undertaken. Classes of Year 10 students at the school where the first author taught were randomly allocated to two groups. Ninety students in four classes were taught basic trigonometry using the ratio method and the other four classes (88 students) were taught by the unit circle method. It happened that the two teachers who taught more than one year 10 class (Teacher 1 and Teacher 2 in Table 1 below) were both allocated one class of each teaching method. This enabled any potential teacher effect to be scrutinised. Each student was assigned to an "ability level" from A (the highest) to F, based on a comprehensive common test taken by all Year 10 students prior to the experiment. A comparison of these rankings showed that the two teaching groups contained a similar number of students at each ability level.

An entire teaching package for each method was prepared by the first author. The teachers were instructed about the methods and the teaching strategies, which included practical work and an outdoor investigation. The emphasis in both cases was on concept development, learning with understanding and enjoyment and skill development. As far as possible the lessons for the two methods were matched in teaching styles and content. The exercises which the students attempted were identical, but ordered differently to fit in with each teaching method. Twenty 45 minute lessons were devoted to teaching this unit of work.

Two tests were administered as both pre-tests and post-tests. Each test contained mathematical items and Likert items to indicate students' attitudes towards the material covered by the test. First, an arithmetic and algebra test was given to determine the level of proficiency of selected algebraic and arithmetic skills which the authors believed to be pre-requisite for successful trigonometry. The items identified students with division misconceptions, students who did not understand the equality nature of the equals sign (Kieran, 1981) and students who could use only informal techniques to solve equations (Kuchemann, 1983).



To find the length of the hypotenuse, x.

Reposition the triangle and match with reference triangle, as shown.

Similar triangles are used to match corresponding sides and set up scale factor

x =scale factor * corresponding side

In this case, x = scale factor* 1
x= 19.863

Figure 3. A difficult example using the unit circle method.

The second test was on trigonometry and was used as pre-test and post-test. It contained 12 questions, each requiring the calculation of a nominated side length in a right angle triangle. About six weeks later, four similar questions were given as part of the end-of-year examination. Other types of questions were treated in the lessons but are not analysed here. Each question on the test was awarded three marks, one for identification of the correct trigonometry function, one for the correct formulation and transposition of the equation to solve (written down or implied by further correct calculation) and one for performing the calculation correctly. If the student initially selected the wrong trigonometry ratio he/she could still gain two marks. Each student therefore received a score out of 36. The students had studied a short unit on trigonometry twelve months previously, using the ratio method. Surprisingly, on the trigonometry pre-test, which was given without warning to the 178 students, all but four students scored zero.

Results

A series of primary comparisons were made to establish which method was more successful. Students' performance on the trigonometry test, and its three constituent skills (their ability to choose the correct function, to formulate the correct equation and to perform the correct calculation) were examined. Two secondary comparisons, improvement in attitude and in solving equations are also reported.

Overall performance on the trigonometry test. Trigonometry performance was defined as the improvement from the pre-test to the post-test. The maximum score was 36. Individual class scores are given in Table 1. The students taught by the ratio method performed significantly better than the unit circle group. This was confirmed by a one-way analysis of variance on the individual scores ($F = 17.79$, $df = 1$, $p < 0.001$). A second measure of success was the higher degree of retention demonstrated by the students on the trigonometry questions on the end-of-year examination. Ratio students obtained a mean score of 8.8 out of a maximum of 12, again shown by an F test to be significantly higher ($p< 0.001$) than the unit circle students' mean score of 7.0. Within each teaching method group, Table 1 shows that the class means show little variation. Factors such as different class compositions and teachers have not influenced trigonometry performance as much as the method of teaching. Further analysis showed that the mean scores of students in every ability group (A to F) was higher for the ratio method than for the unit circle method. Indeed, the lowest mean of the ratio ability groups (27.62 for ability group F) was higher than the highest mean for the unit circle ability groups (25.82 for ability group A).

Choosing the correct trigonometry function. Students taught by the ratio method were more capable of identifying the correct trigonometry functions. The ratio students identified an average of 11.08 of the 12 functions correctly, with a median score of 12. However, the unit circle students identified an average of 8.47 functions correctly, with a median score of 11. A large group of unit circle students identified very few functions. There are several key skills involved. For the ratio method, the student needs to identify the sides in the triangle and then select the correct function. Most students reported using the mnemonic SOHCAHTOA to help them do this. In contrast, the unit circle method requires a complex inter-related set of procedures. The student must re-orient the given triangle if necessary, recall the unit circle diagram and identify which reference triangle matches the re-oriented triangle. Approximately one third of students could not correctly re-orient the triangle and nearly half were unable to draw correct reference triangles. The reference triangle for tan was the most difficult.

Table 1.
Class means (and standard deviations) for trigonometry performance

Ratio Method				Unit Circle Method			
Class	N	Mean	(S.D.)	Class	N	Mean	(S.D.)
Teacher 1	20	31.35	(8.92)	Teacher 1	20	22.60	(9.57)
Teacher 2	23	29.13	(9.59)	Teacher 2	23	19.26	(12.92)
Teacher 3	22	32.32	(6.21)	—			
Teacher 4	21	32.76	(4.98)	—			
—				Teacher 5	22	21.82	(12.64)
—				Teacher 6	25	20.48	(11.05)

Ability to formulate and transpose the equation. The equation was considered to be correctly formulated if it was actually written down or it was implied by the student obtaining the correct answer for the selected ratio. For the ratio method, a correctly transposed equation was expected (i.e. in the form x = . . .) and for the unit circle method the scale factor needed to be determined together with the equation to calculate the side length of the triangle. Questions involving only multiplication were simple to formulate (e.g. find the length of a side opposite a given angle of 20° if the hypotenuse is known). These were completed successfully by 94% of ratio students and 68% of unit circle students. However questions where division was involved (e.g. finding the hypotenuse from the sine of an angle as in Figures 1 and 3) the formulation and transposition was more difficult and the success rate was 77% for ratio students and 38% for the unit circle students. The most common mistake for both methods was to multiply instead of divide. Ratio students were more successful with the division questions than unit circle students were with the multiplication questions. Students from all ability groups were more successful solving the problems involving division using the ratio method. The difference was particularly evident with the lower achieving groups, C to F. These results clearly demonstrate that the ratio method of teaching provides all students and in particular the lower ability students with an effective way to overcome their very real difficulties associated with the solution of more complex equations.

Ability to perform the arithmetical calculations accurately. The difference between the number of correctly formulated equations and the number of correct answers was used as the measure of calculation error. The mean number of errors made was 0.442 (s.d. 1.12) for ratio students and 0.456 (s.d. 0.985) for unit circle students. This difference is not significant (F = 0.01, d.f. = 1, p = 0.931). Calculation mistakes are not method dependent and are a minor factor in trigonometry errors.

Improvement in attitude. At the same time as the trigonometry pre-test and post-test were given, students were asked to rate their liking for trigonometry on a five point scale. A nine point improvement-in-attitude scale (from -4 to +4) was derived by subtracting the post-test attitude from the pre-test attitude. For the ratio students the mean improvement in attitude was 0.722 (s.d. 1.00), significantly greater than the mean improvement for unit circle students of 0.318 (s.d. 1.15) (F = 5.75, df = 1, p = 0.018). Every ability level of ratio method students showed a greater improvement in attitude towards trigonometry than the corresponding unit circle ability level students.

Improvement in solving algebraic equations. The scale factor version of the unit circle method is promoted by teachers and texts because it avoids solving equations where the unknown is in the denominator and formal equation solving methods are demanded. On the other hand, the ratio method requires students to solve such equations. Eight items on solving equations with decimals were on the arithmetic and algebra pre- and post- test. Examples are $0.025x = 5$, $30 = 0.003/x$ and $25 = x/0.004$. As expected because of the practice involved, the ratio students showed significantly more improvement than the unit circle students. Ratio students improved by an average of 1.360 questions (s.d. 2.79) and unit circle students improved by 0.402 questions (s.d. 2.26) (F = 6.17, df = 1, p = 0.014). Again, the greatest improvement was seen with the ratio students in the lower ability groups, C to F.

Conclusion

This study has provided substantial evidence to suggest that the ratio method of teaching introductory trigonometry is a better choice for schools than the unit circle method. Students were better able to master the skills required and, with the greater success, came a greater improvement in attitude to the subject. The advantage the ratio students gained lasted through to the end-of year examination. The mean scores of the four ratio classes clustered together, considerably above the means of the four unit circle classes, showing that the teaching method, not the teacher, was the dominant effect. Students of all ability levels performed better with the ratio method, but the low ability students benefited the most. An analysis of the techniques that made these students in these groups successful is given in Kendal (1992).

Two procedures were critical to success on the test items. The first is an ability to identify which trigonometry function is appropriate for the problem. To identify the correct function many teachers, including those in this study, use a very simple, easily remembered mnemonic SOHCAHTOA. In contrast for the unit circle method there is a set of procedures required for re-positioning the triangle and matching it with a reference triangle. Not surprisingly, since the unit circle method is fraught with multiple opportunities for mistakes, there was a highly significant difference in students' ability to select the correct trigonometry function.

The second procedure involves formulating the correct trigonometric equation and transposing it so that it can be solved. For the ratio method, this is the step where most of the difficulties occur as the students may need to solve complex equations. Proponents of the unit circle method have sought to avoid this by adopting the scale factor method. Sometimes the scale factor is straightforward, able to be readily visualised but sometimes a counter-intuitive division operation is required to calculate it. The results of the experiment show that unit circle students were less successful in identifying the appropriate scale factor in complex cases than ratio students were in transposing the complex equations. Although formulations involving division were difficult for students in both methods, ratio students were more successful. Some transfer of learning, an improvement in the ability of ratio students to solve equations with decimals, was also noted.

Implications for curriculum choices.

There are some clear limitations to this study in assessing the place of the unit circle method in a broader view of school mathematics. In particular, the items on the trigonometry test were skill exercises of only one type, albeit the simplest exercises on the main target of the teaching, solving triangles. Students were not tested for conceptual development, nor to see if foundations had been laid for later extension of the definitions of the trigonometric functions beyond the first quadrant. (Surprisingly, in the unit circle method as advocated by the curriculum guidelines, only first quadrant angles are demonstrated in Year 10.) It is possible that the unit circle students may benefit in the long term in these aspects, but their generally very low facility in what was tested makes this seem unlikely. The shockingly low retention of previous learning, as judged by the pre-test, also indicates caution before choosing any method on the basis of untested long-term benefits.

The unit circle provides concrete meanings for the trigonometric functions, as lengths of intervals which can be directly measured. The problem with the unit circle method is not the unit circle. Rather it is the complex procedure that are required to use the resulting definitions to solve triangles. In recommending the unit circle approach, curriculum developers opted for the more difficult path, with promise but no evidence of any long term benefit. Moreover, the scale factor version of this approach which is designed to eliminate one major difficulty has only replaced it by another of equal or greater magnitude.

Student learning is not a haphazard affair but is controlled by factors such as the method chosen by the teacher to convey the concepts and develop the required skills together with a range of teacher and class influences. We suggest that basic trigonometry be introduced using the unit circle trigonometric function definitions, connecting them to the ratio definitions and then adopting the techniques of the ratio method for the solution of triangles. This approach is adopted by Ganderton, McLeod and Creely in *Mathematics For Australian Schools Year 10* (1991, pp. 258-259) and some other texts. This may enable the younger students to experience real, concrete meanings for trigonometry definitions, lay some foundations for more advanced work and also have the thrill of successful performance in mathematics. It is clear that proposed curriculum change should be backed by empirical research, not only a philosophical debate and innovative teaching ideas.

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Children's perceptual judgement of area

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This paper proposes and justifies a program of research to extend knowledge of how children use judgements to determine area of a rectangle. It looks particularly at the relationship between perimeter and height + width rules used in making judgements. Children's responses to a variety of tasks in a pilot study are described and analysed and inferences drawn for further study.

Our interest in investigating area stems from the number of years spent by the first author teaching mathematics in a number of Queensland High Schools. Each year a new group of Year 8 students would arrive in the mathematics classroom and were able to demonstrate their abilities in calculating the area's of regular plane shapes, but were unable to distinguish between area and perimeter. It seemed that a lot of children had little or no understanding of the area concept.

The first explorations of the literature on area learning were based on the work of Piaget (Piaget, Inhelder & Szeminska, 1960). Piaget's theory of developmental stages, his proposals on conservation of area, and his notion that young children's quantitative concepts were largely one-dimensional in character attracted attention. Piaget's findings appeared to be widely accepted by developmental psychologists (e.g., Silverman & Paskewitz, 1988)

However, further exploration uncovered criticisms concerning the validity of Piaget's views and alternative theories. Piaget's theories appeared to ignore perception and did not provide easy application to instruction. A paper by Anderson and Cuneo (1978) proved to be very interesting. They investigated rectangular area judgements of 5 year olds. They concluded that "... most children in this age group can and do take account of both stimulus variables in making their judgements." (Anderson & Cuneo 1978, p.346). They employed a theory they called Information Integration Theory (IIT) and a functional measurement technique to find that young children's judgements of area followed two-dimensional rules: the additive height + width rule; and the multiplicative height x width rule. They showed that children as young as 5 years were using additive and multiplicative integration rules in judging area of rectangles.

The paper of Anderson and Cuneo (1978) influenced a series of studies that employed the methods of IIT and functional measurement to trace out the course of development of area concepts. These studies reported that adults and older children tended to follow the normative multiplicative rule for area while younger children tended to follow the additive rule.

Anderson and Cuneo (1978) considered that the additive height + width rule was a misuse of perimeter. In this, they were supported by the body of literature that followed them. They suggested that the height + width rule reflected a primitive integration process. They argued that the way in which children integrate both the height and width of a shape to form the area is crucial in children's understanding of area.

The mechanism which underlies the height + width and the height x width rules are not known. The form of integration in these rules (additive or multiplicative), and what stimuli cue this integration in 12 to 13 year-old children, has become the interest of the authors.

Research Questions

The aim of the research is to explore how 12 to 13 year-old children determine the area of rectangles (i.e., to identify and explain children's judgement rules for area). This exploration is based on the proposition that children determine area of a rectangle through three steps. First, values are assigned to the height and width of the test rectangle. Second, these values are added or multiplied depending on whether the subjects use the additive or multiplicative rule, respectively. Finally, the results of this operation are reported and can be measured on a response scale.

Purposes

Specifically, the research has the following purposes (which stem from its primary aim):

(1) to identify the way in which children integrate stimuli cues to determine area of a rectangle (i.e., to identify the function by which separate stimulus cues are integrated);
(2) to explore children's use of the specific additive or perimeter judgment rule, Area = Height + Width;
(3) to develop models and theories to explain use and misuse of judgement rules; and
(4) to draw implications for learning and teaching area.

Significance

Mensuration is a means of quantifying and describing the world around us. It provides us with a link between mathematics and the 'real world'. Measuring "... involves students in active participation in mathematics, and provides experience in handling and describing discrete and continuous quantities." (Department of Education, Queensland, 1987, p. 58).

Linear measurement is mensuration in a single dimension. Mensuration of plane and solid figures involves the use of formulae, and when considered in terms of plane figures, it becomes mensuration in two dimensions (Hubbard, Blakey, Elliott, Hubbard, Low, & Maher, 1972). Thus, the study of area is the study of mensuration in two dimensions. An understanding of geometrical figures existing in one plane is essential if a student is to understand mensuration in three dimensions.

As stated above, this research is based on the body of literature stemming from Anderson and Cuneo (1978) which has studied children's judgements using the height x width and height + width rules. It extends this body of literature by investigating the mechanism behind the Area = Height + Width rule, particularly any relation to perimeter. The literature considers that the perimeter and the height + width rule would produce similar responses thus rendering the two rules indistinguishable. The literature fails to address this aspect of children's thinking concerning area of rectangles. The study aims to fill this gap in the literature.

It should be noted that Silverman and Paskewitz (1988) have expressed concern against taking literally the height + width and height x width rules generated in the Anderson and Cuneo (1978) study of area. They agreed that the height + width rule may reflect the use of a perimeter rule. However, as the two rules, perimeter and height + width produce very similar responses, they expressed disquiet that the two rules may be different but not distinguishable in practice.

The work of Anderson and Cuneo (1978) is extended in this research by using a way of exploring children's thinking on area of rectangles which has not been taken up in the literature: the use of nonrectangular figures. Deleting a rectangular corner from a rectangle produces a figure of equal perimeter but less area. The removal of a scoop out of one side of a rectangle produces a figure with less area but greater perimeter. Children using a perimeter judgement rule might be unable to account in their judgements for variation in physical area, while a height + width mechanism might be able to do so (using similar arguments to Anderson & Cuneo, 1978). Any kind of height + width rule implies that responses to rectangles, rectangles with corners removed, scoops removed and right triangles of equal height and width will be proportional. Under the perimeter rule, the responses would show systematic deviations from linearity.

The intention of the research is, then, to seek to find the mechanism that underlies the Area = Height + Width rule. In this way, the research makes a contribution to the teaching of area in terms of differing ways in which children perceive area. This contribution has immediate implications for mathematics education in both primary and secondary schools. Theoreticians aspiring to understand children's perception and practitioners implementing area education can benefit from the outcomes. The results of the study could be directly implemented into both curriculum design and classroom practices in the domain of area.

Theoretical Background

The existing research into the concept of area is based on two theoretical frameworks.

Logical operational approach

The first framework belongs to Piaget who hypothesised how cognition developed, and he claimed that this developmental process is marked by a series of stages, the order of which is invariant although the age of the commencement and conclusion may vary. Piaget does not claim that a person functions solely at one stage, but that a person may function at one level for one concept and at a higher or lower level for another. Each stage represents a different way of dealing with a particular aspect of the environment, and as such, a child's thinking would reflect the stage he/she has reached.

Piaget is also responsible for a theory of the development of the knowledge illustrated by these characteristic stage behaviours. The theory is based on biological functioning which stresses that cognitive development is the result of the person's adaptation to the environment through accommodation and assimilation (Turner, 1975). In this theory, intelligence, at the operational level, is thinking, and both thinking and intelligence have to be distinguished from learning

Integrated Information Theory

The second theoretical framework is that of Information Integration Theory (IIT). IIT is based on the assumption that virtually any obvious response is the integrated combination of personal responses to different aspects of an object (Wolf, 1995). The methodological counterpart of the theory, called functional measurement, allows diagnosis, in simple algebraic terms, "... of the rules which govern integration of information about perceived stimuli." (Wolf, 1995, p. 49-50).

The guiding idea behind functional measurement is to use algebraic rules as the base and frame for psychological scaling. The algebraic rules provide the validity criterion needed to decide the controversy over rating and magnitude estimation. The algebraic rules provide a breakdown of the observed response into its functional components, as represented by the scale values and weights of the various pieces of information (Anderson & Cuneo, 1978).

Interview tasks

The interview tasks employed by Anderson and Cuneo (1978) were based on children's perceptions of the areas of different sized rectangles. Their study investigated ten children aged 5, 8, and 11 who were asked to judge the area of rectangular cookies. Nine of the cookies corresponded to factorial combinations of 7, 9, and 11cm. To obtain measures of the children's area judgements, the children were provided with a scale with two end points. Two special cookies of dimensions 5 x 5 cm, and 13 x 13 cm were used as end anchors. The children were asked to judge how happy a child would be to receive each of the 9 cookies to eat. The cookies, all of equal thickness, were presented individually, and judgement was expressed on a 19-point response scale with a smiling face at one end and a frowning face at the other. The presentation of the cookies was randomised, and a practice phase preceded the test phase.

Analysis of data

The children's responses are plotted against the length of one of the dimensions of the rectangles. If the resulting plot is parallel (a collection of parallel lines or curves), the students' judgements are considered to be additive. If the plot is fan shaped (expanding lines or curves), the students' judgements are considered to be multiplicative. Parallelism provides joint support for three conclusions: (a) it supports the additive model; (b) it provides validational support for the linearity of the response measure; and (c) it provides estimates of the subjective, psychological values of the stimulus variables on validated linear scales. In relation to the area studies, the parallelism theory illustrates the height + width rule. Similarly, fan shaped curves give rise to multiplicative rules. In an area example, estimated area of a rectangle would be expected to follow a height x width model. Statistical tests for both the models are available by use of analysis of variance. Through the use of within-subject design and numerical response, the integration approach makes feasible analysis at the level of the individual subject (Anderson & Cuneo, 1978).

Pilot study

The pilot study conducted was based on the body of literature stemming from Anderson's work in 1978.

Purposes

The purpose of the pilot study was to determine the following:

(1) that 12 to 13 year olds can perform the tasks;

(2) that the changes made to the size and shape of the rectangles to produce the nonrectangular shapes produce different behaviours from the rectangular shapes;

(3) that the instructions given to the children are easily understood and not at all confusing; and

(4) that Queensland educated children can show abilities to integrate both sides of a rectangle when judging its area.

Participants

Three boys were interviewed in the pilot study, David aged 8 years, and Rick and Simon, both aged 12 years. The individual interviews covered three separate experiments and lasted approximately 30 minutes.

Procedure

The boys were asked to pretend that they were very hungry. In order to get rid of their hunger, they were offered a rectangular piece of chocolate. The boys were asked to indicate on a wooden scale (the same as used by Anderson & Cuneo, 1978) how happy or sad the hungry child would be to receive that amount of chocolate in relation to the anchors. The expectation was that a small piece of chocolate may yield a sad response on the scale whereas a large piece may yield a very happy response.

Experiments

The first experiment contained 16 rectangular pieces of chocolate corresponding to all the possible combinations 3, 6, 9, and 12 cm. The second experiment used identical rectangular pieces of chocolate to the first experiment, but with a rectangular corner 'bitten' off producing a figure of equal perimeter, but less area. The dimensions of the 'bitten' off corner were all one third of the length and one third of the width of the rectangular stimulus. The third experiment again used identical rectangular pieces of chocolate but this time had a 'bite' out of one side producing a figure with less area but greater perimeter. The semicircular 'bite' was to be centred along the longest dimension, with the radius of the 'bite' one third of the length of the shortest dimension. Two trials were conducted for each child.

Results

The tasks were appropriate for the two older boys and provided interesting data. The younger boy found verbalising his thinking difficult.

The analysis of the results was two fold. Initially, factorial plots were drawn. These were parallel and indicated that all the three boys had utilised an additive rule in their judgements of area in all three experiments. However, these factorial plots failed to differentiate between the additive height + width rule and the perimeter rule. The second method of analysis was a simple linear regression of the responses. The responses were the dependent variables, while either the actual area, perimeter, half perimeter (height + width), or half perimeter using altered corners or sides, were the independent variables. The equation of the regression line was determined for all possible combinations for all experiments. These equations were then compared with the regression line equations generated by using the actual maximum and minimum values of the chocolate pieces, thus yielding a 'true' regression equation (correlation coefficient of 1.00). The closer a child's regression equation was to the true equation, then it was assumed that the child was using that method to judge the area of the chocolate pieces.

The true regression equations are given below along with the equations best matched from the subjects.

Experiment 1 - Rectangles. True Area is given by $y = 0.54x$, true Perimeter by $y = 0.42x - 3.6$ and true height + width by $y = 0.83x - 3.6$. David is represented by $y = 0.46x - 7.5$ (perimeter rule), Simon $y = 0.33x + 0.96$ (perimeter rule) Rick $y = 0.48x - 4.81$ (perimeter rule).

Experiment 2 - Corner Bites. True Area is given by $y = 0.6x$, true Perimeter by $y = 0.42x - 3.6$, true Height + Width by $y = 0.83x - 3.6$, and true Height + Width (corner sides) by $y = 1.3x - 3.8$. David is represented by $y = 1.0x - 7.8$ (height + width rule), Simon by $y = 0.69x - 0.90$ (height + width rule), and Rick by $y = 1.4x - 4.9$ (height + width rule).

Experiment 3 - Side Bites. True Area is given by $y = 0.66x$, true Perimeter by $y = 0.42x - 3.6$, true Height + Width by $y = 0.83x - 3.6$, and true Height + Width (bite plus adjacent side) by $y = 1.25x - 2.8$. David is represented by $y = 0.48x - 8.1$ (perimeter rule), Simon by $y = 0.82x - 3.9$ (height + width rule), and Rick $y = 0.42x - 4.7$ (perimeter rule).

Following the presentation of the three experiments to each child, the interviewer asked the child what method they had used to judge each piece of chocolate. David was not able to express himself in terms of a rule that he was following, but his comments suggest that he was looking at how many times the width of the test piece could be superimposed along the width of the largest end anchor, and likewise for the test piece length.

Simon indicated that he was looking at both height and width of the test pieces:

> I used the rule length by width times 2. No wait. I mean length plus width. Like if this is 6 cm and this is 3 cm then 6 + 3 = 9, so the area is 9 cm.

This indicates that Simon was using the Area = Height + Width rule, as the linear regression equation suggests. Rick informed the interviewer that the area rule for the rectangle was length plus width times two.

> You get a value for the length and a value for the width ... add 'em together then times the lot by two.

When queried about the missing sections he said you do the same for them, then subtract these from the original. Rick was using the perimeter rule, which again supports the results of the linear regression equations.

Discussion

The pilot study indicated that the research process contemplated could succeed as linear regression equations in the pilot study was able to extend the work of Anderson and Cuneo (1978) by differentiating between students using the Area = Height + Width rule and the Perimeter rule for judging area.

Implications for Proposed Study

The pilot study has shown that the proposed research has potential in two ways. First, the method of estimating area by using hunger, happiness and sadness appeared to work and provide insight into area judgement uncued by reference to area formulae. Second, the use of altered shapes, and altered areas and perimeters, appeared to provide the data from which it was possible to discover more of the nature of the integration cues for determining rectangular areas. In this, the pilot was particularly interesting in that the nonrectangular tasks enabled differentiation between the perimeter and length + width rules.

A large proportion of the research into the area concept has been conducted within the theoretical framework of the logical-operational approach. The logical-operational approach has two significant limitations - the ignoring of perceptual judgment, and a methodology which tests the supplementary abilities as well as the logical abilities of the child. This study utilises an alternative approach which considers the perceptual judgment of the subject, and has a functional measurement methodology, which has the specific purpose of assessing stimulus integration rules.

The primary aim of this research is to identify the function by which separate stimulus cues are integrated in determining the area of a rectangle. The pilot indicated that the functional measurement methodology developed by Anderson could be highly appropriate for this aim.

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Assessing the Major Trends and Directions of Research Into Students' Judgements of Area

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This paper highlights two major research approaches into studying how students judge area, logical-operations and information processing. The logical-operations approach focuses on determining the order in which students acquire area knowledge based on a framework developed by Piaget. The information processing approach focuses on perceptual judgement. The paper assesses the two approaches in terms of appropriateness for future research.

There are two basically different approaches which have guided the research into students' judgements of area. The first approach emphasises logical operations within a conceptual framework developed by Piaget. The other approach emphasises the role of perception and judgemental processes. Both approaches are considered in this paper.

The paper begins by reviewing studies based on the logical-operational approach to research into children's judgement of area. These studies are considered in terms of pre and post 1975; 1975 seems to be key year in which the nature of research changed from being concerned with the order in which area knowledge is acquired to being concerned with conceptions (understanding) and misconceptions of area, and the form of research changed from interviews to teaching experiments. Then, the role of perception and judgemental processes is considered in a section which outlines the nature of information processing techniques. Here a body of literature contradicting the findings of Piaget is considered in relation to students' perceptual judgement of area. The limitations of the logical-operational approach are then presented, thus giving rise to the appropriateness of the alternative approach and its focus on perceptual judgement. The final section of the paper recommends a possible direction for future research.

The Logical-operational Approach

The logical-operational view originates from initial research into students' understanding of measurement conducted by Piaget and his associates in 1960 (Piaget, Inhelder, & Szeminska, 1960). Piaget's description of the development of area concepts has been the subject of a wide variety of studies attempting to replicate, disprove, extend or explain his conclusions. These studies covered the development of the major concepts of conservation and transitivity and they dealt with the following four basic concerns:

(1) difficulties with validating the reality of individual cognitive operations and describing the stages of their development - this issue has led to studies which have replicated (with slight variation) some of Piaget's procedures and materials;
(2) difficulties in investigating the relationships between different cognitive skills - this second issue has been addressed by researchers by administering a series of different tasks to the same sample of students to establish a developmental hierarchy;
(3) misconceptions students have relating to area - information on this issue was obtained by analysing incorrect answers provided by students; and

(4) difficulties with identifying the nature of the progression between stages of development - this has generally been dealt with through training studies.

If these four types of research questions were dealt with simultaneously, the resulting study was considered to be a fifth type. Often this fifth type of study involved the application of information processing techniques.

Pre 1975

Area research issues gradually changed over time. Piaget and his associates published their findings in 1960. During the next 15 years, most area research centred around replicating and extending Piaget's work. A number of studies (Beilin & Franklin, 1962; Goodnow, 1968; Lovell, Healey & Rowland, 1962; Wagman, 1975) followed the design of Piaget and the Geneva group and focused on determining the ages at which students acquire an understanding of measurement concepts. Most, if not all, studies confirmed those of Piaget, Inhelder and Szeminska (1960) in identifying definite stages through which students pass. Piaget Inhelder and Szeminsk (1960) divided the development of measurement into four stages, the middle two stages being further divided into two substages. The defining feature of each stage was the presence or absence of set operations or mental actions.

Stages I and IIA. For children at this cognitive stage, the measurement of area is not feasible. The longest linear measurement is used to make area judgements. Children can not demonstrate an understanding of a complete covering with a constant unit of measure. Children (up to the ages of 5 or 6 years) are also unable to conserve area or understand the notion of transitivity.

Substage IIB. At about 6 - 7 years of age, children have an obscure perception of conservation and transitivity. Some area measurement is possible, but children fail to understand the importance of the size of the unit. Fractional units are often counted as whole. They equate two quantities which have the same number of covering units, even though the size of the units vary.

Substage IIIA. Children 7 - 8 years of age conserve interior area but not complementary area. They are able to apply middle term transitivity as long as it was as large as one of the objects being measured. Children recognise that a quantity is the sum of its unit covering, but they still have no notion of unit size and completeness.

Substage IIIB. From 8 - 10 years, change of position and subdivision are coordinated and measurement through unit iteration is possible. Usually the unit covering needs to be physically applied. At this level, conservation is generalised to cover complementary areas.

Stage IV. This is the logical operations stage. Beginning at 11 - 12 years of age, development of measurement is complete. A child develops the notion that space consists of an infinite and continuous set of points. Children are also able to calculate area from linear dimensions. An older child would "conserve", recognising that the area remains constant despite the perceptual change.

Other studies focused on whether the development of students through the stages of acquisition of measurement knowledge can be accelerated through training studies. In reviews of the literature, both Beilin (1971) and Carpenter (1976) concluded that most training procedures accelerate acquisition of conservation and transitivity, but found little evidence for the preeminence of a single set of training procedures. Carpenter (1976) argued that training is able to extend existing operations, but not create new logical operations.

Post 1975

After 1975, replicating studies were rarely seen, with the exception of Lu's work (Lu, 1981, 1991; Lu, Zhang, & Cheu, 1985) from China. It is interesting to speculate as to why Lu is the only researcher studying the order of acquisition since 1975. Perhaps Lu has not accepted Goodnow's (1968) findings that nationality has very little influence on conservation ability.

Since 1975, area research generally focused on investigating relationships between different cognitive skills. Bell, Hughes and Rogers (1975) were followed by Carpenter, Corbitt, Kepner, Lindquist and Reys (1981), Foxman, Ruddock, Badger and Martini (1982), and Hart (1981), in undertaking large scale assessments of the performance of students on a variety of area tasks. These large scale studies appear to influence many smaller studies relating to the understanding of area conducted throughout the 1980s to the present.

Taken together, the conclusions of the studies indicate the following:

(1) students find the pre-formula area concepts easier than later area concepts;
(2) students use superimposed-grids for determining linear measurements which are then multiplied as well as for unit counting;
(3) most students are able to use A = l x b as a rule for finding area;
(4) students are often introduced to procedures before they fully understand the concepts of area;
(5) very few students have any knowledge or understanding concerning the relationship of the square and triangle area formulae to that of the rectangle;
(6) in estimating area, students tend to underestimate if the concept is presented in a 'real-life' context but overestimate if the same problem is presented in a mathematical context;
(7) students' test performance is dependant upon the means of testing; and
(8) nonconservers have shown area conservation skills if the transformation of the area is made to appear accidental.

Many students, and adults, as shown by Baturo and Nason (1996 in press), lack an understanding of basic area and measurement concepts. This lack of basic understanding means that their area knowledge is limited, disjointed and difficult to apply It also means that errors and weaknesses with area problems does not necessarily indicate a total lack of area knowledge or understanding; rather, the person may not be able to apply their knowledge or understanding in the given situation.

It is not surprising that with the availability of a large amount of data relating to students' understanding, that the 1980s saw the emergence of studies concerned with student misconceptions. Hirstein (1981) and Hirstein, Lamb and Osborne (1978) highlighted the need to investigate students who were failing the tasks an experimenter was administering. As a result, studies since the early 1980s have tended to focus on the misconceptions displayed by their subjects.

The research on misconceptions has shown that students' problems in understanding the area concept relate to a confusion between area and perimeter. In calculating either area or perimeter, the process of obtaining the shapes' measurements was also found difficult revealing errors concerning which dimensions to consider as well as how to count the units along the selected dimensions. It has also been shown that these errors are more conceptual than arithmetical.

As well as on misconceptions, studies have also focused on effective instruction of area concepts. Van Hiele (1986) claimed that instruction is most successful if it is directed at the student's level. The ideas of van Hiele (1986) are partially based on the notion that geometric growth takes place in identifiable levels of understanding. These levels were developed as a result of "perceived deficiencies with the views of Piaget" (Pegg, & Davey, 1991). Recent studies (Sayeki, Ueno & Nagasaka, 1991; Zaslavsky, 1991) have included teaching experiments which aim to provide novel methods for increasing the levels of understanding in the area concept. Their findings were that area instruction which employed novel tools or methods (such as concrete materials and everyday examples) achieved greater student understanding of the area concept than traditional methods. The conclusion drawn was that the use of concrete materials and everyday examples makes the information more relevant to the students, thus increasing their attentiveness and understanding.

Information Processing Techniques

Since 1978, alternative approaches to the research methods employed in the logical-operational studies have emerged. According to Langford (1979, p. 1), this has been in response to the belief that Piaget "... may have missed whole realms of concepts and modes of thought.". In particular, Anderson and Cuneo (1978) argued that, as perceptual processes and judgement theory were largely passed over in the logical-operational approach, they provided an alternative information-processing basis for the study of students' judgement of quantity. They proposed a theory that provided a combined treatment for both physical quantities and social quantities.

Information-processing techniques, based on those proposed by Anderson and Cuneo (1978), have been found to offer the best opportunity to explain the process of area concept development in students and, therefore, have been used in many recent area studies. Although these information processing techniques emerged from the theories posed by Piaget and the Geneva group, they took almost exactly the opposite stance towards Piaget, "... accusing him of insufficient use of mathematics and formal logic." (Langford, 1979, p. 3). Contrary to Piaget's proposals, which were sometimes vague, the information-processing techniques aimed to be quite precise about how area concepts are used in making decisions and guiding actions in specific situations. The evolving body of literature based on these techniques focused on perceptual processes and judgement-theory. It was based upon the assumption that virtually any response is the integrated combination of inner, or subjective, responses to different aspects of a target object (Wolf, 1995).

Anderson and Cuneo (1978) and Wilkening and Anderson (1982) developed a research methodology, titled the functional measurement methodology, which assessed students' quantitative judgements and allowed diagnosis, in simple algebraic terms, "... of the rules which govern integration of information about perceived stimuli." (Wolf, 1995, p.49-50). They developed a theory called Information Integration Theory (IIT) which places importance on problems of stimulus integration and multiple causation. According to IIT, all behaviours reflect a blend of stimuli, and a response is the consolidated resultant of multiple causal forces. In IIT, analysis attempts to scrutinise an observed response into its causal components, while synthesis determines the integration function combining the causal components.

In IIT studies, students were provided with different rectangular shapes and asked to place their area on a linear scale. This scale position were plotted against one of the sides (length) of the rectangles. If the resulting factorial plot was a series of parallel curves, then the students was considered to be using an additive model (the length + width rule). If the factorial plot was "... a diverging fan of straight lines." (Anderson & Cuneo., 1978, p. 339), the students were considered to be using a multiplicative model (the length x width rule). Statistical tests for both the models are available by use of analysis of variance. Through the use of within-subject design and numerical response, the integration approach makes feasible analysis at the level of the individual subject (Anderson & Cuneo, 1978).

The results of the information-processing research have indicated that the Piagetian conclusions relating to development maybe inaccurate. The studies of Anderson and Cuneo (1978) and Wilkening and Anderson (1982), and those following them, revealed findings that strongly contradicted the assertions of Piaget "... that young students' judgements of quantity are necessarily or generally one-dimensional in character" (Silverman & Paskewitz, 1988, p.75).

A number of investigators (Anderson & Cuneo, 1978; Lautrey, Mullet & Paques, 1989; Rummele, Kuhn & Zoeke, 1990; Schllottman & Anderson, 1994; Silverman & Paskewitz, 1988; Wilkening & Anderson, 1982; Wolf, 1995) found that students' judgements of area obeyed two-dimensional rules. In other words, they found that young students (3 to 5 years old) used additive integration rules in which they judged area in terms of a height + width rule, while older students (by 11 years of age) obeyed the normative, multiplicative rules. They argued that 8 year old students were "... in a 'transitional stage' between the height + width rule and the height x width rule." (Silverman & Paskewitz, 1988, p.77). The studies of both Lautrey, Mullet and Pasques (1989) and the Silverman and Paskewitz (1988) found that not all preschool students follow the height + width rule. Silverman and Paskewitz (1988) had students rate the areas of rectangles, triangles and ellipses. Their findings indicated that only a minority of preschoolers use a one dimensional rule. This strengthened the Anderson and Cuneo (1978) argument for rejecting Piaget's contention that young students can judge a stimulus only according to a single, salient dimension.

Limitations of the Logical-operational Approach

A large proportion of research into the area concept has been conducted within the theoretical framework of the logical-operational approach. The logical-operational approach has two significant limitations that are of concern. The first is that the approach ignores perceptual judgement. In Piaget's writings, perceptions and operations are segregated (Flavell, 1963). Wolf (1995) considered the processes of perception a central component to almost all behaviour, including that of conservation of area tasks. It is doubtful, then, that an acceptable theory can be produced if perceptual judgement is ignored.

The second limitation is related to methodology. A correct response in the standard Piagetian task demands supplementary abilities besides the logical abilities in question. For example, the conservation tasks test not only logical capacity, but control of attention, estimation skills and correct semantics. Information-processing researchers (e.g., Anderson & Cuneo, 1978) argue that the all-or-none character of choice data in logical-operational research has fostered all-or-none conceptualisations of what may actually be continuous concepts.

As a result of these limitations, many generalisations have become accepted in logical-operational research that may be incorrect. An experimental methodology has evolved in this research approach that is inadequate to determine many of the substantive concerns that it faces.

Recommended Future Research

The evidence from the literature (Anderson & Cuneo, 1978; Lautrey, Mullet & Paques, 1989; Rummele, Kuhn & Zoeke, 1990; Schlottman & Anderson, 1994;

Silverman & Paskewitz, 1988; Wilkening & Anderson, 1982; Wolf, 1995) is that some students judge area using a height x width rule while others use a height + width rule. However, the mechanisms that underlie these area judgement rules is unknown. At this time, the best approach to find these mechanisms would appear to be to study the way separate stimulus cues are integrated to determine the area of a rectangle and to identify the function by which this integration occurs. Because the functional measurement methodology and Information Integration Theory (IIT) developed by Anderson use algebraic rules to provide the validity criterion needed for rating and magnitude estimation, they appear to be a highly appropriate approach to studying area judgement. The algebraic rules provide a breakdown of the observed response into its functional components, as represented by the scale values and weights of the various pieces of information (see Anderson & Cuneo, 1978).

One of the interesting aspects of the IIT studies is the relationship between the additive length + width rule and the perimeter rule. The literature considers that the perimeter and the height + width rule may produce similar responses thus rendering the two rules indistinguishable (e.g., Silverman & Paskewitz, 1988). It may be possible to investigate this relationship by using nonrectangular figures (Anderson & Cuneo, 1978; Silverman & Paskewitz, 1988). For example, deleting a rectangular corner from a rectangle produces a figure of equal perimeter but less area and removing a scoop out of one side of a rectangle produces a figure with less area but greater perimeter. In these cases, students using a perimeter judgement rule may be unable to account in their judgements for variation in physical area, while a height + width mechanism may be able to do so.

In summary, the problem of stimulus integration in students' area judgments remains largely unanswered. IIT research has made some advances with the Area = height + width and Area = height x width rules. There seems to be a possibility for further advances in continuing IIT research with nonrectangular figures. The development of models to explain the mechanisms underlying stimulus integration appears to be a fruitful aim of this research.

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First Year Teacher Trainees' Understanding of Geometry

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In the early 80s, Mayberry (1981) developed a diagnostic instrument to be used to assess the van Hiele levels of pre-service teachers. The test was designed to be carried out in an interview situation. The Mayberry study has been replicated under Australian conditions in a written format, testing sixty first year primary-teacher trainees. This paper presents the results of the study, comparing them with the results of the Mayberry students, and relating them to their geometric backgrounds. Responses, in general, show that many of the students who had completed a recognised senior secondary geometry course could not display better than Level 2 understanding.

Being able to assess students' levels of understanding is a part of the ability to instruct students at their van Hiele level of insight. To assist in making such an assessment, there needs to be available a reliable diagnostic instrument. In the early 80s, Mayberry (1981, 1983) in her work with pre-service primary teachers, developed a diagnostic instrument for use in an interview situation. In 1992, a written test based on the Mayberry items was developed at the University of New England, and used in a detailed study of the geometric perceptions of 60 Australian primary-teacher trainees. Analysis of the results by concept and by level led to the identification of strengths as well as inconsistencies in the Mayberry items. Lawrie (1993) reported on four main aspects in Mayberry's work which have the potential to lead to the incorrect assessment of a student's van Hiele level of understanding in geometry. In contrast, several items designed by Mayberry to test for Level 4 understanding, were shown by Lawrie (1994) to be capable of indicating, not only a student's ability in deduction, but also a student's degree of reasoning for any van Hiele level. This paper presents the quantitative results of the Australian study, relates the results to the students' geometric backgrounds, and compares them with the assessments of the American students.

Background

The van Hiele Theory

In the 1950s, Pierre van Hiele and Dina van Hiele-Geldof completed companion PhDs which evolved from the difficulties they had experienced as teachers of geometry in secondary schools. Whereas Dina van Hiele-Geldof explored the teaching phases necessary in order to assist students to move from one level to the next, Pierre van Hiele's work developed the theory involving five levels of insight. A brief description of the first four van Hiele levels as they relate to 2-D geometric figures is given. These are the levels which are commonly displayed by secondary students, and are the levels involved in this study, in the assessment of the students' responses.

Level 1	Perception is visual only. A figure is seen as a total entity and as a specific shape. Properties play no explicit part in the recognition of the shape.
Level 2	The figure is now identified by its geometric properties rather than by its overall shape. However, the properties are seen in isolation.
Level 3	The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised.
Level 4	Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions are used with understanding.

Van Hiele (1986, pp.5-6) maintains that, contrary to the theories of Piaget, progression from one level to the next is not the result of maturation or natural development. He explains (p.39) "The attainment of the new level cannot be effected by teaching, but still, by a suitable choice of exercises the teacher can create a situation for the pupil favourable to the attainment of the higher level of thinking". Students' understanding of a geometry topic undergoes several changes as they continue their studies, the changes reflecting their progress through the van Hiele levels in a particular order. If one level is not mastered before instruction proceeds to the next level, students may perform only algorithmically on the higher level.

Design

For the study, the Mayberry interview items were modified to ensure clarity of the written questions. A preliminary study validated the reliability of the written questions. The test was restricted to one hour, and hence the seven topics tested by Mayberry were divided between two papers; Paper 1 tested the concepts square, right triangle, circle and congruency, whilst Paper 2 tested the concepts square, isosceles triangle, parallel lines and similarity. The sample consisted of the students enrolled in Mathematics Education I, the first of two compulsory mathematics education courses in the Bachelor of Teaching degree at the University of New England. The test was administered in the second week of the course, before instruction in the space segment. The students were randomly allocated to a paper. Before the test, the students were instructed both orally and in writing, to present as much detail as possible when giving reasons for their answers. The students were also asked to complete an information sheet detailing their mathematical background.

The evaluation method developed by Mayberry was used to assess the students' responses, thus enhancing the comparability of the two sets of results. This method registers whether a student has given sufficient correct answers at a particular level to reach the criterion set for the level. Results are expressed as a quadruple showing the highest van Hiele level of thinking displayed by a student for each of the four concepts. For example, a quadruple (3,3,3,2) resulting from Paper 1 would indicate that a student displayed Level 3 thinking for the concepts square, right triangle and circle, and Level 2 thinking for the concept congruency. A 10% sample of the students were interviewed to validate the results of the written test. The responses to individual questions and to clusters of questions were further analysed to ascertain whether the patterns Mayberry perceived in her evaluation, also occurred in the Australian responses. Mayberry applied Guttman's scalogram analysis, calculating a coefficient of reproducibility of 0.95. This high value was taken as "an indication that the van Hiele levels do form a scale, and that, therefore, the levels as tested by these questions are hierarchical in nature" (1981, p.71). A Guttman coefficient of 0.98 for the Australian study was considered confirmation that the transformation of the interview questions to a written test paper had not affected the essence of the Mayberry test.

Results

Every endeavour was taken to replicate Mayberry's evaluation of responses. Her thesis was examined in depth to ascertain her expectations in the responses to the items. However, this was not possible for every item, there being occasions, particularly for the Level 3 and Level 4 items, when insufficient information was to be found in the Mayberry writings. The number of correct responses tabled appeared to exceed Mayberry's statements in the text concerning the number and type of responses given for some items. In this report, to facilitate comparison all results are given as percentages, with the horizontal sums in Tables 1, 2, 4 and 5 being 100%.

The results of the assessment of the students' levels of understanding are summarised below in Table 1, whilst Table 2 shows the comparable results for the Mayberry subjects. Some students failed to identify concepts. Those students are recorded as achieving No Level. The results show that, for both studies, the majority of students were assessed as having no greater than Level 2 understanding, i.e. they were comfortable recognising concepts, and listing the associated properties, but did not understand the relationships between the properties. The tables also show that the most familiar topic for both sets of students appears to be the circle. However, there is no evidence to support the notion that students find the concept of the circle easier to understand than other common 2-dimensional shapes. Rather, the detailed investigation into the Mayberry test items has shown that the better-than-expected results for the circle are attributable to weaknesses in the design of several of the questions (Lawrie 1993, pp.383-385).

Table 1
Highest level reached by the Australian students for each concept
(% of sample)

Concept	No Level	Level 1	Level 2	Level 3	Level 4
Square	0	3	84	7	7
Right Triangle	3	19	55	19	3
Isosceles Triangle	7	27	43	20	3
Circle	0	13	19	52	16
Parallel Lines	0	17	80	0	3
Congruency	0	32	35	3	29
Similarity	0	43	40	10	7

Table 2
Highest level reached by the Mayberry students for each concept
(% of sample)

Concept	No Level	Level 1	Level 2	Level 3	Level 4
Square	0	11	32	26	32
Right triangle	26	21	21	16	16
Isosceles triangle	26	16	11	26	21
Circle	5	11	16	21	47
Parallel lines	26	16	16	37	5
Congruency	0	21	32	21	26
Similarity	5	42	5	21	26

A most common level of working was considered to be a useful descriptor, particularly if wanting to refer to the overall insight of a student. This most common level was determined for each student by calculating the mean of the four assessments listed in the quadruple. In the example given above, the quadruple (3,3,3,2) has a mean of 2.75. This is rounded to a whole number, resulting in the student being said to have a most common level of working of Level 3. Table 3 shows the percentage of students most commonly working at each van Hiele level. Although three students each failed to recognise one of the concepts, all students showed a most common level of understanding of at least Level 1. This table again indicates that the majority of students lacked understanding of the relationships between properties, and of knowledge of formal proof.

Table 3
Australian Students' most common van Hiele working level

van Hiele level	Percentage of students
1	8
2	69
3	10
4	13

Table 4 relates the students' secondary geometric backgrounds to the level of understanding each has most commonly displayed in their responses to the test. Of the Australian students, 64% had completed a senior secondary mathematics course which included a formal or recognised geometry segment (e.g., NSW 2 Unit Mathematics, Queensland Mathematics I), 23.5% had completed a senior secondary mathematics course which did not contain a formal geometry segment (e.g., Mathematics in Society), and 12.5% had not completed any senior secondary mathematics. It is significant that 63% of the students who had completed a course in which the instruction is assumed to be at van Hiele Level 3 and possibly higher, could not display overall understanding of Level 3 knowledge in their responses.

Table 4
Relationship between the Australian students' geometric backgrounds and their most common van Hiele working level
(% of sample)

Geometric Background	van Hiele Level 1	van Hiele Level 2	van Hiele Level 3	van Hiele Level 4
Senior geometry	0	63	14	23
Senior maths but no geometry	8	92	0	0
No senior maths	43	43	14	0

Mathematics is commonly studied in High Schools in the USA (Years 9/10 to Year 12), as separate optional courses, e.g., algebra, calculus, geometry. Mayberry's examination of high school geometry textbooks (1983, p.68) showed that "Level 3 thought appears to be needed to begin the course and that Level 4 thought should be developed during the course." In Australia, most of the mathematics courses offered in senior secondary schools have a composite syllabus, the geometry segment of which generally appears to be designed for Level 3 (and sometimes Level 4) instruction. For example, notes on content of the plane geometry segment of the NSW mathematics syllabus for the 2 Unit and 3 Unit courses (Board of Senior School Studies, 1982, p.14) state that students are expected to have "a knowledge of the various common geometrical figures and their properties" (van Hiele Levels 1 and 2). The content includes the development of the understanding of notions, and of the ability to provide proofs for deductive exercises, i.e., the development of van Hiele Level 3 and early Level 4 competency. In Mayberry's study, 68% of the subjects had taken geometry as a course in High School, and 32% had not. This is similar to the composition of the Australian sample in which 64% had studied senior geometry and 36% had not. For Australian and American students with similar geometric backgrounds, Table 5 compares the highest van Hiele level achieved by each student for each concept.

Table 5
Comparison of highest levels reached by students who have taken senior geometry and those who have not, for Australian and Mayberry students
(% of sample)

Highest level reached		No Level	Level 1	Level 2	Level 3	Level 4
Geometry background	Australian	1	12	50	19	18
	Mayberry	13	16	12	29	30
No geometry background	Australian	3	33	55	9	1
	Mayberry	12	26	33	14	14

Of the students who had completed a recognised geometry course in which the content was at van Hiele Level 3 or higher, only 37% of the Australians and 59% of Mayberry's subjects could demonstrate Level 3 or better understanding in their responses. The differences in the success rates between the two countries may be a reflection of the optional nature of the American geometry course, this attracting students more appreciative of geometry. However, the generally low number of students displaying a level of insight which matches the level of instruction in the senior geometry courses may be an indication that many students have learned by rote to manipulate the 'knowledge' of geometry which they possess but do not understand.

Mayberry's examination of individual questions and clusters of questions led her to summarise many of the patterns displayed by students in their responses to individual questions and clusters of questions (1981, p.89). Most of these patterns appeared also in the Australian results. Overall, the analysis of the responses to individual questions and to clusters of questions in this study, indicated that:

1 some students have difficulty in recognising a figure in a non-standard position;
2 properties of sides of figures are more readily identified than properties of angles;
3 some students working at Level 2, property identification, resort to quantifying a figure when tackling a Level 3 problem;
4 other students, unable to work with a generalised figure, draw a particular geometric figure;
5 many students do not have an understanding of class inclusion, for example, do not recognise a square as a rectangle;
6 most students do not understand the difference between necessary and sufficient conditions;
7 on only three occasions did students fail to reach the first level, all occasions occurring with the triangle concept;
8 the majority of students who had completed a senior secondary geometry course in which the content was Level 3 or higher, were not able to demonstrate Level 3 insight in their responses.

Conclusion

The distribution of the Australian results was similar to that of the Mayberry results. More than half the students from both countries were unable to demonstrate skills for levels higher than Level 2. Low levels of understanding (Levels 1 and 2) were displayed both by students who had completed a recognised senior geometry course, as well as by those who had not. However, a higher percentage of the American students who had completed a senior geometry course were shown as demonstrating ability at Levels 3 and 4. This difference between the results of the students from the two countries needs to be investigated further. First, there needs to be an examination into whether the responses accepted for the Mayberry and the Australian studies were of the same level. As mentioned earlier, every endeavour was taken to replicate Mayberry's expectations in the responses. However, there were occasions with the higher level items, when there was insufficient or confusing information concerning what was an acceptable answer. Investigation also needs to be carried out comparing the content of American and Australian courses in geometry, the time spent teaching these courses, and whether the teaching in the USA is more effective than in Australia, i.e., is the instruction of the majority of the students at their van Hiele level of insight? If this last factor is true, then to what degree is this readiness of the students for Level 3 instruction a result of geometry being offered as an elective course rather than as a compulsory segment of a composite course?

In a paper he wrote in 1959 (cited by Mayberry 1983, p.67), van Hiele stated two implications of his theory:

(a) a student cannot function adequately at a level without having had experiences that enable the student to think intuitively at each preceding level;

(b) if the language of instruction is at a higher level than a student's insight, the student will not understand the instruction.

The results of both studies indicate that many of the students attempting senior geometry may not have been at an appropriate level to understand the instruction. This appears to have led to many senior secondary students learning by rote to manipulate the 'knowledge' of geometry which they possess but do not understand and of which they have not seen the origin. They have been offered a system of relations ready-made, and have not learned to establish the connections between the relations, and do not know how to apply what they have learned.

In Australia, there is much emphasis on the final results for secondary schooling, this often serving as an admission measure for tertiary education. Emphasis on the scoring outcome can have an adverse affect on the amount of time spent in the classroom which is directed towards the discovery of knowledge. This can result in the giving of subject matter or tasks which surpass the comprehension of the students. The teachers sometimes quieten their consciences by pretending that the final score is what is important. Van Hiele (1957, p.241) noted that "both examinations and test-papers tend to push the pupil towards algorithmical insight instead of leading him on towards the far more valuable higher forms of insight."

If students are to be instructed at their van Hiele level of insight, the nature of mathematics as described in the general principles which underlie, for example, the NSW syllabuses, needs to be kept in mind:

> Mathematics is:
> - a search for patterns and relationships
> - a way of thinking
> - a powerful, precise and concise means of communication
> - a creative activity. Therefore it may involve invention, intuition and discovery.
>
> NSW Department of Education, 1989, p.2

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The Relationship of the Concept of Fair to the Construction of Probabilistic Understanding

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This report combines the results of two small studies of probability based on the concept of fair. The outcomes of the first study led to additional opportunities in the second and finally to a hypothesis of conceptual development. In both studies students from grades 3 to 9 were interviewed using a protocol designed to assess their understanding of fair in relation to dice. The theoretical framework used to analyse student responses was the SOLO Model with Multimodal Functioning developed by Biggs and Collis.

Children experience the concept of fairness in a variety of contexts most of which are outside the school curriculum. Two consecutive studies discussed in this paper aimed to associate these experiences with the development of a mathematical notion of fairness as it relates to probability. In particular the investigations focussed on the notion of fairness as it relates to dice and equally likely outcomes. An understanding of equally likely outcomes is an important aspect in the development of an understanding of probability (Australian Education Council, 1994). Determining mathematical fairness can be a higher order cognitive task because not only does it require knowledge of equally likely events but also in the context of the activities used in the second study it requires an understanding of sampling and data analysis.

Many stochastic understandings begin development at an early age (Yost, Siegel & Andrews, 1962; Goldberg, 1966; Fischbein, 1975) and with the concept of fair it would be reasonable to assume that children begin to develop their understanding before they start school. Sibling and family rivalry provides a motivation for understanding the meaning of fair, with children having many opportunities to construct their meanings in relation to sharing and equality (Streefland, 1991). It is doubtful that to date the full extent of these understandings have been explored. As a consequence children may be making meaning of school-presented material in the context of their current understanding of fair and this may lead to a mismatch between the intended learning planned by the teacher and the actual learning constructed by the students (Kapadia, 1988). Although not specifically related to fairness the research of Fischbein and Gazit (1984) describing the intuitions developed by relatively young children through experiences prior to formal schooling indicates a connection between these intuitions and the later development of complex mathematical concepts. They stress the importance of acknowledging these intuitions or cognitive beliefs because they can assist or impair understanding of probability and statistics concepts during schooling.

Very little appears to have been written about the development of the mathematical concept of fairness although an expectation that students have an understanding of the concept of fairness is implicit in a number of studies both of older students (Konold, 1989; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993) and of younger students (Watson, Campbell & Collis, 1993). Konold, et al.'s research (1993) poses questions about the likelihood of possible outcomes when a fair coin is tossed five times. Fairness associated with games has been the focus of numerous projects (Bright, Harvey & Wheeler, 1981; Peard, 1995).

Kahneman and Tversky (1972) described a representativeness heuristic which students employed when asked to predict randomly generated outcomes. For example, if the students were asked to predict the outcomes from tossing a regular die sixty times, using the representativeness heuristic their responses would closely resemble the theoretical outcome of 10 of each number 1 to 6. In exploring the notion of fairness the second study reported in this paper has focussed on the students' prediction of outcomes from a similar perspective and then extended this idea to explore how far results differ from the uniform theoretical distribution before students are willing to accept that the sample is not representative.

A consistent theme in research exploring students' understanding of probability and statistics has been that of students' misconceptions (Kahneman & Tversky, 1972; Shaughnessy, 1977; Cox & Mouw, 1992). The use of this term indicates a deficit view of learning which judges mental constructs as being either correct or incorrect. The authors prefer a constructivist perspective of learning where the student's understanding is not viewed as a misconception but rather as an alternative conceptualisation to that which has been conventionally agreed. Some "misconceptions" could more accurately be described as understandings which are incomplete in some respect.

Theoretical Framework

The theoretical model used for the analysis of the data from these studies is the SOLO Taxonomy (Biggs & Collis, 1982) incorporating intermodal and multimodal functioning (Biggs & Collis, 1991; Collis & Biggs, 1991). Evolving from Piagetian theory, the model postulates five modes of functioning which originate in a fixed order but which continue to develop alongside each other throughout life: sensorimotor (from birth), ikonic (from early childhood), concrete symbolic (from the years of schooling), formal (for those capable of higher education), and post formal (associated with research). The ikonic and concrete modes will be most relevant in these studies because they are associated respectively with intuitive functioning and with the concrete symbolic learning which takes place in school often based on concrete materials.

The model proposes that earlier developing modes continue to develop in conjunction with later modes and provide opportunities for interaction which may facilitate intellectual functioning in general. Results demonstrating multimodal functioning in the acquisition of understanding of fractions (Watson, et al., 1993) and the work of others in the field of stochastics (Berenson, Friel & Bright, 1993; Callingham, 1993; Fischbein & Gazit, 1984; Watson & Collis, 1993) lead to the belief that such functioning will also occur for concepts related to fairness in probability.

Within each mode hierarchical development takes place by means of a cycle of learning (Biggs & Collis, 1982; 1991) which has five levels: prestructural, unistructural, multistructural, relational and extended abstract. Considering responses within a mode of functioning, it is the middle three types of response which are of main concern: unistructural (U), multistructural (M), and relational (R). Recent studies of mathematical understanding have found two U-M-R cycles operating within the concrete symbolic mode in connection with students' understanding of volume measurement (Campbell, Watson & Collis, 1992), fractions (Watson, et al., 1993), arithmetic mean (Callingham, 1993) and data handling (Watson, Collis, Callingham & Moritz, in press). In complex settings, mathematical fairness is a higher order concept which incorporates aspects of both probability and statistics and therefore is likely to require two U-M-R cycles to categorise responses in the concrete symbolic mode. With the concept of fairness intuitive understandings would be expected to be significant because children probably develop notions of fairness from an early age through varied experiences with sharing and games. It has been noted that intuitive reasoning is frequently used by adults in solving problems with probabilistic contexts (Konold, et al., 1993). There is hence an interest in the ikonic mode and the interaction of it with the concrete symbolic mode.

Study 1

Method

An interview protocol was used in the first study with 30 girls in grades 3, 5, 7 and 9 from a South Australian independent school. The dice protocol was one of eight used during a 45-minute interview. Students were introduced to the protocol with questions on whether they played games with dice and whether some numbers were more likely to come up more often than others. Then they were shown three dice; a blue one which had been weighted producing a bias in favour of the number two, an unmodified red one which was similar in external appearance to the blue die, and a white one on which two of each of the numbers 1, 2 and 3 were represented. In an ensuing discussion, students were asked whether the dice were fair or not and how they could justify their decisions.

Results

An analysis of the results from the South Australian sample produced several trends which were significant in the design of the second study and the length of time devoted to the topic of fairness of dice. The results are first summarised with respect to the type of argument used in discussing outcomes expected in relation to tossing dice.

In grade 3 all responses incorporated ikonic belief at some stage. These were related to personal experience (e.g., "whenever I get a 1, I knew I was getting a 1 which is sometimes a pain), myths (e.g., "3 because in fairy-tales there is 3 wishes and there is 3 fairy god-mothers and there's 3 witches and there's all sorts of 3 things"), anthropomorphism (e.g., "the dice isn't really a genius") and a combination of these (e.g., "6 comes up not the most because it's biggest and for me 2 usually comes up"). These responses decreased markedly after grade 3 with only one grade 5 emphasising similar beliefs.

Arguments based on physical characteristics, unsystematic trials (a couple of tosses) or an "anything can happen" view of chance were relatively constant across all grades in the sample. These were hypothesised to be typical of the first cycle concrete symbolic mode (Watson, Collis & Moritz, 1995a). They were often combined with ikonic support and more sophisticated arguments. With reference to physical characteristics, reference to colour decreased and heaviness increased as criteria for determining fairness over the grades interviewed.

By grade 7 half of the students were suggesting systematic trials to determine fairness of dice and by grade 9 most students were adding mathematical arguments to justify their reasons for drawing conclusions about dice. This indicates movement to the second cycle of the concrete symbolic mode. The suggestion of trials in various forms, however, was not restricted to the oldest students. One grade 3 suggested that to determine which dice were fair, "You could play a game all day and see what happens." A grade 5 student said, "You need heaps of throws." Older students made more sophisticated suggestions.

Of particular interest were the responses which dealt directly with the idea of dice being "fair", that is each number having the same chance of coming up. The youngest students were willing to accept both the view that some numbers come up more than others and the view that each number has the same chance of coming up. This may be a result of viewing "chances" as associated with the future and hence "fair", while numbers which seemed to come up more often were observations from personal experience in the past and hence factual evidence. Early childhood teachers will recognise that this phenomenon of being able to hold contradictory propositions as true at the same time is fairly typical of young students' views on the mathematical underpinning of everyday experiences.

Older students were split between two categories of understanding in relation to these questions. In one group were those who believed that no numbers were more likely to come up than others and hence all numbers have the same chance on a fair dice. Whether this understanding was based on personal experience of playing games, experience of tossing dice, or a theoretical understanding of the mechanism associated with the construction of dice is unknown. In the other group were students who acknowledged that, in one way or another, it may appear that some numbers come up more often when dice are tossed. This may be due to some personal experience, say wanting a number and it not occurring. These students, however, believed that in the long run, say if many trials were performed, the results for all numbers would even out and hence fair dice produce the same chance for all numbers. These students were dealing with conflict of sometimes-observed short term results and the understanding of long term trends. It is hypothesised that this represents a more sophisticated understanding of the operation of fair dice.

For those who did not suggest the using of a series of trials to determine which dice were unfair, physical characteristics were the main alternative. Once it was realised that one of the dice had the numbers 1, 2, 3 repeated, it was not uncommon for younger students to declare the other two dice "fair" because they had all of the numbers from 1 to 6 on them. Some (not many) suggested that the die with two each of 1, 2, 3 was still fair because it had two of each number - this was certainly a real possibility and represents a sophisticated intermediate response - the use of trials being an independent suggestion.

Within the scope offered in Study 1, the responses were classified as shown in the left column of Table 1. In the ikonic mode, no distinctions were made in terms of a U-M-R cycle. In the first cycle of the concrete symbolic mode, unistructural responses reflected single physical characteristics or the "anything can happen" view. Also contradictions were accepted. At the multistructural level, other characteristics of the dice were combined to judge either fairness or unfairness (e.g., the distribution of numbers). At the relational level, some initial rather unsophisticated attempts to consider pre-sampling ideas occurred as did the realisation of fairness in other situations than the presence of the numbers from 1 to 6. More sophisticated ideas related to sampling occurred in the second cycle but there was a ceiling effect on the opportunity for responses due to the protocol's structure and the length of time available.

Study 2

Method

During the analysis of Study 1, there were indications that given the opportunity students would have provided more elaborate responses. This led to modification in the design of the protocol used in Study 2 and enabled a more detailed categorisation of responses in the ikonic mode.

In Study 2, 24 Tasmanian students from grades 3, 6 and 9 participated in a similar interview protocol incorporating the biased blue die and the red die. The expanded dice protocol occupied the entire 45-minute interview. A "Horse Race" game was used to provide a context for questioning the students about whether the dice were fair, and a series of graphs were also used for the same purpose. The data presented in the graphs were produced using Prob Sim© (Konold & Miller, 1992) where three of the dice were fair and three of the dice were unfair. The data were selected to show a range of samples of 60 trials where one was fair and looked fair, two were fair but looked unfair, one was unfair but looked fair, and two were unfair and looked unfair. See Lidster, Pereira-Mendoza, Watson and Collis (1995) for a complete set of the series of graphs.

Results

The opportunity to explore ideas more fully in Study 2 made possible a more detailed analysis of responses in terms of the SOLO model. The responses that were categorised as ikonic were identified within three levels: unistructural, "all the dice in the whole wide world are fair!"; multistructural, "the die is fair for me but it wouldn't be fair for the person I was playing against"; and relational, "in some games six is the worst number to roll." The unistructural responses asserted a single belief, whereas the multistructural responses contained two or more connected, but not integrated, beliefs and often considered perspectives other than the student's own. When past experience influenced judgements the responses were relational within the ikonic mode.

A shift from beliefs and views held from experience in the past to an attention to the present situation marks the transition from ikonic responses to unistructural responses in the first cycle of the concrete symbolic mode. When asked how they would determine if a die were fair, many of the grade 3 to 6 students attended to the physical characteristics of the die: "There aren't any chunks out of it," or "The bits missing where the dots are might make it unfair." Other responses classified in this category related to the way that the die is rolled and these came from students across all grades. Students who connected several of the unistructural ideas which were related to physical characteristics of the dice were considered to have responded at a multistructural level.

Many of the alternative or incomplete conceptualisations of fair appeared within the first cycle of the concrete symbolic mode. Some students displaying this level of

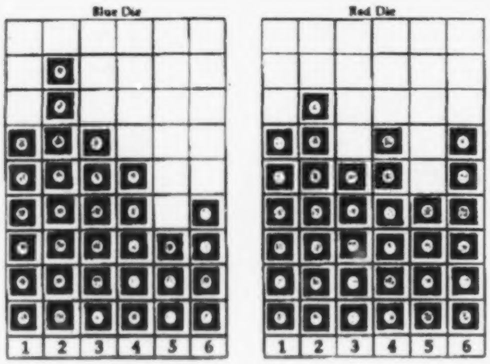

Figure 1. Results of a "Horse Race" game.

cognition were selecting particular outcomes *between* two samples. Figure 1 depicts the outcomes from the blue and red die after the completion of a "Horse Race" game. When students were comparing the samples produced by the two dice after playing the Horse Race game, a grade 3 student, ignoring the fact that some numbers came up more often than others, established that "the dice are fair because they have about the same amount [respectively] of each number." Another student said, "The fives look about the same for both dice and so do the ones." After comparing the number of times one came up on the blue and red die a grade 6 student maintained, "One is the fairest of them all".

Other students compared the outcomes *within* a sample from a single die when looking at the graphs. "This dice is fair because the 1 and 2 are fair and the 3, 4, 5 & 6 are all fair." Some students encountered conflict at this level deciding that a die could be both fair and unfair at the same time. "Fair in one way but not in another way. Basically really fair but it isn't fair for the four," or "This one looks pretty fair except for the two." These responses are multistructural with ikonic support in the form of a belief that dice must be fair. This belief together with a valuing of some numbers over others also led to responses which judged a die to be partially fair. In comparing graphs exhibited by the interviewer, one student commented, "Pink [the pink die] is a bit unfair because 1 and 2 are up but 3, 4, 5 & 6 are a bit fair."

An emerging understanding of sampling appears at the relational level of the first cycle of the concrete symbolic mode. Responses at this level demonstrate a slight shift from the multistructural level with an acknowledgment that a sample from a fair die may look unfair. "Looks pretty unfair but they could be fair." Several grade 6 students in grappling with the notion of trials suggested that the game could be played over and over again to see if the dice were fair. These were also considered relational responses in that they were constructing an integrated scheme by which fairness could be determined.

It is the lack of a more explicit explanation of trialing which differentiates the first cycle relational level responses from the second cycle unistructural level responses in the concrete symbolic mode. At the higher level students described the process of trials easily, demonstrating a consolidated idea of using a sampling process to determine fairness. At the second cycle multistructural level the students considered two or more samples in their judgement of the die but failed to suggest or recognise that a larger sample would provide more reliable information. At the relational level the responses resolved conflict by considering trends across a number of samples of increasing size.

The responses from Study 2 are summarised by SOLO level in the right column of Table 1. Generally at an ikonic level the students relied upon personal experience and belief to judge whether or not a die was fair. The U-M-R cycle within the ikonic mode which was identified in the second study classified responses which began with personal feelings and a view of the die as it related to them personally (U). This was followed by a shift to a consideration of how others might view the die but still within a personal view of the die as it behaved relative to the one using the die (M). Finally the responses referred to past experiences playing games or using dice (R). These responses incorporated experiences of being expected to accept what appeared to be unfair because that was life and "sometimes you had to lose". With respect to Study 1, it is likely that anthropomorphic and mythical responses are at the unistructural level, unless combined together in a multistructural way.

Of particular interest in the first cycle of the concrete symbolic mode was a collection of alternative conceptual frameworks identified at the multistructural level.

These became evident when students were struggling to decide whether results from tossing dice were sufficiently the same or different. It was obvious that for most students this was the first time that they had been forced to consider the possibilities of dice being unfair and what this might look like. This was significant in contributing to cognitive conflict and gave rise to three identifiable alternative conceptualisations. The first was the notion of a die which could be both fair and unfair at the same time. For example, it could be fair for some of the numbers on the die but it was unfair for another or others. The second alternative conceptualisation also related to the die being both fair and unfair but this referred to separate occasions. The die could be judged fair for one sample of tosses but on another occasion the same die would be unfair. The third alternative conceptualisation was of a fairness continuum. A die could be fairer than another but not as fair when compared to a third die. This continuous property of fairness was also applied when comparing samples from the same die, i.e., "this die is getting fairer" or "it is not as fair this time." This illustrates typical multistructural responses as they develop towards the relational level, where the conflict is resolved, for the time being at least, by some form of higher level integration.

In the second U-M-R cycle, the students began to use the idea of sample more effectively as a way of judging a die to be fair or unfair. The observed constructions of this understanding became more structurally complex at each level and at the relational level an understanding based on concrete sampling experience was complete. This understanding acknowledged increasing confidence with increasing sample size and was particularly evident in older students' interpretations of the computer-generated graphs.

Table 1.
SOLO levels for the concept of fair identified in each study

SOLO Level		Study 1	Study 2
Ikonic	U_1	Anthropomorphic dice Witches and myths Personal experiences	Me • what I want / don't want • situation specific
	M_1		Others • what people want
	R_1		Experiences • with games • that's life • some numbers are harder to roll
Concrete Symbolic (First Cycle)	U_1	Edges Anything can happen Fair and unfair both.	How this happens • physical properties • the way it is rolled
	M_1	Looks normal: all six numbers Unfair - only 1, 2, 3. "All" dice fair: same chance	Comparison between and within a dice • Fair/Fairer/Fairest • Both Fair & Unfair (Response with/without ikonic support)
	R_1	Two of each 1, 2, 3 - fair. "Play games all day." Dice may appear unfair but be fair in the long term	Looks unfair but could be fair. (Response with/without ikonic support) Could play game over and over
Concrete Symbolic (Second Cycle)	U_2	Suggest trials	Could do 1000 tosses, use tally marks
	M_2	Carry out limited systematic trials	Information from two or more samples put together but lack of recognition that generally a larger sample is more reliable predictor.
	R_2		Resolves conflict when presented with results of trials within concrete context of the graphs

Discussion

There are some interesting trends in the data from both studies which are consistent with the development of other probability and statistics concepts such as likelihood and sampling (Watson, Collis & Moritz, 1995b). It is hypothesised that there are two distinct themes in the conceptualisation of fair. An emerging understanding of what it means for a die to be fair is followed by an appreciation of the significance of sampling in determining if a die is fair. At the relational level of the first cycle of the concrete symbolic mode these conceptualisations overlapped when some students displayed more complex mathematical thinking when they suggested playing the game over and over to determine if the die were fair. They appeared to have not yet acquired vocabulary to adequately describe their intentions in terms of sampling and trials. The vocabulary was used with ease, however, in student responses at the unistructural level of the second cycle and above.

It was observed that most students in the first study had very little idea of how to test whether the dice were fair and in the second study most were willing to accept that a die was fair in the face of strong evidence that it was most probably biased. A second hypothesis was based on the students' explanations when they appeared to have no experience or awareness of how a die could be made to be unfair. These students could not reconcile two conflicting situations: (1) a *belief* that it is not possible to make a die behave unfairly and (2) an *observation* that this sample looks odd because, for example, you would not expect the "4" to come up that often. Added to this were students' experiences of playing games and rolling dice which produce unequal outcomes even though the dice are *known* to be fair. Hence the conclusion that many students reached was that the results of tossing the die looked strange but the die was probably fair.

Origins of the notion of fair are probably established in experiences with sharing and games played at home where parental explanations for losing would be designed to convince the child that the dice can come up with any results and that it is still fair and acceptable. This, together with a lack of knowledge about the possibility that a die could be unfair or how that could be achieved, provides strong support for the development of alternative and incomplete conceptualisations of fairness. The implications for teaching which arise from this observation are discussed elsewhere (Lidster, et al., 1995).

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Using Algebra to Solve Problems: Selecting, Symbolising, and Integrating Information

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We report an investigation of students' attempts to formulate equations for word problems. Ninety students in Years 9, 10 and 11 were tested twice over 10 months. We trace a progression of stages from naming quantities through describing relationships to writing equations and solving them. Even when all relationships were recognised and correctly symbolised, integrating them into an equation was a common difficulty.

Over the past two years we have been investigating students' performance in using algebra to solve word problems. In Stacey and MacGregor (1995) we reported our findings concerning the effects of problem presentation on students' solution methods. We found that a large number of students, concentrated in particular schools, had not attempted algebraic methods in spite of the instruction to write an equation for each problem and solve it. Their solution methods, usually successful, included informal arithmetic, guess-check-improve procedures, and complex reasoning. However there were many others who had used algebraic symbolism to name quantities, to label parts on a diagram, or to express relationships in a problem, but who did not go on to combine these pieces of information as an equation. To solve the problem they switched to arithmetic calculation or a guess-check-improve procedure. One school where the majority of students were in this category allowed us to use the same test items some months later with the same students. The test results indicate that on the final test most of these students still did not formulate a correct equation and solve it. By comparing their written responses in the two tests, we were able to see how they approached the task of formulating an equation for a problem, in what ways they had improved, and what the difficulties were that prevented success.

Method

Subjects

The students were in Years 9 and 10 (age 14-15) for the initial test, and Years 10 and 11 (age 15-16) for the final test. All classes involved were mixed-ability classes. The school is a coeducational school in a middle-class suburb.

Test

The test comprised six algebra word problems varying in difficulty. Students were asked to write an equation for each problem and solve it. In Figure 1 we show three of the problems, which students in Years 10 and 11 should be able to solve, and we refer to them in our discussion.

Procedure

The testing was administered by the mathematics teachers at the school, and carried out as part of normal lessons. Completed test papers were sent to the researchers for marking and analysis. The first test was given in August 1994 and the second test was given 10 months later in June 1995.

WORK OUT THE ANSWERS BY WRITING EQUATIONS AND SOLVING THEM

1. A group of scouts did a 3-day walk on a long weekend. On Sunday they walked 7 km farther than they had walked on Saturday. On Monday they walked 13 km farther than they had walked on Saturday. The total journey was 80 km. How far did they walk on Saturday?

2. Jeff washes three cars. The second car takes 7 minutes longer than the first one. The third car takes 11 minutes longer than the first one. Jeff works for 87 minutes altogether. How many minutes does he take to wash the first car?

3. The three sides of a triangle are different lengths. The second side is 3 cm longer than the first side, and the third side is twice as long as the first side. The side lengths add up to 63 cm altogether. How long is the first side?

Figure 1. Test items

Results

Test papers showing all written working and answers were obtained from 90 students who had done both tests. Their progress, judged on all six problems, is shown in Table 1. Students' best performance is represented. For example, if a student used algebraic letters to record given information in only one item of a test and used no algebra at all in the other items, this student is counted in the "Partial Use" category for that test.

In Table 1, "No algebra" means that there was no attempt to use any algebraic notation. "Partial Use" means that the student had attempted to use some algebraic notation, even if only to denote an unknown quantity by a letter (e.g., labelling a diagram for Problem 3 with the letter x on one side). Examples of partial use include a wide range of responses such as $x + 7 + x + 13 = 80$ (Problem 1) in which one of the parts is not represented, and correct equations in three variables such as $x + y + z = 80$ which were not further used to solve the problem. "Equation" means that a correct equation was written. By "correct equation" we mean an equation in one variable that could have been used, or was used, to solve the problem (e.g., $x + x + 3 + x + x = 63$ for Prob.3).

Table 1
Progress in students' use of algebra over 10 months ($N = 90$)

	Final test		
Initial test	No algebra	Partial use	Equation
No algebra	53	11	2
Partial use	2	9	8
Equation	0	0	5
TOTAL	55	20	15

For each problem, correct numerical answers were written by approximately 65%

of students in the first test and 75% in the second test. Most answers to all the problems, whether right or wrong, were obtained by non-algebraic methods. As Table 1 shows, there were 35 students (39% of the sample) in the final test who attempted some algebra. Five students formed and solved equations correctly in all three tests. However three of the five did not use standard algebraic techniques for solving their equations. One used a systematic trial-and-error routine, trying different values for x in his equation. The others wrote informal calculations that were not necessarily based on their equations.

It might be argued that since algebraic methods are not necessary for solving the problems, students who were capable of using algebra may not have done so and therefore have been wrongly classified in the "No algebra" category. However algebra was required for the harder problems in the test (see Appendix). The students who demonstrated their capability to use algebra for these harder problems had used it for all the problems, and therefore are correctly classified in the table as users of algebra.

We had expected that in the time between the tests many students would move from using no algebra to being able to formulate equations. However, as Table 1 shows (see first row), 53 students continued to make no attempt to use algebra, 11 moved from "No algebra" to "Partial use" and only two students moved from "No algebra" to writing a correct equation. The second row of the table shows that eight students moved from being partial users to writing an equation, and nine did not progress. There were 20 students at the time of the final test who used some algebraic notation but had not yet progressed to a stage where they could write an equation for these relatively straightforward problems. It is important to point out that five students who are seen in the table as "progressing", that is, moving from using no algebra to using some algebra notation, had solved the problems successfully in the first test but failed to solve them in the second test when they tried to use algebra. It is to be expected that students like these are reluctant to use algebraic methods and come to believe that algebra is unreliable and unnecessary.

Two students used some algebra in early tests but no algebra in the final test. Their apparent regression can be attributed to the effects of teaching. Their responses, and the responses of other students in their class, indicated that they had been trained to use a systematic trial-and-error approach to problem-solving. Although frequently tedious and time-consuming, this method usually resulted in correct answers. Training in using trial-and-error as the easiest or only way to solve a problem may be a reason why so few of the partial users of algebra did not progress, and why so many did not use algebra at all in either test.

Students attempted to deal with the information given in the problem in many different ways. Some of these were helpful ways of recording, whereas others contained specific well-documented errors that obstructed progress towards writing an equation. Stages in this progression, and examples of difficulties at each stage, are shown below. All the examples are chosen from students' written responses to Problem 1, unless stated otherwise.

1. Naming the Unknown Quantities Referred to in the Problem

The use of letters as abbreviated words was a cause of difficulty for three students. For example, to begin Problem 3 one student wrote:

$3\ \text{car} = 1c + 11$
$2c = 1c + 7$
$3c = 87$

He has written "3 car" to mean "time taken for the *third car*", "1c" to mean "time taken for the *first* car", and 3c to mean "total time for *three cars*". He understands the relationships in the problem situation, but has not understood the crucial difference between letters as abbreviated words and letters as representing quantities or variables. It is widely recognised that in certain circumstances students tend to use algebraic letters as shorthand names for objects or as abbreviated words (Booth, 1986). As this example demonstrates, the use of c to mean the word *cars* blocked the route to an algebraic solution.

2. *Expressing the Relationships Between the Parts*

Many students were able to express the relationships in the problem situations, often in correct algebraic notation (e.g., x, $x + 7$, $x + 13$). For Problem 3, diagrams were frequently drawn and labelled appropriately (e.g., x, $x + 3$, $2x$). However there were several students who used a different letter for each unknown quantity (e.g., X, Y, Z) and did not go on to express these in terms of a single variable. Whether one or more letters were used, students who did not write an equation abandoned any further use of algebra and switched to non-algebraic methods for solving the problems.

Several students made errors in notation that blocked further use of an algebraic method. These errors included concatenation for addition and exponential notation for a product. Concatenation for addition was seen in the work of four students; for example, $x7$ and $x13$ to mean "7 more than x" and "13 more than x". Exponential notation for a product was seen in the work of six students; for example, x^2 to mean "twice x".

These errors generally blocked further development of an algebraic procedure. However one student who wrote $x^3 + 20 = 60$ interpreted the exponential notation as a product and solved the equation. Summarising the data above, we see that of the 37 students who attempted to use some algebra, 10 students (approximately one-quarter of the sample) were prevented from writing correct equations by their misunderstandings of algebraic notation for sums and products. Their misuse of notation may indicate a poorly-developed concept of multiplication and its relationships with repeated addition and repeated multiplication (Stacey & MacGregor, 1994).

3. *Writing a Useful Equation that Integrates the Problem Information*

Some students wrote correct equations or expressions for all four relationships but did not combine them into an equation in one variable. For example, for Problem 3 a student labelled the three sides of her diagram of a triangle as

A
B = A + 3
C = 2 × A

and then wrote the equation A + B + C = 63. She did not make any use of this equation and did not solve the problem. Another student wrote, for the same problem,

$S1 = x,\ \ S2 = x + 3,\ \ S3 = 2x$

Although this also looks a useful beginning, where S1 means "Side 1" and correct expressions in terms of x have been written for each side of the triangle, the student wrote as his equation S1 + S2 + S3 = 63. He then abandoned any further use of algebra and solved the problem by a trial-and-error method.

Several students did not know how to write equations in the standard way, although they appeared to understand what relationships were involved and integrated them to some extent. Examples are shown in Figure 2. They were able to solve the problems by non-algebraic methods, and perhaps had used their "equations" in some way to guide reasoning and calculation. It seems likely that with appropriate instruction they would quickly learn how to write equations in the standard way. It is puzzling why they had not learned to take this small last step.

4. *Equations as Descriptions of Procedures Used for Calculating*

Several students calculated answers to each problem by arithmetic reasoning, and then tried to represent these calculations as equations. This method of dealing with algebra word problems has been observed by other researchers (Arzarello, Bazzini & Chiappini, 1993). These equations were not representations of problem structure, but descriptions of the procedure used to calculate a value for one of the unknowns. For example, a student wrote the equation for Problem 2 as

$x = (87 - 18) \div 3$

and others wrote

$x = 87 - 18 = 69 \div 3 = 23$

where x stands for the time taken to wash the first car. It can be argued that, technically, the first of these is an acceptable equation. However in the harder problems on the test, which were too difficult to solve by mental reasoning and arithmetic, students who were limited to writing a description of the solution method had no chance of success. Lacking the support of an algebraic representation of the problem, they were unable to devise a solution.

(i) Sat Sun Mon

$x \rightarrow x + 7 \rightarrow x + 13 = 80$

(ii)

$$\begin{array}{l} x \\ x + 7 \\ \underline{x + 13} \\ \;\;80 \end{array}$$

(iii)

$1 = y$

$2 = y + 7$

$3 = y + 13$

$1 + 2 + 3 = 80$

(iv) $x,\ x + 7,\ x + 13,\ 80$

(v) <------------------------ 80 ------------------------------->

$a = ?\ b = a + 7 \qquad c = a + 13$

Figure 2. Attempts to integrate problem information

Discussion

Our data indicate that major difficulties in formulating equations in the test did not lie in students' failure to comprehend the written information, to understand the problem structure, or to see how the parts were related to each other and to the whole. Most students could solve the problems by non-algebraic methods, providing confirmation that understanding the problem situation was not a difficulty. For the students who tried to use algebra, the main obstacles to success were (a) incorrect use of algebraic syntax, and (b) failure to integrate the given information as an equation or set of equations. There were others who wrote correct equations that could be used to solve the problems, but did not use these equations, apparently not knowing how to use the notation as a tool for deductive reasoning.

As Arzarello et al. (1993) have commented, writing equations for problems is a complex process. These researchers have suggested that the process of naming (i.e., the choice of variables) and understanding the main relations in the problem are the most crucial steps. Our data indicate that, for most students in the sample, naming variables and understanding relations were not difficult for the simple problems we used. Most students who tried algebra could name quantities, and there was little difficulty related to expressing several quantities in terms of one variable. However there were several instances of students who named the three parts in a problem appropriately (e.g., x, $x + 7$, $x + 13$) but did not try to relate them to the total. As we have shown, some students used unconventional formats such as arrow-diagrams, vertical addition, or invented notations to try to denote the idea that the sum of the parts is equal to the total given. They were unable to write an equation to express the structure of the problem situation. Others had not learned that an equation is written to represent the problem situation; they wrote a description of the calculation procedure they had used to solve the problem. These students have perceived the equation as a formula for calculating. They need to know that algebra can also be used to extend and support logical reasoning; its purpose in problem-solving is not to describe a solution procedure that has already been constructed mentally.

In a typical school algebra curriculum, the first problems given to students to solve by algebraic means can also be solved by simple arithmetic, intuitive reasoning, or a simple guess-and-check. Until they achieve a certain level of fluency, students see algebra as an extra difficulty or unnecessary task imposed by teachers for no obvious purpose and not as a useful tool for making problem-solving simpler. This attitude is reasonable, since the problems they have so far encountered (such as the three problems presented in this paper) are not good examples of the power of algebra. Cortes, Vergnaud & Kavafian (1990) state that if students are to learn to formulate algebraic equations for solving problems, teachers need to discourage the search for non-algebraic solutions. We support this view, while reminding readers that it is difficult to find problems that are sufficiently complex to warrant an algebraic solution but easy enough for students to work through with understanding and learn from.

It is generally assumed that when comprehending a mathematical problem and preparing to solve it people construct a mental representation of some kind. Several theories have been proposed about the form and function of these representations. Using the context of elementary-grade arithmetic problems, Kintsch and Greeno (1985) proposed that a problem representation is built in several steps, beginning with a conceptual representation of meaning in the form of a set of propositions (called a *propositional text base* by Kintsch and Greeno). Johnson-Laird (1983) also proposed that the initial stage of comprehension produces a propositional representation. The individual combines this propositional representation with other general and specific knowledge to construct the second representation - an integrated and articulated mental model of the problem situation. We suggest that the initial set of propositions (Johnson-Laird's *propositional representation*) is sufficient for solving problems by the guess-and-check method. The propositional representation provides comprehension of each piece of information given without necessarily relating it to other information in a single model. The problem-solver guesses a value for the unknown, and checks whether it allows each proposition to be true. The process is essentially substitution in expressions. Almost all the problems we have seen in textbooks for the first four years of school algebra can be solved quickly by this method because of the nature of the numbers involved, and consequently many students use it. Students who have a good number sense often reach the answer after three or four guesses. An algebraic equation does not represent the separate pieces of propositional information but arises from the integrated mental model that is produced in the later stage of comprehension. As we have seen, some students do not know how to represent this model as an equation.

In their investigation of algebra word-problem comprehension, Nathan, Kintsch and Greeno (1992) have pointed out that students may understand a problem in everyday terms but be unable to represent its formal aspects as required for an algebraic solution. These researchers suggest that features of the student's cognitive representation of the problem determine what information is available for reasoning. Our data provide no evidence that students' mental models were inadequate or incorrect. The students demonstrated their capabilities in comprehension, logical reasoning, written calculation, mental arithmetic, and problem-solving by non-algebraic methods. There is no obvious explanation for their difficulties in constructing equations for the simple problems they were given. It seems likely that they had not had sufficient experience. We conclude that formulating an equation is not an intuitive way to represent a problem, but needs to be carefully taught.

Reluctance to formulate an equation may be one indicator of the gap between arithmetic and algebraic thinking that is now widely recognised in the literature. Cortes, Vergnaud & Kavafian (1990, p. 28) have pointed out that "most pupils are not familiar with the concept of the equation. For them, an equation is an abbreviated way of writing the terms of the problem: a summary". Our data support this assertion. In certain contexts students are familiar with equations as formulas, that is, as instructions for arithmetic calculation. They know how to use a formula by substituting values and calculating the answer. As Cortes et al. observed, "this tool-like characteristic of the equation" (p. 28) when written as a formula (e.g., $c = ax + b$) is not visible to students when the equation is written as $ax + b = c$. The concept of an equation as a statement about relationships, rather than as a formula, may be crucial to students' ability to use algebra for solving problems. Students' perceptions of what an equation represents and how it relates to a problem are being investigated in the current stage of our project.

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Appendix

The Two Hard Problems in the Test

(i) I think of a number, multiply it by 8, subtract 3, and then divide by 3. The result is twice the number I first thought of. What was the number?

(ii) Chris runs from A to B, a distance of x Km, and immediately cycles back along the same route to A. The total time to go there and back is 3 hours. The running speed is 10 Km/hr and the cycling speed is 40 Km/hr. Write an equation using the fact that the total time is 3 hours. Solve your equation to find the value of x.

An Analysis of Teacher Generated Whole Number Word Problems

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In curriculum statements provided for teachers, teachers are encouraged to present to their pupils a rich variety of word problems. Teachers may draw their word problems from reference texts, "real-life" problem situations or generate their own word problems. An analysis of teacher generated word problems reveals that these word problems not only fail to provide as diverse a range of problems as suggested in the resources made available to teachers, but also reinforce some misconceptions about the four arithmetic operations. The implications of these findings are discussed and proposals advanced to overcome this problem.

Over the last few years there has been a considerable interest in word problems, both as a topic for research (Carpenter, Ansell, Franke, Fennema, & Weisbeck 1993; Carpenter & Moser, 1984; Carpenter & Moser, 1982; Cortes, 1995; English, 1996; in press a; in press b; Gillies, Walker & Bailey, 1995; Greer & McAnn, 1993; Greer, 1992; Mulligan & Mitchelmore, in press), and in mathematics statements throughout much of the English-speaking world. The National Council of Teachers of Mathematics' *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989, p. 5) states, as one of the goals for students to achieve, *that they become mathematical problem solvers* and that they aim to *develop and apply strategies to solve a wide variety of problems ...* (p. 23). A similar approach has been taken in Australia both nationally (Australian Education Council and Curriculum Corporation, 1991) and in the states such as Victoria (Board of Studies, 1994), and New South Wales (New South Wales Department of Education 1989) as well as in England and Wales (Department for Education, 1995). This point is considered below. Of particular interest is the focus upon solving "real life" problems. This point is taken up below. Such "real life" problems are often presented to students as word problems. Hembree and Marsh (1993, p. 152) define a problem as *a task or question stated in words wherein the person attempting its answer must select the operations.* They suggest that word problems can assist students refine their conceptual understanding of mathematics and learn to use appropriate strategies suited to the particular situation. Greer (1992) makes reference to misconceptions often implicit in teacher generated word problems. This point is considered below. The present investigation analyses arithmetic whole number word problems generated by a class teacher. The problems are matched with the general curriculum statements published for the guidance of teachers in Victoria. To provide the means of categorising word problems a system, based on semantic analysis of word problems (Carpenter & Moser, 1982; Maguire, in press), has been used in this analysis.

Categorisation of word problems

Additive Word Problems: Greer (1992) comments that the categorisation of word problems is a research tool and in no way indicates any solution strategy adopted by a child to solve the word problem. The source of students' "real life" problems can be a text book, the mathematics of a given situation itself or the class teacher. Silver (1992) has suggested that further research is needed into this latter source of problems. Carpenter and Moser (1982, 1984), and Carpenter et al. (1993) have distinguished six categories of addition and subtraction word problems. These categories are based on the semantic features of the problems. Table 1 provides examples of each of these categories of word problems.

Table 1. Problem categorisation, representative word problem and semantic criteria used for the categorisation of word problem based on Carpenter and Moser (1982; 1984), and Carpenter et al. (1993).

Problem categorisation	**Representative samples of word problem**	**Criteria of word problem categorisation**
Joining	*Mary has 5¢ in her pocket. On the way home she found 10¢ in the grass. How much money does she have now?*	1. Active. 2. Two quantities are a sub-set of the third. 3. Increase.
Separating	*Richard had 15 marbles. He lost 7 of them in a game. How many marbles has he left?*	1. Active. 2. Two quantities are a sub-set of the third. 3. Decrease.
Equalising-add on	*Mary has 15¢. Susan has 35¢. How much more does Mary need to save to have the same amount as Susan?*	1. Active. 2. Compares disjoint sets. 3. Increase.
Equalising-take away	*Rowan has 25 marbles and Ellen has 17 marbles. How many does Rowan need to give away to have the same number of marbles as Ellen?*	1. Active. 2. Compares disjoint sets. 3. Decrease.
Part-part-whole	*In the purse there is a 10¢ coin and a 20¢ coin. How much is in the purse?*	1. Static relationship 2. Two quantities are a subset of the third.
Comparison	*In his bag Rowan had 25 small marbles and 16 large marbles. How many more small marbles are there than large marbles?*	1. Static relationship 2. Compare disjoint sets.

Carpenter and Moser (1982) have used the criteria of action/static, set inclusive and sub-set relationships, as well as whether there was an increase or decrease in a given quantity to categorise additive word problems.

In the joining problem: *Mary has 5c in her pocket. On the way home she found 10c in the grass. How much money does she have now?* the problem is active in that it involves the action of joining the two quantities in order to obtain a solution. The two quantities *5c* and *10 c* are a subset of the solution quantity. The operation upon these two quantities produces an increase in the amount of money Mary now has. Each of the above word problems can be categorised according the semantic criteria proposed by Carpenter & Moser (1982).

Word problems provide information in the form of statements which are usually followed by a question. In the word problem *Mary had 5¢ in her pocket. On the way home she found 10¢ in the grass. How much money does she have now?* there are two informational statements - *Mary had 5c* and that she *found 10c*. These two statements are reference statements in that they provide a reference for the solution calculation. The question asks "*How much money does she now have?*". This is a solution statement. These points are further developed in the following section.

Multiplicative Word Problems: Many researchers (English, in press a, Greer, 1992; Mulligan, & Mitchelmore, in press; Silver, 1992; Vergnaud, 1983) working with children's categorisation of multiplicative word problems have identified a number of categories. A sample of these is presented in Table 2.

Table 2. Problem categorisation, and a representative example of word problem based on Greer (1992) and English (in press a).

Problem categorisation	Representative sample of word problem
Comparison multiplication	*Richard had written 12 lines of his story. Jessica had written three times as many lines as Richard. How many lines has Jessica written?*
Comparison division	*Paul has already written 46 lines of his story so far. This is twice as many as Tony has written. How many lines has Tony written?*
Cartesian product	*Nancy has three different summer skirts, four different tops and two pair of shoes. How many different outfits can she wear?*
Partitive division	*Mrs Jones bought 15 chocolate bars to give to the children at the party. There were 5 children at the party. If each child was given the same number of chocolate bars, how many chocolate bars did each child receive?*
Equal groups	*Mrs Jones gave three pieces of fruit to each of her four children. How many pieces of fruit did they have between them?*

This table does not include all possible multiplicative word problem categories. Table 2 is representative of word problems considered by Greer (1992) and English (in press a). The sample does not include quotition division, comparison division and non-routine word problems.

The above multiplicative word problems involve an implied operation. This operation can result in either an increase or a decrease in the solution quantity (for example the product or the quotient) without an increase in quantities of the subset or the comparison of the disjoint sets. Maguire (in press) has provided criteria which allow multiplicative word problems to be categorised according to semantic criteria. These criteria are different from those criteria developed by Carpenter and Moser (1984) and have been introduced by this author to classify multiplicative word problems. Multiplicative word problems involve the interrelationship between the three types of statements present in many word problems, viz an informational reference statement, an informational relational statement and a solution statement. Word problems generally consist of three main types of statements, two (or more) of which are informational statements. Informational statements may be reference statements in that they provide information necessary to solve the problem. For example in the word problem *Richard had written 12 lines of his story. Jessica had written three times as many lines as Richard. How many lines has Jessica written?* the reference statement *Richard had written 12 lines of his story* provides reference information about the number of lines written by Richard. The relational statement *Jessica had written three times as many lines as Richard* as well as providing information, establishes a relationship between the number of lines written by Jessica and Richard. The solution statement *How many lines has Jessica written?* identifies the unknown and provides a unit of quantity.

Mathematics Guide

Prior to the current analysis, a Victorian primary school had undertaken a review of its mathematics syllabus. The teaching staff developed a curriculum guide using the state's Mathematics Framework (Ministry of Education (Schools Division), 1983) as a resource. The guide was prepared to assist teachers to develop their individual class syllabi and included numerous references to the different approaches a teacher could take to encourage mathematics learning within a problem solving classroom environment. The guide advocated mathematics learning through the use of problem solving situations. This was reflected in the choice of resource material suggested. Not only was mathematics to be presented in a problem solving format but, in addition, it was suggested that students be given weekly exercises in the use of problem solving techniques.

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Analysis

The present analysis seeks to analyse some 282 word problems posed by the Year 3/4 teacher. The teacher was one who wished to provide for his students a thorough understanding of those class/age appropriate arithmetic concepts underlying the four arithmetic operations (Ministry of Education (Schools Division), 1983). Anecdotal evidence suggested that many students within this class had commenced the year with poor mathematical understanding and experience. The children had been taught by a teacher whose mathematics syllabus was not in accord with the curriculum guide provided.

This analysis deals only with positive integers in mainly single-step arithmetic word problems involving the four operations. The term "quantity" therefore generally refers to a positive whole number and not a fraction. Initially, in Years 3 and 4, the children were presented only with partition and quotition division word problems which resulted in whole number quotients. Later, after the children had gained more experience in using multiplicative concepts (Year 4), quotients with remainders were introduced and later, involving money, quotients with decimal fractions were included.

The teaching situation: The class was a composite class of both Grade 3 and 4 students. Throughout the mathematics sessions both grades were set separate class activities. Each student was provided with worksheets which s/he was to complete by the end of the school week. The teacher planned that one section was to be completed each day. Each section of the worksheet dealt with a different aspect of the class mathematics syllabus. At the beginning of each mathematics session the teacher would introduce new work to the Grade 3 students whilst those in Grade 4 read through their assigned work or completed any unfinished work from the previous session. During this time the Grade 3 children were required to look through the activities whilst the teacher provided an overview. Occasionally some examples were worked through as a class exercise before the children commenced their worksheets. Children were encouraged to work cooperatively in groups of two or three students. After checking that every child had commenced work the teacher moved to the Grade 4 children. When all children had commenced work, the teacher moved from group to group listening, talking with children, challenging them and helping them to resolve any difficulties they were experiencing. Towards the conclusion of the session both grades were brought together for a class discussion of the work completed.

The word problems: The word problems selected for analysis come from the class syllabus for both the first and second semester of 1993. The first of these problems (Grade 3) were devised to extend the children's understanding of addition of two digit numbers together with a selection of subtraction, and introductory multiplication and division word problems. For these problems children were encouraged to use calculators (initially) and manipulatives (MAB) were freely available. The teacher commented that he was more concerned with the children learning to read the problems and map them to an appropriate operations than actually calculating the "correct" answer. The focus of these lessons was the understanding of the underlying mathematical concepts. Both multiplication and division operations were introduced concurrently as the teacher wished the children to understand the interrelationship of these two operations.

The four processes: Whilst the problems given to the Grade 3 children were straight forward, those presented to Grade 4 were generated to extend their ability to reason mathematically. The focus of these questions was not to access the children's ability to calculate but their ability to select an appropriate operation to solve the problem. Calculator usage was permitted for many of these word problems. The initial selection of word problems was included in this analysis as they were generated by the teacher as a revision activity. The second selection gives an example of a longer complicated group of interrelated problems for which the children were permitted to use calculators.

Diversity of Word problems: As shown in Table 3 the number of word problems of different categories differ quite notably in their representation. Whilst some categories are not represented (Equalising subtraction, Comparison multiplication and division, and Cartesian product), there is a tendency to present certain categories (Joining, Separating, and Equal Groups) to a greater extent than others (Equalising Addition, and Comparison). Far from providing "a wide variety of problems", the problems are quite limited in range and often not relevant to the students' out-of-school activities. Addition and subtraction are represented almost entirely by Joining (24.8%) and Separating (23.1%) word problems, with Comparison a mere 2.1% of the total. Multiplicative word problems are almost all represented by Equal Groups (24.8%), Quotition Division (9.6%) and Partition Division (12.4%).

Table 3. Problem categorisation, representative sample of problem type and the number and percentage of such problems evident in the analysis

Problem categorisation	**Representative sample of problem type**	**Number of problems sampled**
Joining	*Gloria skipped seventy-three skips on her first try. On her second try she was tired and skipped only another twenty-six skips. How many skips did she skip on her two tries?*	70 (24.8%)
Separating	*Steven taped a TV show for us to watch. It was eighty-five minutes long. If we have already watched twenty-eight minutes, how many more minutes have we still to watch?*	65 (23.1%)
Equalising-add on	*The computer game Steven wanted to buy cost $48.00. If he had only $23.00, how much more had he still to save until he had enough money to buy the game?*	1 (0.4%)
Equalising-take away		0
Part-part-whole		0
Comparison	*Sarah had seventy-eight stickers in her book. Gloria had only started saving stickers and had only thirty-one stickers in her book. How many more stickers had Sarah than Gloria?*	6 (2.1%)
Comparison multiplication		0
Comparison division		0
Cartesian product		0
Equal Groups	*Each day Helen's mother walked five kilometres. How far did she walk in ten days?*	70 (24.8%)
Quotition division	*Jack had twenty-five marbles. He gave each of them five marbles. With how many friends did he share the marbles?*	27 (9.6%)
Partitive division	*As a special treat Mrs. Jones bought twenty-two chocolate bars. She shared them with the children at the party. How many chocolate bars did each of the eleven children get?*	35 (12.4%)
Non-routine	*During the holidays I had $700.00 to spend. I bought a pair of trousers for $85.00, a jacket for $250.00; four pair of socks for $2.50 each, and a pair of shoes for $125.00. How much did I have left over?*	8 (2.8%)

Real-life problems: Word problems of the type *Peter had 150 chocolate bars. If he ate five chocolate bars each evening, how many evenings would it take him to finish eating all the chocolate bars?* may initially interest some children but could not be considered within the children's reality. It is unlikely that a child would have so much chocolate and would no doubt be quite ill or would lose their appeal well before they were all eaten. Problems not included among those generated by the teacher such as *Two pizzas are*

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divided equally among eight people. How much pizza does each person get? could well be relevant to the children and within their experience. This author's observation of the word problems suggest that many problems were not within the children's experiences and may not be considered relevant to the children.

Misconceptions: Greer (1992) suggested that elementary school teachers share a number of misconceptions regarding multiplicative operations, viz

- that multiplication results in an increase whilst division produces a decrease in a given quantity
- that when dividing, one divides the larger number by the smaller

Observation of the representative word problems indicate these teacher generated word problems reinforce these misconceptions. In those cases involving multiplication the product was larger than any of the quantities in the reference or relational statements. A similar situation is found in division calculations where the quotient was always smaller than the dividend or the quotient. In no division calculation was the divisor greater than the dividend.

Implications

There are two main characteristics of these teacher generated word problems that invite consideration, viz

- lack of breadth
- reinforcement of misconceptions

The limited range of word problems represented in the analysis indicates that the children did not experience as diverse an range of problems as suggested in the resource literature (Board of Studies, 1994; Australian Education Council and Curriculum Corporation, 1991). It is this author's contention that the range of word problems provided was not encompassing enough to extend the children's arithmetical experience of the world in which they live.

The analysis raises two major questions which require further investigation, viz

1. whether primary school children, when presented with a limited range of word problems, are able to cope with problems which deal with rational numbers, decimal fractions or proportion. This question is in agreement with those raised by Silver (1992).
2. whether the misconceptions evident in the analysis
    a) that the multiplication of one quantity by another always results in an increase whilst division produces a decrease and;
    b) that when dividing one divides the larger number by the smaller (Greer, 1992),

   may cause young students difficulty in adjusting to operations of allocating a smaller number of articles amongst a larger number of groups.

Word problems such as *Two pizzas are divided equally among eight people. How much pizza does each person get?* require the division of the smaller quantity by the larger giving either a fraction or decimal representation of the quantity.

One method of overcoming the problem of a limited range of teacher generated word problems may be for mathematics educators to become involved with a cluster of primary schools to conduct continuing professional development in mathematics. This author regards such involvement as including the presence of a mathematics educator in cluster schools to work alongside class teachers, learning with them in an action research situation (Kemmis and McTaggart, 1990). In the action research paradigm the mathematics educator and the teachers work together as equal partners. The role of the mathematics educator could be to introduce new research findings; to act as a conduit for ideas; and to enable the sharing of behaviours and experiences between teachers.

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"Mathematics" and "Maths" - The Same Meaning?: Investigating One Young Child's Beliefs

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This paper discusses beliefs of one young learner of mathematics about the meaning of the terms "mathematics" and "maths". The latter term is generally considered to be a common abbreviation of the former but with the same meaning. Data collected from one child demonstrate that for her the meanings of the two terms are different. Use of a range of interview procedures has facilitated the child's articulation of her beliefs and the accessing of them by the researcher.

Major curriculum materials in the discipline of mathematics (e. g. Australian Education Council, 1991; Board of Studies, 1995; Lovitt & Clarke, 1988) most commonly use the formal term "mathematics", whereas in everyday speech the abbreviation, "maths" is commonly substituted. The familiarity of the latter term and the acceptability of the exchange of the two terms, are illustrated, for example, in a newspaper article (Hunting, 1995) in which the title and the summary statement, designed to catch the attention of readers and inform as to the main message of the article, use the term "maths", whereas the text which talks about children's learning deploys the term "mathematics".

"Maths" is a recognised abbreviation of "mathematics" (Wilkes & Krebs, 1982), with an assumption of the same meaning. However, the present research shows that young children may not hold an understanding of this relationship between the two terms. This finding is of significance to teachers and others in the education community as it reminds that children, even at the young age of eight or nine years, are individuals who construct their own meanings which should be acknowledged. By giving attention to children as individuals an educator can come to know the children better and can respond appropriately to them by taking their beliefs into account. The research illustrates that one should not assume the meanings children hold, but should take a stance of "listening" to individual students. This listening may occur through verbal, written and pictorial communication modes.

The Focus of This Paper

The research reported in this paper is taken from a larger study which includes an investigation of young children's beliefs about mathematics. Eight children, of eight to nine years, were interviewed on ten occasions over a five month period. The interview data were analysed thematically (van Manen, 1990), beginning with the use of the qualitative data handling computer program NUD•IST (Non-numerical Unstructured Data Indexing Searching and Theorizing) (Richards & Richards, 1990), to organise data into manageable categories and to record index trees for each child. One theme, not originally planned as an issue for analysis, but having emerged from the data, is the meanings expressed by some children for the term "mathematics", as distinct from "maths". This paper explores this theme or issue from the perspective of the beliefs of one child, Cara, as expressed in her interview data. Differences in meaning were found for some other research participants also, but are not explored in this paper. The discussion includes consideration of Cara's meanings for "maths", the term she was more familiar with, but gives emphasis to her understandings of the term "mathematics", as it is the difference in meaning between the two terms and her idiosyncratic beliefs about "mathematics" that were unexpected. The value of the data collection approach and the possible impact of the approach on the articulation of the children's beliefs are discussed also in the paper.

Background

Cara was chosen by her teacher as a low-achieving grade three female research participant. She was interviewed on ten occasions over a two term period, for about thirty minutes each time. Data were collected through a range of means, for example: written responses using sentence starters or a focused question; verbal responses expressing such things as reactions to photographs, video clips, and children's drawings; pictorial responses through drawing; and semi-structured discussion.

In the main research study, of which this paper reports on an element, a key research question was: "What meaning do children give to "learning" and "mathematics"?". The intention was to access and analyse children's constructs for these terms, partly as background to the remainder of the study. It was within this section of the study that the theme or issue of different meanings for the terms "mathematics" and "maths" arose. Each of the children had constructed their own idiosyncratic understandings, some believing the terms had the same meanings and some believing they were different. Cara was one of the children for whom the construction of different understandings for the terms became evident.

The discussion below attempts to portray Cara's understandings as one example of the findings of the study, to illustrate that terms interchanged unreflectively by adults are not necessarily understood in the same way by a young child, and to illustrate that a young child's meaning for a particular construct can be complex, requiring multiple instances of data collection and careful analysis to gain more than superficial insights.

Findings

The research was begun with the intention to use the words "mathematics" and "maths" interchangeably. As the researcher and interviewer, I soon learnt this to be naivety on my part, a naivety not precluded by results from trailing of research procedures, nor by reports I had read of previous research. Thus the findings which I detail below were not only unexpected but showed me early on the value of intensive interviewing in seeking the young child's perspectives. I begin with a discussion of Cara's meaning for "maths", the term with which she was more familiar and seemingly more comfortable.

Cara's Meaning for "Maths"

In contrast to her beliefs about "mathematics", as discussed below, Cara seemed to be confident in responding to tasks where she was asked questions which sought her understanding of the nature of "maths" and maths activity. It seems that her responses regarding "maths", rather than those regarding "mathematics", should be compared to those from previous research which discuss children's understandings of mathematics.

It seems common for children to associate number or computation with mathematics (Frank, 1988; Garafolo, 1989; Spangler, 1992; Stodolsky, Salk, & Glaessner, 1991). Cotton (1993) reported from his study with 5 to 9 year old children that "the older children saw mathematics as nothing more than number" (p. 15).

The third interview with Cara provided results which contrast in part with these findings from previous research. In response to two word association type activities Cara related the words measuring and estimating to "maths". These spontaneous responses suggest that these processes were perhaps the aspects of maths of which Cara was most conscious, or with which she was most familiar. Cara also referred many times in interviews to measuring and estimating in relation to activities in school, home, and work contexts. Later in her third interview Cara added other aspects when defining maths. In response to a word wheel asking "What is maths?", Cara gave an expanded view of maths. Cara read from her writing on the wordwheel:

> Measuring, estimating, playing with things, guessing things, using things, doing things, games, learning things like maths

At times Cara gave different responses. For example, when asked what she would tell an alien maths was, thus being asked virtually to define maths, she listed: estimating, times tables, dividing by, equals and half millions. The conversation continued, with the interviewer asking if there was anything else she would tell the alien so that he would have a good idea what maths was. The only things mentioned were pluses, takeaways, and Applied Maths, which Cara thought was something to do with halving. This response gave precedence to number concepts thus showing contrast with the other examples cited.

The above examples, although given in brief, suggest a lack of consistency in Cara's beliefs, or, perhaps more accurately, multiple aspects within her beliefs. Cara demonstrated that she saw maths as physical activity through responses such as measuring, playing with things, and using things, as cognitive activity through responses such as estimating, and guessing, and as content as expressed in responses such as times tables, pluses, and equals. It is clear that these responses contrast in part with the findings from previous research.

Cara appeared confident in giving the responses discussed above. The same confidence did not appear when she was asked about "mathematics", as is demonstrated below. Interview excerpts are referred to in an attempt to seek insights into Cara's meaning for the term "mathematics". The possible impact of the interviewer upon the construction of meaning by the child is discussed also.

Cara's Meaning for "Mathematics"

In my first interview with Cara, I asked her to draw a situation in which she felt she was learning mathematics well. In response she said: "I don't know anything about mathematics". However, she said she did know what maths was and went ahead to complete the task.

From this point, which was the beginning of the first interview I conducted, I was aware of the possibility of children having differing meanings for the terms "mathematics" and "maths". The second interview provided an opportunity for validation of Cara's first interview response.

When reflecting on Cara's response in her second interview to the task of drawing a person doing some sort of mathematical activity I became truly alerted to the possibility and complexity of differing understandings.

> Interviewer: Okay Cara, the first thing I'd like you to think about is, or what I'd like you to do for me, is I want you to draw a person doing some sort of mathematical activity
> Cara: Mathematics?
> I: Yeah
> C: On sports we done mathematics we done, you had to have a partner that was just a tiny bit taller than you and you have to go on their back and go around the wicket and back around and back
> I: Mhm
> C: And you had to, you know the way you do a pyramid
> I: Mhm
> C: You had to do that just with two people like one person down the bottom and one person on top
> I: Right, so what sort of mathematics were you doing there?
> C: Er prac, getting warm up for gymnastics
> I: For gymnastics. And you said you were doing maths there at that time?
> C: No it wasn't maths, it was seeing if we could do things without falling

At this stage of the interview I was not alert to the significance of Cara's first reference to "mathematics", that is, of her taking on the meaning for the word "gymnastics" when I said "mathematics". I did not recognise her subtle shift. As the interview progressed a difference in her meaning for the terms "mathematics" and "maths" began to become more apparent:

> I: Right, so I said can you think of, can you draw someone doing some sort of maths activity
> C: Maths
> I: Maths activity. What did you think I said?
> C: Mathematics
> I: What's the difference between mathematics and maths?
> C: Mathematics can do, er you, I don't know
> I: What is mathematics?
> C: I don't know

My lack of full awareness of the possibility of miscommunication resulted in unconscious continued interchange of the terms "mathematics" and "maths". Cara's response caused me some confusion which it appears was conveyed to Cara. My confusion, combined with a sensed lack of acknowledgment of her view, seemed to cause Cara to change what she was saying. Cara's following statement suggests that she received from my responses the message that "maths" and "mathematics" are the same and that she responded by adjusting her story to show an understanding of this:

> I: Well why did you choose that activity when you were doing pyramids and things when I said to think of someone doing some mathematics activity?
> C: W.e.l.l er I thought you meant um . . . measuring

At the end of the conversation regarding the gymnastics activity I asked Cara: "So was that maths?". She answered: "Er, well it was really sports". So Cara was fairly sure that she hadn't been doing "maths" in that situation but she was not certain as to whether she had been doing "mathematics", as she was not sure of the meaning of the term. From the vividness of this discussion regarding gymnastics I, as the interviewer, took on an understanding that Cara had associated the words "mathematics" and "gymnastics" rather than "mathematics" and "maths". In addition, I was alerted to the possibility of differing understandings and perspectives of the terms "mathematics" and "maths" among the eight children.

In response to this incident I added to my planned procedures a simple task through which I could directly address the issue of children's meanings for "mathematics". In her response to the sentence starter "I think mathematics is ....", given in her eighth interview, Cara showed that she had begun to form a meaning for the term "mathematics". Cara responded to this task by writing her first response but then the exchange became verbal.

> C: I think mathematics is great
> I: Why did you say that?
> C: 'Cos sometimes you can get it right, sometimes you can get it wrong (giggle)
> I: Alright. And what do you mean, sometimes you can get it right?
> C: Like when, what is mathematics? Like if you do something wrong you don't get a tick and if you do something right you get a tick

Cara's second statement indicates that she may been seeing "mathematics" and "maths" as the same thing. However, by asking herself the question "What is mathematics?" Cara suggests that she remained unsure of the meaning of "mathematics", that she was questioning her own understanding of this concept.

I: Did you just say what is mathematics? Is that the question?
C: Yeah
I: Well what would your answer to that question be? What do you think it is?
C: Err, learning things, er doing things,
I: Mm
C: About maths

Once again Cara associated "mathematics" and "maths", suggesting not only that they are the same but that they are an active process, made up of learning and doing. Cara continued by giving examples which reinforce this notion of action:

I: What sort of things?
C: Like, um, um, umm, like, like you don't know what dividing by is and you learn how to do it. And um it's like learning how to swim, learning how, how width the pool is and how long it is
I: Mhm. Is that mathematics is it?
C: Yeah that's what I think
I: So how's that different from maths?
C: Width and length is mathematics I think
I: Is it also maths?
C: Well, yeah
I: Is maths the same as mathematics or is maths different from mathematics?
C: I think maths, maths is different, maths is different from mathematics
I: Can you explain that difference to me?
C: Um,
I: I'm pretty interested in this, so I sort of want to know what you think
C: I think that maths is different from mathematics because mathematics is doing something else than maths. Maths is like times tables, and well sometimes you can learn things from maths but I think mathematics might be good to learn

In this segment of the conversation Cara began by giving examples of activities which suggested that she saw "mathematics" and "maths" as the same. However, when asked directly whether they were the same or different she stated that she thought they were different. This change of view suggests some possible confusion or uncertainty. The last statement by the interviewer suggests also that Cara was tiring of, or beginning to have some difficulty with, the conversation, perhaps aware of her own uncertainty and therefore finding the probing frustrating. This is endorsed by Cara's following statement in which she begins by saying she doesn't know what "mathematics" is:

I: Can you still tell me a bit more about what mathematics is? It's good to learn . .
C: I don't know what mathematics is, I think it's something that you learn and then you can do and then you can like learn it when you grow up, more
I: So is it more when you're grown up that you learn mathematics is it, and children at school learn maths?
C: No, children at school learn mathematics and when you grow up to be in that school I think you learn more mathematics
I: The school across the road? [Reference made to the secondary college]
C: Yeah
I: So you're still, you are learning mathematics now are you?
C: Yes, I think
I: You think so?
C: No we're learning maths
I: Oh you're learning maths
C: We've only done mathematics once

I: Alright, when did you do that? Tell me about the once you did it
C: Ages ago. Term three
I: Tell me about it
C: We were in with Miss T and we were doing a worksheet and then she was telling us a bit about mathematics and I forget what she said [Miss T, a specialist teacher, taught the class the measurement section of the mathematics curriculum]
I: Alright. So it was just that once with Miss T was it that you did mathematics?
C: (Non-verbal: Yes)
I: Right. So you don't usually do mathematics at this school?
C: No I don't do it anywhere

Cara again stated that she did not know what mathematics was, but thought it was something you learn and then do and then learn more. She was unsure as to the timing of this, suggesting that "mathematics" is mostly learnt in the secondary school. She believed that at primary school she learned "maths" but because of one experience which the teacher may have called "mathematics", she thought she had learnt "mathematics" at one time. The conversation moved on to consider the use of "mathematics" by adults and continued to explore Cara's understanding of the relationship between "mathematics" and "maths":

I: Do you think adults use mathematics at all?
C: Yeah, when they're measuring
I: Like your dad?
C: No, when they go shopping and they're measuring how heavy is a ton of bananas or about three or four bananas and they're doing that, I think that's mathematics
I: Is that also maths?
C: Yes
I: And that's mathematics too?
C: Mhm
I: In your classroom if you did some weighing would you be doing mathematics or maths?
C: Maths
I: Not mathematics?
C: Yep

In the above excerpt, Cara's tendency to associate "mathematics" with people older than herself is apparent again. She indicated during the interview that she saw "mathematics" either as something you learn as student in the upper primary grades or at the secondary level, or as an activity done by adults when measuring as part of shopping. For these people and activities she equated the terms "mathematics" and "maths" but generally did not do so in relation to her own experiences in the classroom.

The interview continued with an attempt to have Cara draw together what she had said through this whole interview segment:

I: Now is there anything else that you can write on here for me as well as I think mathematics is great?
C: Uh uh (No)
I: What else could you write after we've had this discussion? What would you write there? I think mathematics is . .
C: (No response)
I: Don't know?
C: Mm mm (No)

This limited response from Cara suggests that the beliefs she had been expressing in the interview were at the formation stage. Cara's beliefs may have been in the process of construction and thus were open to fluctuation, uncertainty, or even contradiction, as illustrated in the quotations. In this eighth interview Cara attempted to communicate her ideas on what mathematics is, but lacked definition and certainty. Some of her statements appear to hold contradiction; for example, we see that she swayed between "mathematics" being "maths" and it not. However, we see also her attempts to make sense of what she observed, for example by suggesting that as "mathematics" is learned mostly in secondary school, it is used or done by adults rather than by children.

In this eighth interview Cara gave a sense that for her "mathematics" was a verb, rather than a noun, that when she thought of "mathematics" she thought mainly of action rather than of content. This was endorsed by Cara in a brief conversation at the end of her tenth and final interview:

I: What did you say you thought mathematics was?
C: Learning maths
I: You thought it was the same did you?
C: Yes. Just learning maths

During the interview I indicated that I thought Cara meant that "mathematics" and "maths" were the same, but in retrospect I believe that she may have been seeing "mathematics " as the "learning" and "doing", that is, the activity within the domain of "maths". This reinforces two statements she made in the eighth interview when she said: "(Mathematics is) err, learning things, er doing things" and that adults use mathematics when they are "measuring things".

Summary and Conclusion

It was found that a young child does not necessarily assume the same meaning for the terms "mathematics" and "maths", thus challenging the often unreflective exchange and assumption of common understanding of the equivalence of these terms by adults. The research provides insight into the value of reflecting on, and being conscious of, the use of the terms, especially with young children. It demonstrates also the value of the use of a broad range of procedures in accessing children's beliefs as the meanings children have constructed may be complex with subtle differences in meaning between apparent synonyms.

The data from one child, Cara, demonstrated that beliefs about the nature of "maths" may encompass a broad spectrum of concepts or areas, and do not necessarily focus on number. This element of the research was presented in summary form in this paper, for the purpose of providing background and context to the discussion of Cara's understandings of the term "mathematics". These understandings of the meaning of the term "mathematics" were less easy to access due to Cara's uncertainty seemingly due to construction of meaning continuing to occur during the time of the research. It appeared that she may have seen "mathematics" as the activity element of "maths" but it cannot be concluded with certainty that this statement represents her belief. When Cara responded to the tasks investigating her meanings of the two terms, one noticeable difference was her level of confidence in responding. She appeared much more familiar, comfortable, and confident with the term "maths" than with the term "mathematics". However, when responding in regard to either term her responses were not always consistent.

It may be that Cara's apparent uncertainty in beliefs about mathematics was due to an evolution of her beliefs during the time of the two-term interview period. Cara's attempts at sense-making seem to have occurred throughout the interviews suggesting that the interviews themselves may have formed part of the process of Cara coming to construct her views.

The data presented illustrate that on any one occasion a child may not give the same response as on another occasion, thus suggesting a broad view, an unstable view, or perhaps one in the process of construction. Changes in response also, for example, may be influenced by the way in which a response is elicited, recent experiences or feelings towards different elements of "maths", the holding of only partially formed views, or the making of an unconscious or conscious choice to mention only selected elements on any one occasion. As discussed above, the response of the interviewer, even if unintentionally influential, also may cause children to alter or redirect what they are saying.

The interview excerpts cited above show the power of the semi-structured one-to-one interview for allowing views and issues to arise, to become illuminated slowly, and possibly to develop further within the time of the interview. The research validates the use of a broad range of procedures combined with multiple occasion one-to-one interviewing.

The research highlighted also that a young child of eight or nine years may have developed individual and idiosyncratic views about the meanings of "mathematics" and "maths". Accessing views through means such as those discussed in this paper allows the teacher or other interested person to "listen" to the child's personal perspective, to come know the child better as an individual, and to be better informed when catering for the child in a mathematics class.

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Young Students' Interpretations of Physical Angle Situations

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Each student in a sample of 144 Year 2 students was presented with models of three physical angle situations chosen from a set of nine. A bent straw was used as an abstract angle model. Students were questioned about how they would use the abstract angle model to represent each situation and what similarities they recognised between each pair of situations. The major factor influencing students' interpretations of the angle situations was the number of visible arms of the abstract angle. Implications are drawn for the teaching of angles.

Length and angle are arguably the two most basic concepts which we use to analyse our spatial environment, and they are also fundamental to the study of geometry. But, whereas most students master length early in primary school, angle causes difficulties well into secondary school (Mitchelmore, 1983). One difficulty is knowing what definition of angle to use. For some time now, mathematics educators have recommended treating angle in terms of turning, at least in the elementary stages. For example, our state mathematics syllabus for Years K-6 refers to angle as "the amount of turning between two lines about a common point" (New South Wales Department of Education, 1989, p. 79). It is argued that the development of the angle concept consists of establishing a particular relationship between the two arms of the angle; and a turn is a familiar, physical action which relates the two arms (Wilson & Adams, 1992).

A concept may be regarded as the end-product of a classification process which starts with the recognition of similarities between different experiences (Mitchelmore & White, 1995a). A definition of any given concept, then, draws out the predominant similarity in all the different situations in which it occurs. In the case of the New South Wales syllabus definition of angle, an amount of turning between two lines about a common point is regarded as encompassing the similarities in all situations involving angles. For example, the angle in a hill may be seen as the amount of turning between the imaginary horizontal and a line which gives the slope of the hill. It would therefore seem imperative to look at common physical experiences as a possible explanation for the complexities of the angle concept and for students' difficulties in learning it.

There has been very little research on how students actually conceptualise angles in their everyday experience. Davey & Pegg (1991) surveyed students from Grades 1 to 10 and reported that their descriptions of an angle went through a sequence of four stages: (1) a corner which is pointy or sharp; (2) a place where two lines meet; (3) the distance or area between two lines; and (4) the difference between the slopes of the lines. Clements & Battista (1989) found that Grade 3 students most commonly described an angle as a sloping line, a place where two lines meet, the two lines themselves, or a turn (the last only by students who had studied Logo). Such studies suggest that students do conceptualise their surroundings in terms of angles, and that the way they do this develops in sophistication as they get older. However, their knowledge of angle may be fragile and fragmented. For example, we have found (Mitchelmore & White, 1995b; White & Mitchelmore, 1995) that Year 4 students are often unaware of angle-related features linking different physical angle situations. Specific irrelevant features of the different situations appear to hinder abstraction of a general angle concept. In particular, it would seem that many students do not recognise turning in all contexts.

This paper reports the first part of a larger study exploring students' development of the concept of angle across Years 2 to 6. The specific research questions being investigated are as follows:

- What features influence Year 2 students' recognition of abstract angles in different physical situations?
- What angle-related similarities between different physical situations do Year 2 students recognise?

Method

Sample

The sample was gender-balanced and consisted of 144 Year 2 students from six schools in Sydney.

Materials

Based on earlier research (Mitchelmore, in press; Mitchelmore & White, 1995b, 1996; White & Mitchelmore , 1995), a series of nine physical angle situations were selected for study. There were four moveable situations (a small wheel, a door, a pair of scissors, and a Spanish [hand] fan) and five fixed situations (signposts, hills, road junctions, tiles and intersecting walls). (We reserve the terms "dynamic" and "static" to categorise angular similarities—see Table 3 below.) Each moveable situation was represented by a single physical model. Each of the fixed situations was represented by a set of three models representing a "neutral" configuration (angle 0° or 90°), an angle of 45°, and a "middle" angle (22.5° or 67.5°). Adjustable models of the fixed situations were specifically not used, in order to avoid suggesting a turning interpretation.

Twelve combinations of 3 physical situations were selected in which each of the 36 possible pairs occurred once and only once; each of the 9 situations occurred in 4 combinations. Each of the 12 combinations was then administered to 12 students. Thus, 48 students responded to each situation and 12 students responded to each pair of situations.

A drinking straw which could be bent at various angles was used as an abstract angle model. A second straw fixed at 45° was also used.

Interviews

A trained research assistant administered the interviews, which had been refined from the ones used in earlier work. For each situation, students were first asked to show how the abstract angle model could represent the given situation, and then to demonstrate an angle of 45° (shown on the abstract model) on the physical model. After they had responded to all three situations, they were asked what similarities they recognised between each pair. Probes were used at various points to clarify how students matched the abstract and the physical models.

Results

Matching abstract angles to physical situations

Students were asked to use the abstract angle model to show a general movement of the moveable situations or the 45° configuration of the fixed situations. The percentages of students who used the abstract model to represent the angular features of each situation in the standard manner are shown in Table 1. The term "standard manner" refers to the generally accepted angular modelling in which the vertex and the two arms of the abstract model match a point and two lines on the physical model. For the moveable situations, the vertex is the point of rotation and the opening of the arms represents the amount of rotation. For the fixed situations, the two arms represent two specific lines and the vertex is simply the point where these meet.

Table 1
Percentages of sample using abstract angle model to represent physical angles in standard manner

	General movement				45° configuration				
	Wheel	Door	Scissors	Fan	Signposts	Hills	Junctions	Tiles	Walls
Percentage standard	18	90	100	98	24	43	73	96	96

The only moveable situation to cause significant difficulties was the wheel. Very few students represented the turning motion of the wheel by an opening of the abstract angle model. All the remaining students (82%) made a rigid rotation of the angle model, either with the arms opened to a straight angle or with one arm bent out of the plane of rotation to represent the axis of rotation.

Among the fixed situations, the greatest difficulties occurred with the signposts and hills. In both these situations, students generally represented the sloping feature correctly but often did not represent a second (reference) line needed to make an angle. A 1-line method of representing slope was used by 72% of the students for the signpost and 53% for the hill. Even those students who did use a 2-line method tended not to represent the deviation from the neutral position, as most adults probably would. Thus, of those using a 2-line method, only 27% represented the signpost using a vertical reference line and only 44% used a horizontal reference line for the hill. Most of the students preferred to use as a reference line a line actually present in the physical model—the horizontal base of the signpost model or a vertical edge of the hill model.

Responses to the junctions situation showed many of the characteristics of responses to the signposts and hills situations, but to a lesser degree. Only 20% of the students used a 1-line method of representing road junctions.

Matching physical situations to abstract angles

Students were asked to set the moveable models, or to select the appropriate configuration of the fixed models, to represent a 45° angle shown on the abstract model. They were then asked to explain precisely how the arms and vertices of the abstract angle model were shown on the physical model. Students' responses are summarised in Table 2.

Table 2

Percentages of sample demonstrating and explaining abstract 45° angle on physical models

	Moveable situations				Fixed situations				
	Wheel	Door	Scissors	Fan	Signposts	Hills	Junctions	Tiles	Walls
Correct angle size	77	88	100	91	47	42	60	64	64
Standard explanation	21	44	28	17	43	45	44	100	91

Responses to the moveable situations were scored as the correct size if the angle was in the first quadrant. On this criterion only the wheel caused any problems, most of the incorrect responses falling in the range 90-180°. Responses to the fixed situations were scored as correct if the student selected the correct one of the three configurations. Almost all the errors consisted of choosing the middle configuration, very few selecting the neutral configuration. Students seemed sure that the 45° abstract model represented a non-neutral situation but were rather uncertain as to the deviation from the neutral position. Their uncertainty was most acute for the two slope contexts, signposts and hills.

The extent to which students used standard angle modelling to match the arms and vertex of the abstract angle to features of the physical model varied enormously. The easiest situations were the tiles and walls, where the two arms and the vertex are clear and unambiguous. The arms are also obvious for the scissors and the fan, but the vertex is not: Many students failed to place the vertex on the pivot. This nicety would probably fool many adults; in fact, 89% of the students placed the vertex on or near the pivot for the scissors and 79% for the fan. For junctions, many students placed the abstract angle model in the roads but did not align the two arms with the sides of the road (93% placed at least one arm parallel to one side of the road). For the remaining situations, many students could only match one arm of the abstract model to a feature of the physical model (the red line on the wheel, the top or bottom edge of the door, the sloping post, or the hillside). It was again noteworthy that those students who matched both arms of the abstract model to the signposts and hills models did not often use the neutral position as a reference: 70% related the signpost to the horizontal and 90% related the hill to the vertical.

Recognising angular similarities between physical situations

The similarities which students professed to recognise between two physical situations were classified as follows:

- Angle-related dynamic reasons, referring to the angular movement implicit in the two situations. Most such reasons referred to turning or opening, but some students could only say that the two situations showed a similar movement.
- Angle-related static reasons, referring to various features of the geometrical configuration implicit in the two situations. Two main categories of static reasons were found: (a) references to lines and points, such as "they both have two lines and a point" or "they both have angles", and (b) an explicit correspondence between the two configurations, such as demonstrating the match between the three tile corners and the slopes of the three signposts or stating that "the wheel can be turned to show the slope

of each hill". In both cases, these reasons were only accepted if students could demonstrate the correspondence correctly.

- Irrelevant similarities (such as "they are both made of wood"), inappropriate angle-related similarities (such as "the block on which the wheel is mounted has corners, like the walls") and appropriate angle-related similarities which could not be explained correctly.

Table 3 shows the number of students who recognised appropriate angle-related similarities between the various pairs of situations and correctly explained them, separated into dynamic reasons and static reasons. Only 5 students (scattered around the table) gave both dynamic and static reasons.

Table 3

Number of students recognising angle-related similarities between each pair of physical situations

	Wheel	Door	Scissors	Fan	Signposts	Hills	Junctions	Tiles	Walls
Wheel	-	6	5	6	0	0	2	1	0
Door	0	-	10	9	0	1	1	0	1
Scissors	0	0	-	9	1	3	0	3	2
Fan	1	0	2	-	2	1	2	0	2
Signposts	0	0	0	1	-	0	0	0	0
Hills	2	3	2	0	5	-	0	0	0
Junctions	3	3	5	3	6	5	-	0	0
Tiles	1	1	3	2	5	1	10	-	0
Walls	3	0	11	8	4	4	10	10	-

Note. The maximum in each cell was 12. Entries above the diagonal give the numbers of students recognising appropriate dynamic similarities. Entries below the diagonal give the numbers recognising and correctly explaining static similarities.

Dynamic reasons were never given to relate fixed situations and rarely to relate fixed and moveable situations. Dynamic reasons were much more often given to relate moveable situations. More than three quarters of the students related the scissors, fan and door dynamically, but only about half related these three situations dynamically to the wheel. Students almost always expressed dynamic similarities between the scissors, fan and door as "opening", whereas similarities between these three situations and the wheel were most often called "turning" and almost never "opening". It would appear that students in Year 2 regard the motion of a wheel as rather different from that of a fan, door or pair of scissors. The difference is consistent with the difficulty they have in identifying abstract angles in the wheel.

Static reasons were almost never given to relate moveable situations. Static reasons were predominantly given to relate fixed situations, but there was a considerable variation in their frequency. Almost all students related junctions, tiles and walls statically, but rather less than half related these three statically to signposts and hills. Static reasons were given more frequently than dynamic reasons to relate moveable and fixed situations, although the difference was not very great. (The large number of students who recognised a static similarity between walls and the scissors and fan is an anomaly which we cannot explain.)

Discussion

Many of the results from the present study (the frequency of 1-line representations of turns and slopes, common difficulties in representing the size of a rotation, and the relative ease of representing the abstract angles in the scissors and tiles situations) confirm the results of our earlier studies of Year 2 children. However, the wider selection of situations used in the present study makes it clearer that the major influence on angle conceptualisation is the number of arms of the abstract angle which are visibly present in the physical situation. The nine situations used may be classified according to the percentages of students who correctly matched the physical and abstract angles as follows.

- No line visible (wheel): 20%
- One line visible (door, signposts, hill): about 40%
- Two lines visible (scissors, fan, junctions, tiles, walls): allowing for some imprecision, between 80% and 100%

The significance of the number of visible lines is further shown by the percentages of students' responses showing the recognition of a similarity (either dynamic or static). The recognition rate was about 70% where both situations had two lines visible, compared to about 30% otherwise. The number of lines of the abstract angle which are physically visible seems to be far more important than whether the situation is moveable or fixed.

The salience of the lines visibly present in a situation is strikingly confirmed by some unintended results of changing the models used to represent the hill situation. In previous studies, we have used an adjustable hill model consisting of a sloping road hinged to a base. When this road was held up by a vertical support, many students used a vertical line as the second arm of the abstract angle. When the road was held up by hand, many represented the obtuse angle formed by the sloping road and the run-off road. In the models used in the present study, the hill was raised above the base so that there was no visible intersection between the sloping face and the base. But there were vertical supports, and students often used one of these for the second arm of the angle. Although even Year 2 students all seem to know that a horizontal hill does not slope, they show no tendency to measure the slope by its deviation from this neutral position. The same finding applies to the signpost situation.

Implications for teaching

The results of this study suggest that it should be possible to start teaching about angle as early as Year 2, provided the examples are limited to situations like the scissors, fan, junctions, tiles and walls models where both arms of the abstract angle are clearly present in the physical situation. Most Year 2 children seem to recognise angle-related similarities between such situations and to be able to represent the implicit angles by a simple abstract angle model. A study of these similarities and methods of representation could clear up their minor difficulties observed in this study (such as imprecision in representing the pivot of hinged objects and the arms of a road junction, and recognition of dynamic similarities between some of the fixed and moveable situations), as well as introducing the basic concept of angle. In this approach, angle would not be defined as "the amount of turning between two lines about a common point". If a definition must be given, perhaps "an opening between two lines" would be adequate for this stage.

The extension of the angle concept to situations where one or both arms are not visible should clearly be left until a later age. Before this can be done, students probably need to investigate turning and sloping as concepts in their own right. Experiences such as quantifying various amounts of turning and identifying the significance of the horizontal and vertical directions in space are likely to be especially valuable. Students could then be encouraged to identify the invisible lines which are needed to construct the abstract angle in turning and sloping situations, and hence to relate turns and slopes to other physical angle situations. It is difficult at present to predict a suitable age for such extension activities, but the replication of the present study in Years 4 and 6—completed since this paper was written—is likely to provide some answers.

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Odds : Chance Measurement in Three Contexts

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What are students' views of odds? Students were asked to interpret a newspaper headline, "North at 7-2". Three different perspectives were distinguished: (1) a probability view often using traditional part-whole ratios, (2) a frequency view involving scores and frequency of wins, and (3) a social view, usually involving betting and money exchange in part-part ratios. Each view followed a developmental sequence, with interaction between them.

Odds are indeed odd in relation to traditional probabilities in two respects. The first is that in addition to expressing uncertainty as theoretical ratios or expected relative frequencies, it is possible to express odds in the social context of betting. These three expressions are described by Kapadia (1986) as: (1) the classical probability view, based on symmetry and equal likelihood, (2) the frequency view, based on long term relative frequency, and (3) the subjective view. The subjective view is rarely taught in the schools, except perhaps in relation to odds. *A National Statement on Mathematics for Australian Schools* suggests activities for junior high school students to "understand and explain social uses of chance processes" (Australian Education Council [AEC], 1991, Statement C1, p. 175) including:

- discuss the term 'odds-on' and note that statements of odds which appear in gambling contexts reflect statements of subjective probability as well as statements of return on money invested (e.g., recognise that if the odds are 100 to 1 then the return on a win is very high but the chance of a win is correspondingly small), and
- investigate 'odds' to determine how the bookmaker makes a profit (the related probabilities add up to more than one).

This significant element of social uses of chance processes appears to have been overlooked in *Mathematics - A curriculum profile for Australian schools* (AEC, 1994). Peard (1991, 1994) found experienced gamblers use sophisticated strategies for working with odds which do not use conventional ratio based mathematics. His work suggested a strong involvement of work-place mathematics for handling odds, quite distinct from performance on more traditional school-based probability tasks.

The second respect in which odds are odd is that they involve part-part ratios rather than part-whole ratios; for example the outcomes of a coin toss are expressed as "50-50" rather than "50/100" or "1/2". Singer and Resnick (1992) investigated students' strategies for comparing two boxes with various ratios of coloured marbles, and found that students preferred part-part comparisons rather than part-whole comparisons, concluding "for these problems, at least . . . children's representations are generally based on the parts" (p. 244). If part-part comparisons are preferred, then odds would be preferred to conventional part-whole ratio probabilities, with major implications for the curriculum. There is no evidence yet, however, that Singer and Resnick's conclusion is transferable to odds.

Pilot work by Watson, Collis and Moritz (1994) gave preliminary indications of both context and level of quantification being evident in student responses to a newspaper headline implying odds for a football match. The headline was interpreted by many students at younger levels as a score, suggesting limited exposure to odds. A variety of expressions, including theoretical chance, relative frequency and betting, continued to be used at all levels of the development of appropriate quantification of the reported odds. They proposed a developmental model using the SOLO taxonomy (Biggs & Collis, 1989, 1991; Collis & Biggs, 1991), which was revised in later research related to more general items on likelihood (Watson, Collis & Moritz, 1995).

The SOLO model with multimodal functioning identifies five modes of thought: sensori-motor, ikonic, concrete symbolic, formal and post-formal. The focus in relation to odds is the concrete symbolic (CS) mode, which usually develops across the school years, where symbols are used to represent concrete situations. Within each mode, learning occurs in cycles. Three levels are distinguished within a cycle: unistructural, multistructural, and relational. Each level represents an increasing order of complexity as summarised briefly below. The prestructural level precedes the cycle, in this case most likely representing the ikonic (IK) mode.

(P) *Prestructural responses*, which use none of the elements required to identify the mode in question;

(U) *Unistructural responses*, which use only one relevant aspect of the mode;

(M) *Multistructural responses*, in which several disjoint relevant aspects are processed, usually in sequence;

(R) *Relational responses*, in which an integrated understanding of the relationships between the different aspects is exhibited.

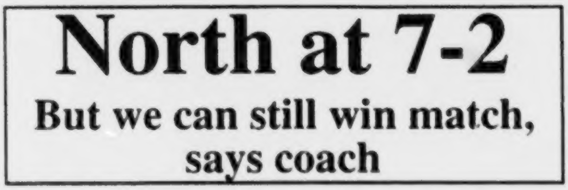
Recent studies have described two U-M-R cycles operating within the CS mode in connection with students' understanding of fractions and decimals (Watson, Collis & Campbell, 1995). These two cycles seem to be associated with (i) the development of a particular concrete concept (U_1-M_1-R_1), usually involving measurement or quantification, followed by (ii) its consolidation and application (U_2-M_2-R_2).

The development of the concept of likelihood was described in two U-M-R cycles by Watson, Collis, & Moritz (1995). In the first U-M-R cycle, U_1 responses recognised uncertainty, M_1 responses qualified the uncertainty, and R_1 responses quantified chance in a single setting, generally as a part-whole ratio, although sometimes as a part-part ratio. The second U-M-R cycle described application of the concept of ratio measurement of chance to problems in multiple settings, such as two boxes of marbles problems for comparing two ratios. In relation to this study, it was hypothesised that a first U-M-R cycle would describe the development of a ratio concept to measure chance, while the second U-M-R cycle would describe applications of this concept to the contexts associated with odds, such as distinguishing between part-part and part-whole ratios, and identifying the direction the odds favour.

Method

A media survey (Watson, 1994; Watson, Collis & Moritz, 1994) using newspaper extracts covering different topics in Chance and Data was administered to 312 Grade 6 students and 393 grade 9 students in Tasmanian government schools during 45 minutes of class time. The survey included the item analysed in this study, shown in Figure 1. Only students who attempted this or later items were included in the analysis.

Subsequent samples of students were surveyed under similar conditions, including 43 grade 7 students and 72 grade 9 students from a private girls' school in South Australia, and 30 grade 8 students and 22 grade 10 students from a coeducational school in the English midlands.



North at 7-2

But we can still win match, says coach

What does "7-2" mean in this headline about the North against South football match? Give as much detail as you can.

From the numbers, who would be expected to win the game?

Figure 1. Item related to odds from media survey about Chance and Data.

Results

Responses generally interpreted the headline within one of three different contexts of expression: probability expressions, frequency expressions and social context expressions. Some students, however, failed to respond or gave irrelevant responses (prestructural for this task), while other responses had aspects of multiple contexts. Within each context of expression, a hierarchy of responses was distinguished; the hierarchy was based on the level of sophistication of the quantification involved. Discussion of each approach follows, with examples of each level within that approach. Tables 1 and 2 provide summary of responses.

Prestructural responses

Some students failed to respond to the task of interpreting the numbers, by (1) non-response, (2) indication that they did not know, or (3) an irrelevant response. The first of the following examples has no relevance to the task. The second response is an example of tautology characteristic of prestructural responses. The last two examples gave some indication of knowing to read "7-2" as "7 to 2" from ikonic experience, but did not frame this within the required CS mode of uncertainty.

P: The weather. [Grade 6]
P: I think just 7-2 and that it's the coach saying to win. [Grade 6]
P-IK: It means like 7 to 2. [Grade 6]
P-IK: 7am - 2pm. [Grade 6]

Probability expressions

Probability expressions involved use of the word 'chance' or 'probability' as the context for interpreting the headline '7-2'. At the unistructural level, responses offered a single idea of uncertainty expressed as chance.

U_1: That's North's chances of winning. [Grade 6]
U_1: It means their chances are 7 to 2. [Grade 6]
U_1: 7-2 means the chance of North winning isn't very high.[Grade 6]

The last response indicated some qualification to the idea of chance, but there was no clear indication that this was an interpretation of the numbers.

Responses at the multistructural level qualified the uncertainty by interpreting the numbers in a primitive quantitative manner.

M_1: 7-2 means that North football team has more chance of winning than South. Because 7 is a larger no. than 2, and 7 stands for North + 2 stands for South, so North has a better chance of winning that South. [Grade 9]
M_1: North has a 7% chance to win and a 2% chance to lose.[Grade 6]
M_1: It means North only has 2 chances of winning whereas South has 7 chances of winning. [Grade 6]
M_1: It means that North has a 7 in 2 chance of beating South. [Gr. 9]

Responses which quantified the chance with a ratio measure were classified at the relational level. Often these responses showed an unconventional measure apparently constructed by the student but which sought to express the chance as a ratio.

R_1: Their chance to win the game, e.g. 2-2 would be an even chance. [Grade 6]
R_1: It means that 7 out of 2 chances of winning the football match, or 3.5 out of 1. [Grade 9]
R_1: That there is, I am not quite sure but I think, 7/10 chance that ~~North~~ [correction] South will win and 2/10 chance that North will win, there is 1/10 chance that there'll be a draw. The odds are stacked against North. [Grade 9]

Responses in the second U-M-R cycle applied the general concept of a quantified measure of chance to the situation of odds. Responses at the U_2 level expressed a consolidated ratio measurement of chance, interpreting '7-2' as a part-whole ratio.

U_2: 7-2 means they have a 2 out of 7 chance to beat South. [Grade 9]

U_2: It means North has a 2/7 chance and the South has 5/7 chance. Basically it means in seven games, South are likely to win five, and North to win two. [Grade 9]

U_2: We are talking about odds I think. The team in question has an estimated chance of 2 in 7. Eg. If you had 7 dice-rolls, then an estimated 2 would come out favourably. [Grade 8]

The last two responses expressed the chance also in the frequency context. These could have been classified at the U_2 level in the frequency approach, although in both cases, the first expression involved chance.

Progression to the M_2 level involved incorporating an additional element specific to odds, namely identification of odds as a part-part ratio, and conversion to an appropriate part-whole ratio.

M_2: 'Experts' believe North has a 7/9 chance of winning and South has 2/9. This means that North is lost likely to win, but South still has a small chance. [Grade 9]

M_2: The odds on North winning are 7-2. If South wins it would be a 2 in 7 chance of it happening. Chance of winning: North 77.77, South 22.22. [Grade 9]

The second response offered a U_2 level expression of part-whole ratio in the second sentence, and then goes on to calculate percentages assuming a part-part ratio. This multiple confused serial response is characteristic of the multistructural level.

Responses which could relate a ratio measurement concept of chance with expression of the appropriate part-part ratio and expression of the direction favoured specific to interpreting odds were classified at the R_2 level.

R_2: North hasn't got a good chance, like a ratio 7:2. A 2/9 chance to win. [Grade 9]

R_2: Out of nine in a scale from one, there is a 7 chance that the other team will win and 2 chance that North will win. [Grade 9]

Frequency expressions

Responses categorised as frequency expressions were those which interpreted the numbers as points in the score of a football game, or as wins and losses in a series of games. Responses which interpreted the numbers as the current score were considered to be frequency expressions which were ikonic prestructural. Although they may be defensible alternative interpretations in another context, they do not recognise the concept of the numbers expressing uncertainty in the intended context.

P-IK: It means 7 goals 2 points. [Grade 6]

P-IK: That North lost by 5 points. [Grade 9]

Responses expressing the numbers as a predicted score were considered unistructural in that they offered a single idea involving uncertainty.

U_1: That is the prediction. [Grade 6]

U_1: The coach thinks they'll win by 7 and the others will only get about 2. [Grade 6]

At the multistructural level, responses qualified uncertainty as a past score which might inform predictions, or as a prediction of a primitive ratio of the scores.

M_1: It means that in their last game they lost 7 goals to 2 goals. But they think there is still a chance. [Grade 9]

M_1: They have a 7-2 chance of winning the match. Every two goals they have, South would probably have 7. [Grade 6]

Both responses above identified uncertainty with the word 'chance', and then qualified this by interpreting the numbers quoted in a frequency expression.

Responses at the relational level integrated their expression of frequency with the newspaper context (which effectively ruled out a current score interpretation) by relating the numbers as quantifying past outcomes of games.

R_1: It means North football team has won two games out of seven. [Grade 7]

R_1: North have lost 5 games against them, but they can win another one, it's not impossible. [Grade 9]

Relational responses expressed the numbers as a frequency of wins as opposed to the points score offered at previous levels, indicating development towards a view of frequency over a longer-term. This development has notably involved the change from a part-part comparison to a part-total comparison which is inappropriate for odds.

The first U-M-R cycle plots the development the concept of quantifying uncertainty as a ratio of long term frequency. The second U-M-R cycle involves the application of this concept to the particular context of odds to include appropriate part-part interpretation of predictive long term frequency in favour of the appropriate team.

At the U_2 level, responses expressed a consolidated ratio concept from the first U-M-R cycle, and included a single idea specific to odds. The first two examples involved a predictive aspect but as a part-whole ratio, while the last example specified the appropriate part-part ratio yet gave no indication that the frequency is used to inform a future uncertain event.

U_2: Out of 3 1/2 games, they would probably win 1. [Grade 9]

U_2: It means if they play 7 games, they'll win two. [Grade 9]

U_2: They have played 9 games and only won 2 and lost 7. [Grade 9]

Responses at the M_2 level included a few ideas specific to odds, such as specifying the appropriate ratio and using this to inform prediction for the future event.

M_2: Out of 9 games North would probably win 7 and South would probably win 2. [Grade 9]

M_2: ~~North has 7 chances~~. [correction] If they played 9 games, North would win 7 times, South would win 2. [Grade 9]

At the R_2 level, responses integrated a number of ideas specific to odds, including clear indication that the odds favour South.

R_2: It means that for every 9 games they play, they are likely to win 2 out of 9. [South selected as expected to win] [Grade 9]

R_2: These are the odds that North will win, these aren't very good. It means that out of 9 games, North will lose 7 and win 2.[Grade 9]

The second response mentioned "odds" which indicates the betting context, then interpreted the numbers as a long-term expected frequency. Responses at the R_2 level often involved multiple contexts of expression with an overview of the different contexts and a flexibility in which context is expressed.

Social context expressions

Expressions in the social context involved "odds", "votes", "favourites", or money terms. Again at the prestructural level there was an intuitive acknowledgment only of the setting.

P-IK: I think it's a bit like betting on the horses. [Grade 6]

At the unistructural level, responses expressed a single idea involving the role played by the numbers in the headline.

U_1: The number of how many they say are for South football. [Grade 6]

Responses at the multistructural qualified the context, generally specifying terms more clearly, such as identifying '7-2' with the term 'odds'.

M_1: 7-2 means the odds stacked against them. [Grade 6]

M_1: It means that North are 7-2 favourites. [Grade 6]

At the relational level, responses expressed the odds with a ratio measurement description.

R_1: It means that people have given North the odds of 7-2, or 3.5 to 1 chance of winning the match. [Grade 9]

R_1: 7/2 means that the odds are against them. (7-2 is that fraction because fractions aren't used in betting. The real number would be 3.5 - 1.) [Grade 9]

The responses in the first U-M-R cycle identified the social context, and offered interpretations parallel to the frequency and chance contexts of expression in the development of a ratio measure. Responses in the second U-M-R cycle applied this concept, usually to describe exchanges of money. Again development is recognised by the acknowledgment of the appropriate part-part ratio, and by the correct direction favoured. At the U_2 level, responses applied a part-whole ratio interpretation, or an interpretation which did not clearly specify the part-part aspect. Some U_2 responses offered votes interpretations, which included consolidated ratio.

U_2: For every $2 you put in, you get $7 out (if they win). [Grade 9]

U_2: These are the odds of winning. If you put in 2 dollars, you are paid 7 back. That means for every vote, 7 are for North and 2 are for South. [Grade 9]

U_2: It means that 9 people or something were asked who would win the match, and 7 people said North would and 2 people said South would. [Grade 8]

Responses which clearly specified the money transactions for a part-part ratio were classified at the M_2 level. Both examples below illustrate unresolved conflict characteristic of the multistructural level. In the first example, the direction favoured was confused, suggesting the expected winner has an overly good return. In the second example, the direction appears again confused, and the additional frequency part-whole interpretation seems inconsistent with the betting scheme described as a part-part ratio.

M_2: 7-2 means that if you place $2 on the game (probably with bookmakers), you will receive $7 plus your original $2 back if North win. These prices (7-2) are called odds. [Part (b): North expected to win] [Grade 9]

M_2: This terminology is commonly used in gambling. 7-2 means if you put $7 bet on something, you will get your $7 back plus an extra $2. I suppose the heading says for every 7 games they play, they win 2. [Grade 9]

At the R_2 level, responses clearly described the betting situation, specifying the team favoured and the appropriate ratio of money involved.

R_2: 7-2 are the odds or chance the team has of wining. For every 2 dollars you bet, if they win you get $7 back plus your 2 dollars. [Part (b): South more likely to win] [Grade 9]

R_2: It means that the odds are 7/2. For example, if you were betting money on them then you would bet $2.00 and if they won you would get $7.00 back. If you have odds like 2/7 then the bookmakers think that team will win because you have to pay more money than you will get back. [Part (b): South is expected to win because 7-2 isn't very good odds] [Grade 7]

Summary across contexts of expression

Table 1 shows a summary of responses by contexts of expression. The summary description for each SOLO level notes the parallel features found in the different contexts. This parallel nature is often exhibited in responses which offered expressions from multiple contexts.

Table 2 shows the SOLO levels of responses by grade level. The score interpretation of the headline was dominant at younger grade levels. There is a trend for older students to achieve higher levels, as might be expected.

Discussion

Students responded to the interpretation task with three views of probability: a traditional chance expression, a frequency expression, and a social context expression. The social expression involves connotations of gambling, and so it is sometimes avoided by teachers on moral grounds. It is, however, an important part of the curriculum for two reasons. The first is that gambling is so pervasive in Australian society that it is important to be educated about the mathematics of the social event, as curriculum documents acknowledge (AEC, 1991), and to address moral issues, albeit sensitively. The second reason the social context of probability is important is that it illustrates a subjective view of probability, where people make subjective estimates of the likelihood of outcomes. This view helps students understand that choices and judgements by others based on probability estimates may differ according to prior information, which may be important for later understanding the effect of prior information in studying Bayesian conditional probability.

Teachers need to encourage learning across different contexts, such as building on the R_1 frequency expression of "they have won 2 out of 7 games" to include prediction of uncertainty in the chance context, "they have a 2/9 chance". To do so, teachers need to be aware of the wide range of student understandings about odds, and assist students to build upon their understandings to construct appropriate meanings for odds.

The headline was interpreted as a score by a large number of students, although this tendency appeared reduced with older students in favour of an odds interpretation (see Table 2). The open-ended question did not restrict answers to one context, and readily permitted the defensible score interpretation. These results indicate a baseline of the extent of students' exposure to odds, against which future research may compare to measure the impact of curriculum and social change.

The development of quantification of chance throughout the first U-M-R cycle supports the model of development proposed by Watson, Collis and Moritz (1995), although responses in the current study tended to be at lower levels, probably reflecting students' lack of exposure, or possibly due to the complexities of concepts associated with odds. The second U-M-R cycle highlighted specific features of odds: prediction, the part-part nature of the ratio and the direction favoured. Future research could profitably consider further the issue of part-part comparisons, using the contexts of both this study and that of Singer and Resnick (1992).

Table 1
Overview of levels of response by context of expression

SOLO Level	**Summary description**	**Chance context**	**Frequency context**	**Social context**
P-IK	No expression of uncertainty or prediction	"7 TO 2"	Score current	Betting
U_1	Simple uncertainty expressed	Chance; Likely	Predicted score	Numbers of how many for South
M_1	Uncertainty qualified or attempted quantification	% chance 7 in 2 chance	For every 2 pts, 7 pts (predictive) Past score predicts	Odds: no clear ratio idea
R_1	Uncertainty quantified in ratio measure	3.5 to 1 chance	Game score: won 2 out of 7	Votes/rating/ratio Odds, ratio clear
U_2	Consolidated ratio as predictor OR part-part ratio	N has 2/7 chance	If play 7, win 2 Played 9, won 2 or 7	Bet \$2, win \$7
M_2	Consolidated ratio as predictor AND part-part ratio	N has 7/9 chance	If play 9, N win 7 N win 7 every 9 games played	Bet \$2, win \$2 + \$7 (North favoured)
R_2	Predictive ratio with correct favoured direction	N has 2/9 chance	If play 9, N win 2	S: Bet \$2, win \$7 + \$2 (South favoured)

Table 2
Response levels by grade

SOLO	Tasmania				South Australia				England			
Level	Grade 6		Grade 9		Grade 7		Grade 9		Grade 8		Grade 10	
-	28	9%	30	8%	0	0%	3	4%	0	0%	3	14%
P	24	8%	2	1%	2	5%	0	0%	0	0%	1	5%
P-IK	149	50%	88	25%	29	67%	31	43%	18	60%	2	9%
U_1	38	13%	46	13%	3	7%	5	7%	1	3%	2	9%
M_1	30	10%	76	21%	2	5%	10	14%	1	3%	9	41%
R_1	8	3%	22	6%	2	5%	8	11%	1	3%	4	18%
U_2	24	8%	62	17%	4	9%	11	15%	9	30%	1	5%
M_2	0	0%	12	3%	0	0%	2	3%	0	0%	0	0%
R_2	0	0%	17	5%	1	2%	2	3%	0	0%	0	0%
Total	301	100%	355	100%	43	100%	72	100%	30	100%	22	100%

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Natural communication in mathematics classrooms: What does it look like?

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This paper recounts how preparation of case study material for a multimedia resource raised questions about traditional power relations in the control of communication in mathematics classrooms. It aims to provoke discussion on whether more natural patterns of interaction should be promoted, who should shape communication channels, and the roles of researchers and teacher educators in transforming classroom interaction. Ethical issues related to forms of presentation of the data are raised.

Introduction

This paper intends to provoke further deliberation and debate about relationships between teachers' control of classroom interaction and the learning of mathematics. It arises from our involvement in a research and development project that focuses on the use of an interactive multimedia resource for mathematics teacher education (Sullivan & Mousley, 1994). Each CD-ROM disc in the program presents a case study, with a video of a whole mathematics lesson linked to interviews with the teacher, the lesson transcript and notes, readings pertaining to particular aspects of the pedagogical style and specific components of the teaching, graphic representations of the classroom interaction, and other features.

As we were developing the CD-ROM discs and their accompanying manuals, seeing the classroom interaction represented in a number of media raised some problematic issues about communication in mathematics classrooms. This paper focuses on just one issue—teachers' control of communication—and uses one case study from the resource to illustrate our reflections and discussions.

Communication

Effective communication was one of the six components of quality teaching identified in an earlier survey (Sullivan & Mousley, 1993) when we grouped phrases used by questionnaire respondents into what seemed like categories useful for describing good teaching. In reporting this stage of the project, we summarised the phrases about classroom communication as follows:

> Communication relates to opportunities for talking, explaining, describing, listening, asking, clarifying, sharing, writing, reporting, and recording. It includes class organisation structures such as co-operative groups, and is characterised by an orientation of teacher and students to accepting communication as a two-way process … (It) is vital in organising for learning, engaging students, and nurturing their all-round development, as well as in mathematical problem-solving … Most importantly, it is a key aspect of building understanding. (Sullivan & Mousley, 1994, pp. 428-9)

It was not surprising that communication surfaced as an important feature of quality teaching. Attention to language factors in mathematics education has long been recognised as a focus for the improvement of pedagogy (see, for instance, Del Campo & Clements, 1990; Mason & Pimm, 1986; Mousley & Marks, 1991; Pimm, 1987). In Australian mathematics education research, there seems to be a recent surge of interest in analysing and improving patterns of classroom interaction, as noted by Ellerton and Clarkson (1992):

> ... especially over the past four years, Australian mathematics educators have investigated ways and means of establishing mathematics learning environments in which learners interact with each other and with their teachers in ways which will improve the quality of mathematics learning. (p. 153)

Data Which Raised the Issue

The case we wish to draw on for this paper was filmed in a Year 6 class. The students were asked to make as many different 'box-shaped' buildings as possible using 24 blocks, to record and compare their solutions in groups, and then to report their findings orally. In handling the data collected in this classroom, there were three incidents that increased our awareness of the problematic nature of classroom communication patterns and of dilemmas that teachers face as they organise discussions and thus position themselves and students within the discursive field.

The first incident arose just after the lesson had be filmed and particular components of the high-quality teaching had been identified. Before the video was committed to CD-ROM disc, we were keen to validate the existence of each of the components. We asked colleagues to watch the videotape and write naturalistic summaries of what they observed. We noted that there was a distinct division in the way that people commented on teacher-to-student communication. Some people suggested that the teacher had focused much more on some pupils than others and should have been more inclusive. Others, however, commented on her nurturing manner of allowing students to join the dialogue when they felt they had something positive to contribute. (For discussion of these disparate positions, see Mousley and Sullivan, 1994).

The second surprising event occurred when, in preparing the multimedia resource, we transcribed the lesson's audio component, noted the numbers of verbal interactions each child contributed, then graphed these. Our intention here was to link electronically each bar of the graph to video snippets of that child's interactions, so that clicking the indicator on a bar would allow users to examine the interactions more closely, enabling a study of, for instance, features such as different types of questions used, the quality of particular comments of classroom participants, or the teacher's responses to individual students. Since the lesson had seemed inclusive of all children, the unevenness of the interactions in the graph produced (Figure 1) was not foreseen.

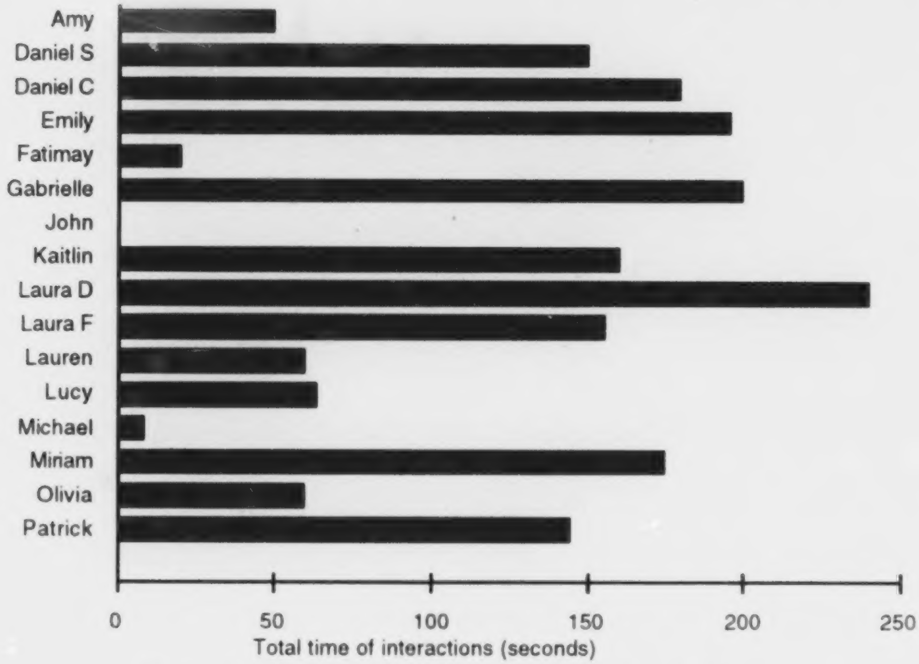

Figure 1: Number of individual children's interactions

The total times of each child's interactions (again mainly with the teacher) were then graphed (Figure 2).

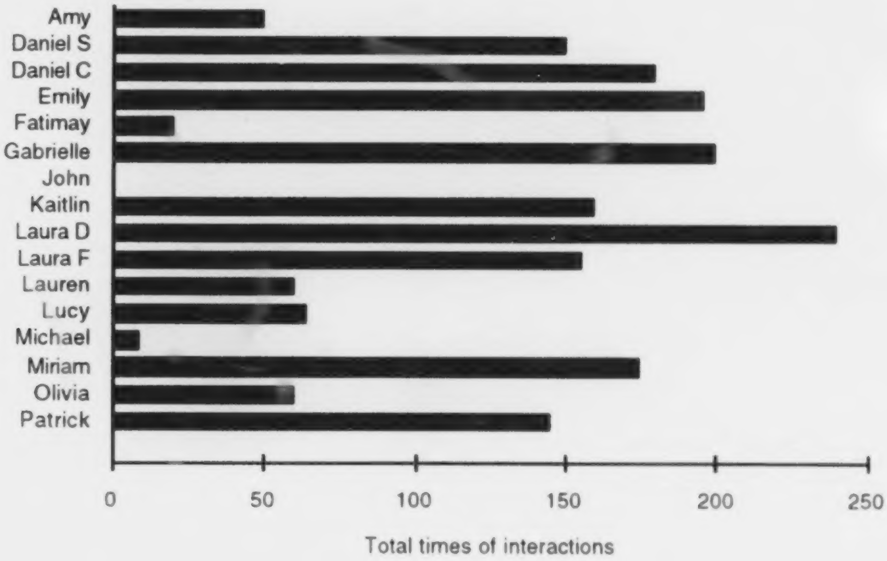

Figure 2: Total times of individual children's interactions

The graph of the number of questions asked by each participant in the classroom (Figure 3) also demonstrated inequability.

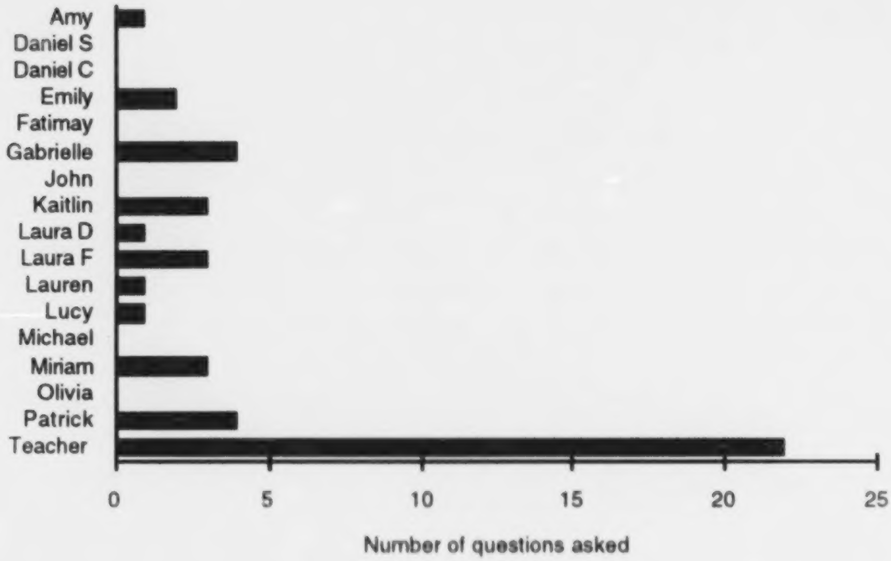

Figure 3: Number of questions asked by lesson participants

It should be noted here that the verbal interactions were those captured by two video cameras whose operators had been instructed to focus on the teacher and the particular children she was dealing with at any time. The data therefore do not include much student-to-student discussion. The data was also from one lesson only, and a further lesson in the same classroom produced different results. It is clear, however, that there were significant differences between the number of verbal interactions contributed to the by individual participants, the amount of 'talking time' used by different participants, and the number of questions they asked in order to clarify concepts and procedures—and these features aroused our interest.

The third incident occurred when a group of graduate students trialing the program noted that the teacher had directed a question to a child who had played no previous part in the class discussion. The girl had appeared flustered and had given a very hesitant answer. The students observed that later questioning by the teacher in a one-to-one situation revealed that the pupil had had a good understanding of the concept and the problem being discussed. This led to a discussion on whether (and if so, how) students should be encouraged to take a relatively equal part in overt exchanges in mathematics classrooms.

The three incidents are related in that each raises questions about the differential status of participants in mathematics classrooms. The creation and sustenance of communication patterns, the claiming or conceding of interaction forms and rights, and the collective or individual nature of responsibility for participation all become topics open to exploration. Further, the incidents suggest the need to debate potential roles for teachers to play in changing patterns of classroom interaction.

Patterns of Communication

In non-institutional social situations, the frequency of people's contributions to discussions, their total time allocation or the number of questions they ask are not controlled, so we rarely feel worried about imbalances or attempt to redress unequal participation. At a dinner party, for instance, there may be some deference to personal status as well as to authority on particular topics, but people generally place themselves within dialogues according to other factors such as their confidence in the situation, their knowledge of the subject matter, the contributions they wish to make (or to reserve), and the roles they wish to play within the group at that time. Contributions are expected to be uneven, and it is generally accepted that people can be fully involved in discussions by just listening.

Mathematics lessons, however, are not natural social situations. Traditional patterns of control of communication have developed in schools, just as they have in other social institutions (the doctor's office, the witness stand) where one participant has a predetermined communication agenda. Edwards and Westgate (1987) claim that most teachers see close and persistent control over classroom communication as a precondition for reaching their educational objectives.

> From this perspective, classrooms can be aligned with other contexts in which a participant has, or claims to have, prior or superior knowledge of the matters in hand. Where the other participants accept that claim then the talk will be organised by reference to that hierarchy ... the 'expert' will 'control knowledge' by asking the questions, evaluating and shaping the answers in the light of what he or she needs to get the other(s) to say, discarding those which are thereby irrelevant or redundant, and terminating the exchange when enough information has been obtained for the practical purposes of that encounter. (Edwards & Westgate, 1987, p. 46)

Other factors in traditional patterns of communication control may be crowded conditions, the judging of even productively noisy classrooms as poorly supervised, fear of possible consequences of the unpredicatibility of open questions and conversations, entrenched cycles of interaction between teachers and students (T initiation, S response and T comment; T initiation, ...), and the boundaries teachers maintain between learned and ignorant—and hence between producers and receivers of knowledge.

While dominant models of classroom interaction and their implications have been described for years (e.g. Bernstein, 1974; Flanders, 1970; Mehan, 1979), it is only recently that the reverse notion—how discourses are shaped by individuals—has been a focus of research attention. Our use the term discourse here goes further than social uses of language patterns in that it includes the positioning of people within and by such activity. Participants in any social situation are positioned by others within a wider network of traditional power relations, often with institutional bases (Foucault, 1979; Weedon, 1987). These networks and the social interactions within them shape the ways that individuals and social groups see, and therefore portray, themselves. Thus it is through discourse that human identity is developed, maintained or changed.

However, the relationship between discursive activity and identity is a dialectical one, enabling any study of a discursive field to include examination of the way individuals position others and shape its social processes. The ways that classroom participants choose and use words, make meaning, and constitute both themselves and the institutional world through acts of speaking are open to examination (Davies & Harré, 1990). Thus while teachers and traditions of classrooms are dominant factors in determining patterns of verbal interaction in mathematics classrooms, it is possible to observe students working to produce their own identities within, and sometimes against, institutional constraints.

Competing Perspectives

Clearly, there would be advantages and disadvantages in bringing all students into discussions. Making sure that children contribute equally, for instance, would help teach the social skills relating to group discussions—skills of participating and including others. A teacher requiring quiet children (and particularly traditionally subordinate groups such as girls) to contribute would model strategies for facilitating inclusion as well as conveying a clear expectation that all students will be involved in the range of classroom activities. Encouragement to contribute could result in a growth of confidence after successful attempts are made, leading to more willingness to contribute in future. Responses from all students would enable teachers to judge individual students' understanding and hence adapt further teaching. As articulation of ideas helps to clarify them, expecting students to contribute to a discussion would be likely to lead to improved learning.

However, the notions that there are only some acceptable indicators of participation and that teachers should control all classroom dialogue arise from a didactic model of education where teachers set learning objectives in terms of measurable forms of predictable performance—and then structure, control and evaluate classroom activity in terms of these objectives. Some other models of education that are beginning to impact on pedagogy position teachers not at the centre of activity, but as facilitators of a variety of learning processes. In these models the above functions of social interaction typically become the responsibility of all classroom participants. Using more natural patterns of interaction, students communicate with each other without regular deference to the teacher, even in whole-class discussions. Individuals are not pressured to participate and are not put on the spot through discursive imposition. While such models of educational practice do not deny that curriculum content reaches students through the agency of teachers, they do require flexible patterns of communication within a context of new social relationships and practices.

The didactic model of mathematics education is a powerful one. It is learned in our own schooling, reinforced during teacher training (in both universities and schools) and valued in most professional settings. It was strong enough, for instance, to make the research team question whether graphs such as those above should be included among representations of the high-quality mathematics lessons included in the resource. ('Did teachers agreeing to be filmed give us the right focus on what could be so easily criticised? Why make the unevenness so obvious?') The influence of the didactic model was also strong enough for the issue to provoke extensive discussions amongst the researchers, their colleagues, and students using the draft materials.

We realised that to omit such data would limit possibilities for analysis of patterns of communication in mathematics classrooms and would also mean that we were not drawing fully on the capabilities of interactive multimedia for provoking research into classroom interaction. Exploring the issue of oral participation within competing views of education is not just a matter of counting numbers of interactions or suggesting ways that quieter students can be encouraged to communicate, so linking each component of the graphs with the snippets of video they represent and with written articles that support the styles of communication illustrated would allow a detailed examination of how, and with what consequences, pedagogical processes are structured in particular ways. It would also provide a resource for studying how verbal interactions impact on what is seen as 'mathematics' and what is legitimised, more broadly, as knowledge. As Walkerdine (1990) notes,

> ... the issue of silence and speaking is not a simple matter of presence or absence, or of suppression versus enabling ... what is important is not simply whether one is or is not allowed to speak, since speaking is about saying something. In this sense, what can be spoken, how, and in what circumstances is important. It tells us not only about its obverse, what is left out, but also directs attention to how particular forms of language, supporting particular notions of truth, come to be produced. This provides a framework for examining how speaking and silence and the production of language itself become objects of regulation. (p. 54)

A focus on this aspect of the other media linked with the graphs could also provoke a study of how the teachers and children who were filmed position themselves and each other expertly, not in dichotomous sites of control but on a continuum, speaking and inviting speech for specific purposes then moving to positions of less control once those purposes are achieved.

Conclusion

We did not set out in writing this paper to present a research report but (in the tradition of post-structuralism) to tell a story of emerging research themes. Concerning communication in classrooms, the story is one of initial surprise, contradiction and reflective discussion; of questioning but not finding answers. In relation to new technologies, it is a story of our developing understandings of the power of a variety of linked media in a new form of research product to engender such questions and hence to provoke educational inquiry. In consideration of research methodology, it is a story about ethical uses of data and the way that particular forms of presentation—while creating opportunities for productive study—may make subjects vulnerable to unforseen critique.

For mathematics teachers the questions raised by this component of our research project are about whether natural communication should be limited in classrooms, and who should be designing communication channels, as well as which parties should be talking, when, and for what purposes. They also relate to how substantive messages about learning (as opposed to schooling) and about the nature of 'doing mathematics' are imparted in different discursive fields.

For mathematics educators and researchers this raises the issue of how to develop a conscious awareness by teachers of their own actions, the reasons for these, and the wider consequences of their teaching behaviours—and then how to facilitate the application of new understandings to the active trialing of new discursive practices in mathematics classrooms. As Walkerdine (1990) states, 'Change is not a matter simply of deconstruction. A new reading permits the possibility of struggle to work for transformation of that sociality, those practices, and of subject-positions produced within them' (p. 57).

Notes 1: This paper was presented at MERGA 18, 1995 in Darwin. However due to an oversight it was not published in that conference's proceedings.

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Children's Representations and Conceptual Understanding of Number

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This paper reports the developmental phase of a 2-year longitudinal study of 120 Grade 2 children, which is designed to explore the relationship between their representations and understandings, of key number processes. The study focuses on the structure of children's representations of estimating, counting, grouping, re-grouping, partitioning and multiplicative processes. The research methodology, including videotaped interview tasks used during trialing in March and April 1996, and preliminary findings will be presented for discussion and critical analysis.

Recent studies have focused on the importance of children's construction of number concepts and processes examining specific areas such as counting, addition and subtraction, or multiplication and division as separate areas of study. This research has provided insight into the diversity of children's numerical strategies but it has not examined closely, the relationship between children's representations of number and their development of number concepts and strategies. Few studies have provided an in-depth analysis of children's representations of essential and related features of number such as estimating, counting, grouping, regrouping and partitioning. Furthermore, there is growing need to examine children's construction of number concepts and strategies over time through longitudinal studies that monitor growth and change.

Studies focusing on numeration have primarily examined children's counting and grouping strategies (Bednarz & Janvier, 1988; Denvir & Brown, 1986a,b; Fuson, 1990; Hiebert & Wearne, 1992; Kamii, 1989; Steffe, Cobb & Richards, 1988; Steffe, 1991; Wright, 1991). Although counting has been the focus of much of the work on early number learning, other researchers have directed attention to the development of multiplicative reasoning (Confrey, 1994; Mulligan & Mitchelmore, in press). Rubin and Russell (1992) asserted that children's counting, grouping, estimating and notating skills are essential elements in developing representations of the number system. They described these elements in terms of "landmarks in the number system". These landmarks appear to be related to additive structure, multiplicative structure, the generation and analysis of mathematical patterns and mathematical definitions. Rubin and Russell also suggested that people who are adept with number operations e.g. computing, comparing, and estimating, have a non-uniform view of the whole number system.

Other recent studies analysing the growth of number concepts and processes, indicate that children's representations of problem-solving situations are closely linked to their conceptual understanding and the way they construct mathematical relationships (Davis, 1992; Davis, Maher, & Noddings, 1990; Goldin & Herscovics, 1991; Hiebert & Wearne, 1992; Maher, Davis & Alston 1993; Mulligan, 1996). Researchers have also highlighted children's representations as evidence of the idiosyncratic and creative ways in which they structure mathematical relationships (Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin, 1994).

Some researchers suggest that conceptual understanding is built on the notion of constructing connections between representations of mathematical ideas. This involves building relationships between quantities that are represented physically, pictorially, verbally and symbolically (Hiebert & Wearne, 1992). Moreover, they suggest that building connections between external representations supports the growth of internal conceptual understanding and more coherent internal representations of mathematical ideas.

However it is not entirely clear how children use their representations in building conceptual understanding. If children's representations of their counting and computational strategies are a reflection of their developing concepts, or if their representations actually enhance the development of mathematical concepts and processes, then their representations may provide valuable evidence of how conceptual understanding grows.

The aim of the present study is to analyse and describe, in the greatest possible detail, children's internal representational capabilities evidenced in solving tasks focused on the structure of the number system and related numerical processes. At the same time, the study examines the relationships children form between various numerical processes such as counting, grouping, additive and multiplicative strategies. A 2 - year longitudinal study will provide a more coherent picture of how young children construct numerical representations, and whether these representations change and develop to accommodate more complex numerical structures over time.

Imagery in Representations of Number

The role of imagery in the representation of mathematical ideas has been highlighted in recent studies investigating representations of number. In the past, personal visuo-spatial representations of number (number-forms) were first described by Galton (1880). Seron, Pesenti, Noel, Deloche and Cornet (1992) studied adults' representations of number claiming that "some automatically 'see' the numbers they are confronted with in a precise location in a structured mental space, others 'associate' specific colours with given numbers" (p.164). A wide range of representations were found categorised into two main types: continuous lines, scales, grids (coded as number forms), and coloured codes. Associated images and simple analogical representations were also observed. Most subjects asserted that their images of number emerged between 5 and 8 years of age, or that their representations were a direct result of teaching. They concluded that number-forms are used to code the number sequence, and that the function of this phenomenon should be examined in numerical contexts and in number operations and calculations.

In a cross-sectional study of 166 children (K-6) and 79 high ability children (Grades 3-6) it was found that the children's internal representations of numbers were highly imagistic, and that their imagistic configurations embodied structural development of the number system to widely varying extents, and often in unconventional ways (Thomas, Mulligan & Goldin, 1994). Children's active processing of internal images were found to be static or dynamic in nature: static meaning a fixed representation, and dynamic as a representation that is changing and moving. Formal notational symbols were organised imagistically, such as spiral formations or flashing numerals, with children visualizing number in a dynamic way. In the cross-sectional sample, 3% of the children displayed dynamic images of the number sequence whereas 10% of the high ability children used dynamic images.

Following this, an exploratory study of 77 high ability Grade 5 and 6 children investigated links between their understanding of the numeration system and their representations of the counting sequence 1-100 (Thomas & Mulligan, 1995). Analysis of children's explanations, and pictorial and notational recordings of the numbers 1-100 revealed three dimensions of external representation: (i) pictorial, ikonic, or notational characteristics, (ii) evidence of creative structural development of the number system, and, (iii) evidence for the static or dynamic nature of the internal representation. Children used a wide variety of internal images of which 30% used dynamic internal representations. Children with a high level of understanding of the numeration system showed evidence of both structure and dynamic imagery in their representations.

Other studies have also indicated that imagery plays a vital role in the development of conceptual understanding. Recent work (Brown & Presmeg, 1993) in which individual students' thinking was probed in clinical interviews indicated that students use imagery in the construction of mathematical meaning and involved very abstract and vague forms of imagery. Students with a greater relational understanding of mathematics tended to use more abstract forms of imagery such as dynamic and pattern imagery, while students with less relational understanding tended to rely on concrete and memory images.

We question then, how children's representations influence their conceptual understanding, and whether particular forms of imagery are more powerful than others in developing numerical processes. However, children's representations of numerical ideas cannot be investigated in isolation from the way children structure numerical relationships.

Structural Development of Number

Numeration involves the development of an increasingly sophisticated counting scheme and system of notation for recording the numbers generated. Initially a system of units, the counting scheme must drive the representation of assemblages partitioned into groupings of ten. The 'ten', while retaining its signification of ten units, becomes itself an iterable unit. This in turn may be operated on by the relation 'form a group of ten', to construct the new unit of 'hundred'. Arrays provide one conceptualisation of the multiplicative process, and can illustrate recursively the relations 'multiply by 10', 'multiply by 100', etc. in the numeration system.

Understanding number concepts and operations is based on the development of this structure as well as the development of a child's counting, grouping and notational capacities. For example, in multiplication and division concepts the development of a structure for forming equivalent groups and multiple counting strategies is crucial for conceptual understanding. The processes described here are not just verbal or notational; the role of imagery in representing these structures is essential.

A study by Mulligan (1996) examined how third graders spontaneously represented and calculated solutions to multiplicative word problems as they worked in a mini-classroom situation in which they were required to represent and explain their solution strategies. Children's internal conceptual structures of multiplication and division, described in earlier studies in terms of intuitive models (Mulligan & Mitchelmore, in press), were inferred from elements such as grouping, partitioning, counting and patterning of children's external representations, and explanations of these representations. Representations of multiplication strategies were classified as: (i) modeling the problem by *direct counting* one by one, without taking into account the structure of the equal sized groups; (ii) *repeated addition* of equivalent groups by rhythmic, skip counting or adding, and (iii) the *operation of multiplication,* represented as a binary operation. Intuitive strategies of division resembled those found for multiplication: (i) a rudimentary level, *direct counting,* where children shared and counted one by one; (ii) *repeated subtraction,* where the child started with the dividend and successively took away the numbers in each group, (iii) *repeated addition,* where the child started at 0 and built up to the dividend by counting or repeated adding, and (iv) the *operation* of multiplication as the inverse of division. Thus counting and calculation strategies wer also evidenced in children's drawn and written representations

Thus, our research now focuses on two aspects: types of imagery, and structural characteristics of number, in an analysis of children's representations of number. This research project seeks to ascertain whether the characteristics of children's representations are indicative of their level of conceptual understanding of number. More specifically, the study investigates the structure of children's internal representational systems for counting, grouping, regrouping, partitioning and estimation, the coherence and organisation of children's externally-produced representations, and their range of numerical understandings. This paper reports the developmental phase of the study. The research design and methodology, including interview tasks used during trialing, and preliminary findings will be presented for discussion and critical analysis.

Research Methodology

The research is an exploratory, descriptive study based on constructivist methodology widely used in contemporary mathematics education research. The qualitative analysis is highly descriptive but incorporates some quantitative measures of performance and coding of responses. Ten NSW Department of School Education primary schools from the metropolitan region of Sydney have been randomly selected for participation the study, from which a random sample of 120 Grade 2 students involving 12 students from each school (balanced for gender), have been chosen. Students will be interviewed individually, four times - once each in April and in November of 1996 and 1997, based on the design used in similar longitudinal studies (Carpenter & Moser, 1984; Mulligan, 1992). A pilot study was completed in March 1996 at a Sydney metropolitan primary school involving 10 Grade 2 children. The chief investigator and research assistant audiotaped and videotaped all interviews. Interview procedures and tasks were trialed to accommodate the range of abilities of the children and to allow sufficient time for children to draw representations.

Task-based interview procedures (Goldin, 1993) were used involving estimating, counting, grouping, regrouping and partitioning, subgrouping, splitting and other multiplicative tasks. Counting and grouping tasks focused on multiple and double counting strategies, counting as iterable units, partitioning and estimating (Mulligan, 1992; Steffe, 1991; Wright, 1992). Numeration tasks were based on structural elements of the number system, visualisation and number sense (Hiebert & Wearne, 1993; Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin, 1994). Further questions focused on multiplicative structures such as arrays, combinatorial thinking, and splitting, were adapted from those used by Outhred (1993); Mulligan (1992), and Confrey (1994) respectively. Calculators will also be used in selected tasks. Some exemplars used during the trialing phase are presented in Table 1.

Analysis of Children's External Representations

Children will be asked to represent their solutions by modelling, drawing and symbolising their mental images. They will be asked to explain their representation and their solution process. The researcher will also probe for other available representations that the child can explain or represent. All interviews will be audiotaped and transcripts will be transcribed and coded for performance, three dimensions of representational characteristics, and for evidence of conceptual understanding. A selected sample of interviews will be videotaped. Results will be analysed for performance and solution processes initially for the whole sample to identify some general trends, and then according to individual performance and representational dimensions across the two-year period. Measures of association between the level of representation and the child's performance as an indicator of conceptual understanding will be identified and described. Profiles on each child will be compiled over the two-year period; analysis of the changes in the individual development of number concepts and strategies, and the corresponding representational capabilities will be analysed.

Table 1 Task-Based Interview Items

1. Numbers in Context

How old are you? What year is it? Close your eyes and tell me what you see.

How old will you be in 2 years time? How old will you be in 5 years time?. What do you think the numbers mean. Close your eyes and tell me what you see when you give the answer. Draw and explain your response.

2. Counting sequence; visualisation tasks

Ask children to close their eyes and to imagine the numbers from 1 to 100, to draw, explain and write about what they see. Check for other sensory stimulation e.g. movement, position, colour, or feelings associated with numbers. Repeat the activity to find what the children imagine for numbers past 100; before 0; between 0 and 1; the smallest and largest number they can think of.

3. Numeration; grouping; rate

You can trade 2 small stickers for a large sticker. How many large stickers are worth the same as 4 small stickers? How many large stickers are worth the same as 2 large stickers and 4 small stickers?

4. Estimation; grouping; regrouping; addition. Show a collection of pencils in packets of 10 and as loose pencils. Ask, without counting, how many they think is in each packet? Why? Ask students to show 26 pencils. Use the pencils to make the number which is 9 larger than 26. Children who were successful with above tasks are asked if they can find another way to show 35.

5. Regrouping, addition -ones, tens, hundreds. Show the number 145 using bags, rolls and separate lollies. How many lollies will you have if you add 10 more to this number?

6. Structure; groups of groupings; determine pertinence of groups. Show a picture of (1) 35 (2) 80 (3) 140 dots randomly drawn. Ask child to guess how many dots there are drawn on this sheet of paper? Repeat for (1), (2) and (3).

I am going to ask the same question to a friend who will be here after you. Could you do something so that, when I show them the sheet, they will be able to tell me very quickly how many dots there are? What would you do? Show the same picture but with groups of ten encircled with a black line and the group of ten tens encircled in red. Look what someone did for you. What do you think of it?

7. Estimation, addition: dot patterns and ten frame tasks.

Flash dot pattern for 8 and ask child to calculate number of dots. Reveal and ask child to count and add two dot cards together.

Show ten frame for 7. How many more dots do we need to make 10? Show ten frame for 10 and for 6. How many more dots do we need to make 20?

8. Multiplicative tasks

(a) Splitting: If you had $24, how would you share it fairly between 2 children (same age). If 2 more children came along, how would you share it fairly with them.

Repeat with $50 (only if $24 easily completed).

(b) Equal grouping: There are 2 tables in the room with 3 children seated at each table. How many children are there altogether (in the room)?

(d) Combinatorial: There are plain chips or salt and vinegar chips in small, medium or large packets. How many different packets of chips can you choose from?

(e) Quotition: There are 12 children and 3 children are seated at each table. How many tables are there?

(f) Array Structure: counting and grouping. Find total number in an array. Draw the array from memory. Show a 3 x 3 array of dots (stickers etc will suffice). Estimate. How many dots are there? Count the array. Find the total. Is there any other way of working out how many, more quickly?

(g) Sub-grouping. Provide model of cakes. Show how to share 2 cakes fairly between 4 people. How much do they get each?

Interview transcripts, together with the external pictorial and notational representations produced by the children will be coded and analysed. Intercoder reliability will be measured by an independent coder. The external representations will be considered with respect to three dimensions (based on operational definitions and coding procedures already developed in the pilot studies): (a) the type of representation (pictorial, ikonic and notational), from which we seek to infer characteristics of each child's internal imagistic representation; (b) the level of structural development of number concepts and processes evidenced in the representation; and (c) evidence of the static or dynamic nature of the image. For example, pictorial recordings will be defined as pictures drawn, or oral descriptions of objects given by a child. Ikonic recordings may include drawings of tally marks, squares, circles or dots that represent number. Notational recordings will be distinguished by the predominant use of numerals and symbols drawn in various formations such as an array.

The level of structural development of number will be interpreted from structural elements (i.e. counting, grouping, regrouping, partitioning and patterning) found in the recordings. For example, children showing evidence of a more developed multiplicative system may record a multiple counting sequence, and marks or pictures in a partial or complete ten by ten array structure. The static or dynamic nature of the image will be defined according to whether the recordings and children's explanations of their representations describe fixed or moving (changing) entities. Children's idiosyncratic representations of number will be explored fully in order to establish their influence on the child's general development of numerical strategies.

Implications

It is intended that the project will provide new knowledge about how children use imagery and structure in their pictorial, ikonic and notational representations of number. The analysis of representations will contribute to a new theoretical model for interpreting children's internal cognitive capabilities which will be particularly significant for assessing and assisting gifted and talented children, and those experiencing learning difficulties. The project will provide teachers with a practical range of innovative task-based assessment strategies and structured learning experiences that can be directly applied to improving mathematics teaching and learning, particularly for Early Numeracy Programs currently being trialed in Australia.

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Creating and Maintaining Knowledge-Building Communities of Practice During Mathematical Investigations

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In this paper, we describe a Pilot Study in which we investigated how a computer-mediated collaborative learning environment called CSILE could be used to mediate the establishment and the maintenance of a knowledge-building community of mathematics practice within two elementary school classrooms. We conclude that if CSILE is to successfully mediate knowledge-building during mathematical investigations, then it needs a human-computer interface which: (1) preserves the concreteness and the spontaneity of the students' synchronous, face-to-face investigative work and (2) helps to recreate or maintain the sense of community of practice that is engendered during face-to-face investigative inquiry.

Introduction

Within education, there are current trends towards thinking about learning and knowing as social as well as individual activities (Cobb, 1994; Lampert, Rittenhouse & Crumbaugh, 1995; Pea & Gomez, 1992). Acquiring knowledge is understood as a broadly social practice engaged in with peers and more knowledgeable others (Brown& Paliesar, 1989; Driver, Asoko, Leach, Mortimer & Scott, 1994; Lave & Wenger, 1991). Parallel to this, there have been recent trends within the discipline of mathematics towards viewing the doing of and thinking about mathematics as a social process of debate (Lakatos, 1976) or of shared meanings (Kitcher, 1984). These notions about learning and knowing and about the nature of mathematics suggest that to understand mathematics, one must understand the activities or practice of persons who are makers or users of mathematics, deviating from the more conventional view that understanding mathematics is equivalent to understanding the structure of concepts and the principles in the domain (Stein, Silver & Smith, 1996).

The above viewpoints are reflected in most mathematics education reform documents (e.g., National Council for Teaching Mathematics's *Professional Standards for Teaching Mathematics*; Australian Education Council's *A national statement on mathematics for Australian schools)*. These documents place a great emphasis on changing the nature of classroom discourse to include authentic mathematical activity, collaborative mathematical thinking and "talk in the spirit of disciplinary work" (Lampert et al., 1995; Stein et al., 1996).

In order to provide students with authentic mathematical activity, most mathematics reform documents are suggesting that ***instructional programs*** need to focus far less on computations and much more on the development of number sense and on problem solving, mathematical modelling, and conjecture and proof. Major changes also are being advocated in ***classroom practices*** such as shifts away *from* (1) classrooms being simply a collection of individuals *towards* classrooms as mathematical communities of practice in which students engage in collaborative mathematical practice - sometimes working with each other in overt ways, and always working with peers and teachers as part of a shared community with shared norms for the practice of mathematical thinking and reasoning; (2) classrooms in which the teacher (or the computer or the textbook) is the sole authority for verification of answers *towards* classrooms where logic and mathematical evidence are used as the basis for verification; (3) merely memorizing algorithms and other procedures *towards* mathematical reasoning; (4) emphasis on mechanistic answer-finding *towards* conjecturing, inventing, problem-finding and problem-solving; and (5) treating mathematics as a corpus of isolated concepts and procedures *towards* connecting mathematics, its ideas and its applications in culture and society.

A great deal of cognitive instructional research seems to indicate that many existing cultures of schooling are quite antithetical to these changes (cf.Scardamalia & Bereiter, 1995). Thus, many educational commentators (e.g., Brown, Ash, Rutherford, Nakagawa, Gordon, & Campione, 1993; Marshall, 1988; and Scardamalia & Bereiter, 1995) are suggesting that in order for changes such as those being advocated by the reform documents to occur, new cultures of schooling need to be developed. One such new culture of schooling conceives of schools as *knowledge-building communities.* A knowledge-building community (e.g., a research team in the scientific community) is a group of individuals dedicated to sharing and advancing the knowledge of the collective. According to Bereiter (1994b), what defines a knowledge-building community is not formal association or physical proximity but rather a commitment amongst its members to invest their resources in the collective pursuit of understanding. Thus, in knowledge-building schools, the students are engaged in producing knowledge objects (e.g., ideas, theories, interpretations etc.) that can be discussed, tested, compared, hypothetically modified and so forth and the students see their main job as producing and improving such objects, not simply the completion of school tasks.

Some classrooms in Canada, the USA and Australia are being restructured into knowledge-building communities. Information technology has been recognized as an important tool for mediating the discourse in these classrooms (Brown et al., 1993; Brown & Campione, 1993; Scardamalia & Bereiter, 1995). However, because most existing information technology-based educational materials (e.g., CAI, ITS, educational games, simulations etc.) tend not to foster either collaborative learning or knowledge-building (Bereiter, 1994a; Nason, Ohlsson & VanLehn, 1995; Ridgway, 1988), new types of educational software such as CSILE have had to be designed for these knowledge-building classrooms.

CSILE (Computer Supported Intentional Learning Environment)

A standard CSILE installation has 6-8 networked computers per classroom, connected to a file server, which maintains the communal database and governs accessibility. The database consists of text and graphical notes, all produced by the students and accessible through database search procedures run locally or over the Internet. CSILE activities do not stand apart from the regular school program the way most computer activities usually do. They are integral to the whole classroom process. When off-line, the students are planning knowledge-building projects, seeking information from a variety of sources, and engaging in small group and whole class discussions of questions, ideas and findings. When on-line, the students are entering and following the plans, entering new information through text and graphic notes, and carrying on more pointed discourse on questions, ideas and findings. Anyone can add a comment to a note or attach a graphic note subordinate to another graphic note; indeed, constructive commenting on other students' notes is proactively encouraged and has been shown to be effective (Woodruff & Brett, 1993). Authors are notified when a comment has been made on one of their notes. Only authors can edit or delete notes.

The CSILE database, by capturing the learning processes of the students, can provide the above information for the teachers too, for in these classrooms, each computer workstation is connected to a multimedia database that contains the ongoing work of the class. In most classroom-based systems of this sort, each student's files are segregated into a separate folder or "account". CSILE takes the opposite tack and places all files (which in CSILE are called "notes") in a common area, viewable by all. Furthermore, facilities are provided which allow students to connect their notes to the notes of others. This permits people to more easily share information, answer each other's questions, provide advice, and so on. The strength of CSILE is that it objectifies the knowledge of the classroom and makes advancement of that knowledge a social activity. All questions, theories, ideas, information, and discoveries are preserved on the database for the analysis of the entire class. Unlike face-to-face conversation, which is transitory, CSILE provides a permanent record of the community's interactions. Furthermore, it eliminates the need for turn-taking, allowing all students to work simultaneously regardless of their location. CSILE databases can be accessed across the Internet, allowing anyone on the net to participate once they are given accounts and password information. These features permit a type of highly intensive peer collaboration across time and distance that would be impractical and chaotic without computers.

At present, CSILE is being used in a variety of curriculum areas. Anecdotal evidence from teachers using CSILE and from formal evaluation studies indicates that CSILE is excellent for nurturing collaborative learning and communities of practice in social studies, art, history, geography, language arts and science (Brett & Woodruff, 1995; Scardamalia & Bereiter, 1995). Thus, in these subjects, CSILE has been successful in facilitating the process of knowledge-building i.e., the construction of knowledge objects. For example, after completing CSILE-mediated natural science investigations, a group of Grade 7 Canadian elementary school children from a challenging urban setting were able to theorize and generate productive and challenging questions which were rated as being of high significance by practicing scientists (Woodruff & Meyer, 1995).

However, mediating such levels of collaboration, communities of practice and knowledge-building with CSILE in the domain of mathematics is proving to be much more difficult (Brett & Woodruff, 1995; Scardamalia & Bereiter, 1995). When they study mathematics, students usually seem to focus either on the personal (e.g., completing the task) or the utilitarian (e.g., using mathematics as a tool for resolving a real or imagined real-world issue) aspects of the discipline and overlook that mathematics also can produce abstract and/or creative knowledge objects (Baturo & Cooper, 1993). Thus, even when involved in CSILE-mediated mathematical activity, students tend not to collaboratively build mathematical knowledge objects (such as mathematical ideas, theories, interpretations etc) which can be further discussed, compared, tested, modified, rejected and so on. This problem is the focus of the study being reported in this paper.

The Study

In this study, we explored how CSILE could be used to mediate the public construction of mathematical knowledge objects during mathematical investigations. The participants in this study which was conducted in August-December 1995 were three groups of elementary school students at a inner-Toronto school[1] (Grades 3, 5 and 6), their two classroom teachers and researcher Nason. Nason introduced each of the investigation problems to the three groups of students and followed up each investigation by further visits to the classrooms and by CSILE-mediated discourse with the students and their teachers via the CSILE computer network.

The decision to ground these studies in a series of mathematical investigations was based on ontological, epistemological and motivational factors. In mathematical investigations, students are presented with open-ended problems or situations. They are not expected to produce "the right answer" but instead are required to explore possibilities, make conjectures and convince themselves and others of what they have found. The emphasis is on exploring the pieces of mathematics in all directions (Pirie, 1987). Mathematical investigations thus enable students to engage in authentic problem solving, mathematical modelling-type activities and in mathematics-in-the-making rather than being mere absorbers/consumers of mathematics (Onion, Burns, Thorpe, & Williams, 1990; Pirie, 1987). Furthermore, mathematical investigations can: (1) promote enjoyment of mathematics, (2) make math experience accessible to students by demystifying its subject-matter, (3) restore and develop students' faith in their own common sense, (4) provide situations where student-student and student-teacher discourse arise naturally, and (5) increase the students' willingness to 'have a go' since the threat of being wrong no longer hangs bleakly over them (Pirie, 1987). Mathematical investigations thus provide a most conducive context for the process of mathematical knowledge-building.

The major criteria for the selection of problems/issues that were investigated in this study were: (1) their appropriateness for the ages and interests of students, (2) their potential for mathematical knowledge-building, (3) whether the problem/issues could be adequately modelled by the graphic/iconic tools in the software; and (4) their potential for generating data which would inform the further development of the CSILE-mediated mathematics learning environment. One of the problems investigated was "Fiona the Frog" (Adapted from Onion, A., Burns, S., Thorpe, J. & William, D., 1990). In this problem, the children were presented with a picture of Fiona stuck down the bottom of a 10 metre deep well (See Appendix 1). They were told that each hour, she climbs up 1 metre and then falls back 0.5 metre. Their task was to investigate: *How long is it before Fiona is out of the well?* When they felt they had solved this problem, the investigation was extended to the following two questions:

(1) What about different depths of wells?
(2) Can you generate a rule for all depth of wells?

A simpler version of the problem in which Fiona climbs up 2 metres and then falls back 1 metre was available for the Year 3 children and for the upper grade children who reached an impasse with the original problem. Simpler versions were created for all of the problems/issues being investigated.

We conceived of the mathematical investigation process as comprising two components: *inquiry* and *justification*. For the most part, the *inquiry component* of each investigation was conducted via face-to-face discourse in whole-class and/or small group contexts (cf., Lampert et al.). The *justification component* was addressed when the students created discussion notes and subdiscussions on the CSILE database. The CSILE-mediated discourse focussed on: (1) the students' understanding of the problem being investigated; (2) what they did; (3) their solution(s) to the problem (and why they thought their solutions were good); (4) how might they change the problem; and (5) what had they learned from doing the investigation. They were also encouraged to make comments and ask questions about other students' notes and negotiate with other students about how the investigation should proceed. We hypothesized that these activities would help reinforce the community of practice that had been nurtured during the inquiry component. We also hypothesized that the asynchronous nature of these activities would facilitate reflection and further knowledge-building by all students but especially amongst those who need time for reflection and/or feel intimidated presenting "first-draft versions" of their ideas in synchronous, face-to-face situations.

During each of the investigations, data was collected and analysed from the following sources: the*CSILE Database notes, interviews of both the students and their teachers, ethnographic observation of the classrooms* and *samples of the students' work*. The CSILE data was qualitatively analysed in order to assess changes in the students': (a) patterns of collaboration, (b) quality of their mathematical knowledge-building, and (c) understanding of the nature and discourse of mathematics. The interviews of the students focussed on growth in the levels of subject-matter knowledge, and changes in practice, reflection and collaboration during the mathematical investigations. The interviews of the teachers conducted focussed on their perceptions about: (a) students' experiences during the investigations, and (b) changes in students' practice, reflection and collaboration during the mathematical investigations. The ethnographic observation of the classrooms focussed on: (a) how the teachers and the students utilized CSILE during the investigations, and (b) changes in students' practice, reflection and collaboration. Samples of the students' work were collected and analysed in order to ascertain: (a) growths in their levels of subject-matter knowledge, and (b) the quality and nature of the products of their mathematical investigations.

Findings

During the *inquiry component* of most of the mathematical investigations, we observed high levels of motivation and knowledge-building by all three groups of students. For example, in the "Fiona the Frog" investigation, all groups of children were able to create and utilize pictorial models, conduct animated discourse based around their pictorial models, and generate not only the solution to the original problem but also general rules for similar problems (e.g., wells with different depths and/or Fiona having different rates of climb and fall). The Year 3 children produced verbal rules; many of the Year 5 and 6 students were able to generate tabular and/or algebraic rules.

However, we found that the students were unable to maintain and/or transfer their sense of community and their high levels of mathematizing and knowledge-building into the electronic medium during the *justification component* of the investigations. On reflection, we felt that the limited success of the CSILE-mediated discourse could be attributed to three factors. First, when the students entered the CSILE environment, they had to leave behind any concrete/pictorial representations of the problem they might have been using during their face-to-face investigative inquiry, or at best, crudely draw those with CSILE's current graphics tools. Second, they had to turn the synthesis of a primarily oral/iconic process into a written text based form, with all the awareness of communication that writing takes. Because of these two factors, the students tended to have great difficulty in describing what they had done, justifying their solutions, explaining how the problem could be changed and in articulating what they had learned etc. Third, when they entered the CSILE environment, they had to leave behind the whole class/small group community which had been the impetus for much of the enthusiasm generated in the face-to-face work. It seemed that maintaining that sense of community required a powerful sense of purpose and direction, as well as skill in one's own metacognitive evaluation.

Discussion

From our findings, we concluded that if a knowledge-building software tool such as CSILE was to mediate knowledge-building successfully during mathematical investigations, it needed an interface which: (1) preserves the concreteness and the spontaneity of the students' synchronous, face-to-face investigative work and (2) helps to recreate or maintain the sense of community of practice that is engendered during face-to-face investigative inquiry.

We summized that the interface should include the following components: (1) a set of iconic mathematical tools, (2) a set of mathematical investigation knowledge-building scaffolds, and (3) a compact graphics design tablet and software package(e.g., *artPad II-Dabbler2*).

We feel that a set of iconic mathematical tools such as:

(1) Base 10 Dienes diagrams which can be composed and decomposed;
(2) a place value chart generator;
(3) a number line generator;
(4) balloon, dot, real-world and hopper/loop markers;
(5) geometrical shape drawers;
(6) slicers (e.g., for halves, thirds etc);
(7) tree and array model builders;
(8) an integrated spreadsheet and graph-making tool;
(9) a magnifying/contracting zooming tool;
(10) a stamping tool for the storage and reproduction of drawings, models and diagrams created by the students; and
(11) a tool that enables students to retrieve notes (including both "doodlings" and solutions/justifications) from previous mathematical investigations

would help to preserve the concreteness and spontaneity of the students' face-to-face investigative work and do much to maintain the sense of community. This belief is based on our observations of and discussions with the teachers and students involved in the study, insights gained from the research and development of iconic mathematical tools in the Australian Research Council-funded MENO Project (Nason & Martin, 1994), and from Kurt VanLehn's OLAE project (Martin & VanLehn, 1995) in which he found that giving students access to previously worked problems greatly assisted student learning by analogy.

In this study, we found that CSILE's present set of knowledge-building scaffolds did not provide an adequate structure for the consolidation and reconstruction of the students' inquiry actions or thinking. We feel that the interface needs both *inquiry* scaffolds(such as OUR PICTURE OF THE PROBLEM, WHAT WE HAVE TRIED SO FAR, and WHAT WE NEED TO KNOW) and *Justification* scaffolds (such as OUR PICTURE OF THE PROBLEM, WHAT WE DID, OUR SOLUTION(S) AND WHY WE THINK THEY ARE GOOD, HOW MIGHT WE CHANGE THE PROBLEM, and NEW LEARNING). The ideas for these scaffolds have been derived derived primarily from CSILE's present set of scaffolds, from research and curriculum literature about mathematical investigations (e.g., Lampert et al., Onions et al., Pirie) and from discussions conducted with the teachers in this study. We believe that the Inquiry scaffolds would facilitate smooth transitions between inquiry and justification during an investigation and together with the iconic tools help to maintain the direction and impetus of a mathematical investigation. We believe that the Justification scaffolds would nurture not only the consolidation and the refinement of knowledge objects built during the inquiry component of an investigation but also the building of new mathematical knowledge objects by both the teachers and the students whilst they are engaged in the process of justification.

In this study, we also observed that much of the animated and productive discourse that occurred during face-to-face investigative inquiry seemed to revolve around doodlings and drawings done by the students on sheets of paper or on the chalkboard. However, the students experienced great difficulty in transferring these doodlings and drawings onto the CSILE windows with the mouse and the keyboard and that what appeared in the windows seemed to lack the richness, insight, creativity and inclusiveness of their hand-made doodlings and drawings. Their CSILE notes thus tended not to encourage significant discourse. Compact graphic tablets and software packages (such as *artPadII-Dabbler2*) enable students to draw, doodle, paint or sketch as easily and as naturally as they do with pen, pencil, crayons or chalk. The inclusion of a graphic tablet and software package within the interface thus would make the process of transferring doodlings and drawings onto the computer screen much easier (or it would encourage groups of students to do their original doodlings and drawings on the computer screen). We therefore believe that the inclusion of a graphics tablet and software package into the interface would do much to maintain the spontaneity and concreteness of the students' synchronous face-to-face investigative work. Furthermore, in conjunction with the iconic mathematical tools, we believe that it would make it easier for students to conceptualize, modify and present their picture of the problem and comment on other students' pictures and/or conceptualizations of the problem. This should contribute much towards maintaining the sense of a knowledge-building community.

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Appendix 1

Mathematics Investigation 4A: Fiona Frog (from Onions et al., 1990)

Fiona is at the bottom of a well 10 metres deep.

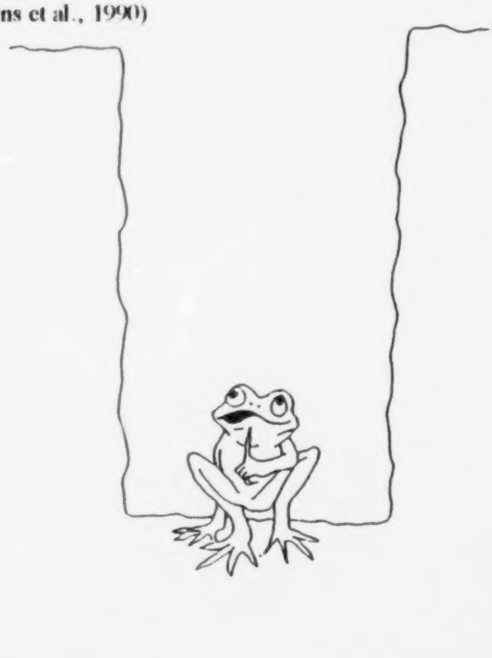
Each hour, she climbs up 1 metre then fall back 0.5 metre.

Your task is to investigate:

How long is it before Fiona is out of the well?

Extensions

1. What about different depths of wells?
2. Can you generate a rule for all depth of wells?



Ethics, Morality, and Mathematics Education

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This paper explores the possible effects of gendered ethical systems or moralities on participation in mathematics education. In particular, it suggests that the prevailing morality in the classroom significantly affects the attraction, retention, and success of female students.

Introduction

This paper discusses the influence of personal ethical systems or moralities on participation in mathematics education. It is a summary of the argument underlying a current doctoral study. The intention of this study is to explore one possible aspect of the resistance of women to post-compulsory mathematics education. The consideration of morality is somewhat unusual within current mathematics education research, but, this paper will argue, it is relevant to the issue of increasing the participation of girls and women in mathematics education.

The terms "morality" and "ethics" are sometimes used differently. "Morality" brings with it a particular, usually religious resonance. It suggests a stern set of duties that require us to subordinate behaviour, particularly sexual behaviour, to a universal "moral law". "Morality" thus has come to reflect a certain set of JudaeoChristian values, which may not be found in other ethical systems. However, while the term "morality" has acquired religious/sexual overtones in ordinary language, it is generally used within western philosophical discussion without these connotations, so that it is synonymous with personal "ethics". Further, it is the term commonly used in the feminist literature on morality. It is for these reasons that the term "morality" rather than "ethics" is generally used in this paper.

Webster's Dictionary (1981) defines morality as "principles or considerations of right and wrong action or good and bad character". This definition is based on concepts of duty, obligation, and principle. Such a construction of morality has been the basis of ethical argument since Classical Greece. Morality has been thought to have one origin, a *human* one. However, the work of Carol Gilligan has suggested that there may not be just one origin. She has identified two distinct moral perspectives, gender-related but not gender-specific. These she named Justice and Care morality- they are the moral perspectives associated with men and women respectively. The first section of this paper describes these two different moral perspectives more fully. The second part of this paper discusses the influence of personal morality on participation in mathematics education, with particular attention to the attraction and retention of female students. The second part also provides evidence to suggest that a classroom environment that embodies a Care (rather than a Justice) morality will produce more successful female mathematics students.

Part One: Morality

Gendered Morality

In 1977, Carol Gilligan identified differences in the ways in which men and women approach moral dilemmas. She reported what she called a "different voice" amongst women talking about morality. In 1982, her challenge to conventional moral theory was published. Gilligan's work provoked considerable controversy as, until this time, gender had not been considered relevant to moral theory. This is shown by the fact that all the major moral theorists (such as Piaget, 1965; Erikson, 1968; Kohlberg, 1969) had worked with all-male samples. They had unhesitatingly presented their findings as universal. Kohlberg (1981), for instance, conducted a heavily funded study of human moral development, in which he followed the development of eighty-four males for twenty years. He did not once interview a female. This possible flaw in his methodology attracted no comment until the work of Gilligan (1977).

Kohlberg (1971) described the development of moral judgement from childhood to adulthood. On his six-stage scale of moral development, men reach the pinnacle of moral maturity, while most women tend to cluster around the halfway mark. Women's judgements generally exemplify the third stage, where morality is conceived of in personal terms and goodness is concerned with helping and pleasing others. This conception of goodness is considered to be adequate for women in so far as their lives are centred in the home. Kohlberg implied that only if women entered the traditional arena of male activity would they recognise the inadequacy of this moral perspective and progress like men to the higher stages where relationships are subordinated to rules (stage four), and rules to universal principles of justice (stages five and six). Thus, Kohlberg's theory positioned women as deficient in comparison to men.

Rather than seeing women in need of change so that they would more closely fit the theory, Gilligan (1982) considered that the theory might be in need of change so as to more closely reflect reality. Gilligan described morality as a tension between the moral perspectives of Care and Justice. Women, she said, were more likely to construct a moral problem as a failure of response, rather than in terms of whether or not "rights" were adequately addressed. Since women were more likely to conceptualise the self as interdependent with others, rather than as separate and autonomous, Care morality centred more on the maintenance of relationships, rather than on individual rights, as in Justice morality.

Justice morality is that theorised by Kohlberg (1969,1971) as "universal" morality. It is the more familiar of the two because it is the basis of western legal systems. The moral priority of Justice is fairness or equality. A morally mature person is able to make decisions independent of the opinions and values of others. Within Justice morality, violation is of personal or property rights, rules, or standards of behaviour.At first sight, Care morality is less familiar. This may be because it has been relegated to the private rather than the public domain because of its association with women, as Martin (1994) argues. The priority of Care morality is the maintenance of connection to others. A morally mature person should be able to respond to the needs and circumstances of others *in their terms.* Within Care morality violation is of connection or relationship; for instance, abandonment, exclusion, ridicule, sarcasm, silence (as in being ignored). These are acts of violence; they do harm.

It follows that detachment, the mark of mature moral judgement within Justice morality, becomes *the* moral problem within Care morality. Conversely, situational morality (attention to the needs and circumstances of individuals) is the mark of mature moral judgement within Care morality but *the* moral problem within Justice morality (Gilligan & Attanucci, 1988). In short, those with a Care perspective have, relatively speaking, more of a focus on others; those with a Justice perspective have more of a focus on themselves. Each sees the moral priority of the other as a problem.

Justice and Care moralities have been presented here as quite distinct entities for the purposes of clear explanation. It is however important to note that they are not completely separate either conceptually or in practice. For instance, while some laws protect property and personal rights, others (such as those that ensure the provision of health services) are designed to provide care. Individuals, too, usually demonstrate a combination of Justice and Care considerations, which differs by gender. Women usually mention more Care than Justice considerations, while men usually mention more Justice than Care considerations (Gilligan & Attanucci, 1988). However, in the North American populations studied, about 30 per cent of women, but virtually no men, demonstrated a Care-only morality. Thus, if women were not present in the samples, all responses could be analysed and scored in terms of Justice. When women are included in the samples, the existence of a sizeable group whose responses cannot be coded in terms of Justice morality forces confrontation with Justice as a "universal" moral theory (Gilligan & Attanucci, 1988).

Gilligan's work is not without challenge. One major criticism is that her theory is essentialist. Using the definition of essentialism given by Martin (1994), someone who claims to have identified an essence claims to have found a feature, characteristic or trait that differentiates all possessors of it from all non-possessors of it. By this definition, any essentialist theory of gender would be able to differentiate all men from all women. However, Gilligan specifically stated that morality was gender-related, not gender-specific (Gilligan, 1977, 1982). It follows that all men cannot be differentiated from all women on the basis of Gilligan's theory. The space limit on this paper leaves no more room for a discussion of criticisms of Gilligan's work, but Kerber (1986), Gilligan (1994), and Martin (1994) provide good overviews of both valid and refuted criticisms, in particular those focussing on race, ethnicity, class, culture, romanticism and essentialism.

This section has focussed on the links between morality and gender. The focus on the next section is on morality and its links with types of reasoning, particularly in mathematics. Ideas drawn from the work of Polya (1954) and Lakatos (1976) introduce this section.

Part Two: Mathematics Education

Morality and Reasoning

Early concerns about the moral climate of the mathematics classroom were voiced by the mathematician Polya (1954) and the philosopher Lakatos (1976). For example, Lakatos commented that "it has not yet been sufficiently realised that present mathematical and scientific education is a hotbed of authoritarianism and is the worst enemy of independent and critical thought ... in mathematics this authoritarianism follows the deductivist pattern." (Lakatos, 1976, p. 142). In criticism of the deductivist pattern he says that "deductivist style presents ... proofs out of the blue in an artificial and authoritarian way. It hides the global counter examples that led to their discovery" (Lakatos, 1976, p. 144). Lakatos valued independent and critical thinking, which he viewed as incompatible with authoritarianism. If we accept that authoritarianism is a certain sort of morality, Lakatos has claimed a link between morality and reasoning. In particular, he claimed that a certain type of morality (authoritarianism) was opposed to a certain type of thought (independent and critical).

Lakatos (1976) used the technique of "conscious guessing", which he contrasted to "the deductivist style" mentioned above, to develop independent and critical thought. This technique involves conjecture about relationships among quantities and shapes. Proof eventually arises from a continuing zigzag between conjecture and counter-example. However, conjecture and counter-example "do not appear in the fully fledged deductive structure; the zigzag of discovery cannot be discerned in the end-product" (Lakatos, 1976, p 42).

Polya (1954) defined the necessary moral qualities of the mathematician as honesty and courage. For Polya, the willingness to constantly confront belief by experience was essential to the practice of mathematics. He believed it was necessary to continually check observation (experience) against generalisation (belief). This process of reasoning, and the willingness to undertake it, Polya termed the "inductive attitude". He identified honesty and courage as the necessary personal qualities that would enable the adoption of the "inductive attitude". In summary, both Lakatos (1976)and Polya (1954) described certain kinds of reasoning they believed to be desirable in mathematics. Lakatos identified a moral climate that he believed mitigated against this; Polya identified moral qualities that he believed encouraged it.

Lakatos (1976) and Polya (1954) wrote twenty and forty years ago, but their ideas were successfully put into practice by Lampert in 1990. Lampert experimented with teaching a course designed to generate the kind of mathematical thinking valued by Polya and Lakatos. She chose problems that had the capacity to engage all the students in the class in making and testing mathematical hypotheses. The hypotheses were embedded in the answers the students gave to the problem; comparing answers elicited hypotheses and provoked discussion of their relative merits, creating the kind of zigzag between inductive observation and deductive generalisation that finally creates the proof. This type of thinking, she noted, involves risk; risk that one's assumptions are open to revision, that one's insights are limited, that one's conclusions are inappropriate. Letting other people in on one's conjectures increases personal vulnerability (Lampert, 1990). Accordingly, she created a moral climate in the classroom designed to maximise honesty and courage in her students.

The kind of mathematical thinking valued by Lakatos (1976) and Polya (1954) is echoed today in educational policy both in Australia and New Zealand. A *National Statement On Mathematics For Australian Schools (1990)* states that

> "The systematic and formal way in which mathematics is often presented conveys an image of mathematics which is at odds with the way it actually develops. Mathematical discoveries, conjectures, generalisations, counter-examples, refutations, and proofs are all part of what it means to do mathematics" (p. 14).

The corresponding publication in New Zealand, *Mathematics in the New Zealand Curriculum* (1992), also reflects this position.

This section has presented the arguments of Polya (1954) and Lakatos (1976), which were echoed by Lampert (1990), that what they value as mathematical thinking is encouraged by the presence of certain moral qualities, and discouraged by others.

The next section will focus on the links between types of thinking in mathematics and gender. It will again draw on the work of Gilligan, (1977, 1982), who identified two different styles in reasoning, each associated with one of the two moral perspectives.

Morality, Reasoning, and Gender

In 1985, Buerk investigated links between types of reasoning in mathematics and the two moral perspectives discovered by Gilligan (1977, 1982). One of Buerk's motivations for undertaking this study was that she "realised that these different perspectives on morality were relevant to (her) work with women who avoided mathematics" (1985, p. 63). In her early work on morality, Gilligan (1982) had identified two different types of moral reasoning - "separate"(found most commonly within Justice morality), and "connected" (found most commonly within Care morality). Buerk chose excerpts from Gilligan's book (1982), juxtaposing phrases that characterise "connected" reasoning with those characterising "separate" reasoning. These are shown, in part, in Table 1.

Table 1: Descriptors of "Separate" and "Connected" Reasoning (Buerk, 1985).

"Separate" Reasoning	"Connected" Reasoning
gets right to solution in a structured, algorithmic way, stripping away any context	tries to experience the problem, relate it to the personal world, clarify language, create context, remove ambiguity
uses mode of thinking that is abstract and formal	uses mode of thinking that is contextual and narrative
geared to arriving at an objectively fair or just solution upon which all rational persons can agree	geared to looking at limitations of any particular solution and describing the conflicts that remain
legal elaboration of rules and fair procedures	tolerant in attitude towards rules and more willing to make exceptions

(See *Women's Ways of Knowing* by Belenky, Goldberger, Clinchy and Tarule (1986) for an expansion of "separate" and "connected" reasoning.)

Buerk gave a group of mathematicians at Ithaca College, New York, these combined sets of phrases. The mathematicians identified the phrases characteristic of "separate" reasoning as the way that mathematics is communicated in the classroom and in textbooks. They identified "connected" reasoning as the way in which they do mathematics. They "stressed the creative side; attention to the limitations and exceptions to theories, the connections between ideas, and the search for differences among theories and patterns that appear similar" (Buerk, 1985 p. 64). In this, they concur with Polya and Lakatos. The "connected" reasoning list is similar to both Polya's "inductive attitude" and Lakatos's "conscious guessing", in which the mathematician constantly checks conjecture with experience, zigzagging towards proof.

Becker (1995) also applied the ideas of Gilligan, and their extension in the work of Belenky et al (1986), in teaching women mathematics. She argues that this zigzag between hypothesis and observation is an essential feature of "connected" knowing in mathematics. It is important for women students to see "how we start a problem, make an error, and begin our solution over again" (Becker, 1995, p. 168). The *National Statement (1990)* states that this is important for all students - "School mathematics should show the intuitive and creative nature of the process, and also the false starts and blind alleys, the erroneous conceptions and errors of reasoning which tend to be a part of mathematics" (p. 14).

Many women prefer "connected knowing" as their primary mode in approaching certain problems (Buerk, 1985, Becker, 1995), and this approach to mathematics is encouraged by the *National Statement (1990)*. However, as Lampert (1990) pointed out, "connected knowing" involves risk. Therefore it is important to provide, at the best, a moral climate that is recognised by the students as supportive, and at the very least, a moral climate that is not harmful. The next section will discuss the beneficial and harmful aspects of both moral perspectives as they might be experienced in the mathematics classroom.

Morality in the Mathematics Classroom

The differing experiences of boys and girls in the mathematics classroom have been well documented (Leder, 1992). In this section, some evidence will be presented and reinterpreted in the light of Gilligan's (1982) theory of gendered moral differences. Gilligan argues that, generally speaking, a predominantly Justice Morality is found amongst males and a predominantly Care morality amongst females. Within the Justice perspective harm is done by infringement of rights, or violation of rules or standards. Within the Care perspective, harm arises from non-response and violation is of connection or relationship; for example, isolation, ridicule, sarcasm, silence (as in being ignored, or excluded from the group). In the mathematics classroom, evidence for the presence of Care morality in females might be found in the observation that women feel anxiety about being left behind, cut off from others, as when an experienced and competent tertiary educator talks about being 'lost' in a mathematics class: 'I feel panic when a classmate who earlier didn't understand, now does" (Leder, 1995, p. 15). For women, being ignored or disconnected from others means some degree of harm; so in a typical mathematics class where females get less attention, less criticism, less frequent praise for correct answers, have work monitored less frequently, and have less contact with teachers (rephrased with the emphasis on females, Leder, 1995), they may well experience the mathematics environment negatively. Both the Justice and Care perspectives have a positive (beneficial) and negative (harmful) side that can be seen in the classroom. The positive side of Justice morality can be seen in the actions of teachers who set into place strategies to redress perceived inequities (for an example of such a programme, see Morrow and Morrow, 1995). The negative side of Justice morality is seen when Justice slides into privilege; the claiming of special treatment. When attempting to exercise privilege, students use the principles and techniques of Justice morality to serve their own interests. This may be seen in the mathematics classroom in such behaviours as students who produce a correct answer may take this as an indication that they no longer have to listen to others (Lampert, 1990); students who are bored may provoke a vote to force a move onto a new topic (Lampert, 1990). The positive side of Care morality is seen for instance in cooperative working, peer tutoring, and mathematically productive discussion, as evidenced in a university-level mathematics course taught by Rogers (1995), in which over two-thirds of the students actively participated in the group activities, and in which retention and achievement was high. The negative side of Care morality in mathematics education is seen when Care slips into patronage. Walkerdine (1989), in a study of fourth-year British maths students, found that twenty boys and eight girls were entered for 0-levels despite girls outperforming boys in the school's prepatory exam. Teachers' arguments for entering girls for the less difficult Certificate of Secondary Education was that they felt they should "protect" the girls, not "push" them. In 'taking care' of the girls, the teachers were ignoring their rights to equal access to status-conferring, not to mention job-conferring qualifications.

The Effect of Classroom Morality on Attraction, Retention, and Success of Students

It seems that courses that embody a Care morality might produce more successful women mathematicians than traditional courses do. Rogers (1990) reported on the mathematics department at The State University of New York College at Potsdam. In 1983, 60.4 per cent of all Potsdam's degrees were awarded to women compared to 43.8 per cent nationally, and for five of the last seven years the highest achieving students in the entire college were female mathematics majors. Female enrolment and achievement in this course are far higher than the national average. However, this is not due to factors such as strong female role models; fourteen of the fifteen staff are male. The significant factor may be that the staff have a Care morality, as portrayed by one teacher who remarked to Rogers as they returned from a student function together "You've got the idea haven't you, I hope, that our students are more to us than just mathematical minds? ... We're really interested in people as people" (Rogers, 1990, p. 41). According to Leder (1995), Potsdam's success in attracting females to mathematics courses well beyond the compulsory level was achieved by creating a class environment that was supportive. It may be that the women at Potsdam feel supported in tolerating the states of "risk and vulnerability" that Lampert (1990) described as being essential to mathematical practice.

Rogers, who reported on the Potsdam experience, also reported on a university-level mathematics course designed to focus on both Care morality and "connected" knowing in mathematics (Rogers, 1995). This course achieved as good results, overall, as two other university courses taught by the traditional lecturing method. However, students in this course obtained more of the highest grades overall and far fewer students failed. This course enjoys high demand, lower levels than usual of attrition and absenteeism, and higher and more consistent achievement.

Alternatively, it may be that a skilled balance of Care *and* Justice moralities provides a productive environment for women. One of the most popular teachers at Potsdam seemed to demonstrate this. Comments of his that indicate Care has not slipped into patronage are, for example, "I do not wish to be the students' 'pal' when I teach ... they (students) cannot forever remain children - some things will not be forgiven (after a certain time)" (Rogers, 1990, p. 38). Justice did not slide into privilege, either - he says "I am quite accessible" (Rogers, 1990, p. 38), and his students describe him as "kind of nurturing ... friendly about his intimidation ... like, "yes, you've made a mistake, but you haven't done anything that any of my other students haven't done" (Rogers, 1990, p. 41). This teacher's classes are always heavily subscribed.

Another possible explanation is that the morality of the teacher may not, in fact, be especially important. Perhaps the group work and discussion in the courses reported by Rogers (1990, 1995) create an opportunity for students to provide a Care environment for each other, regardless of the teacher's intentions. Group work, class interaction, and discussion, with their opportunities for connection and relationship, have the potential to provide a moral climate that appeals to girls and women far more than the traditional silent, individualistic mathematics classroom.

Conclusion

This paper attempts to provide one answer to the question of why women do not continue with mathematics at post-compulsory levels. It puts forward the argument that the answer, at least in part, depends on the "moral climate" of the classroom. It discusses the theory of Gilligan (1977, 1982) that morality is gendered in western societies, and argues that this has significant effects in mathematics education. In particular, it argues that a classroom that encourages discussion, group work and cooperation is experienced by most women as supportive: whereas, in contrast, the moral climate of the traditional, silent, rule-based classroom is experienced by most women as harmful. This accounts, at least in part, for the avoidance of traditional mathematics education by women. This paper also offers evidence that there are higher attraction, retention, and achievement rates for women in courses which allow and encourage behaviours characteristic of the gender-related morality of Care, and thus suggests that a consideration of gendered moral perspectives may be of significance in mathematics education.

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Integrating Science and Mathematics Concepts: A Student Teacher Perspective

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There is an emphasis in primary preservice teacher education that Key Learning Areas should be integrated. In this study primary preservice teachers' abilities to formulate mathematical links to teaching activities related to science and technology topics was investigated. Two factors seemed to affect the quality and number of links suggested by preservice teachers; their age and the amount of science, but not mathematics, they had studied in the senior secondary school. Overall, student teachers had great difficulty in perceiving and describing mathematical concepts that could be integrated with science activities.

Science and mathematics are naturally related in the real world so there have been many recommendations that the teaching of these subjects should be integrated (Lehman, 1994; McBride & Silverman, 1991; National Council of Teachers of Mathematics, 1989). At the primary school level it has been suggested that such integration would increase achievement in both disciplines and produce more positive attitudes toward both subjects. Moreover, relationships between concepts would be emphasised and, as a result, more meaningful learning would be promoted (Lehman, 1994). Since science provides students with concrete examples of abstract mathematical ideas, integrating the two subjects should improve students' learning of mathematical concepts by providing appropriate, motivating contexts. In addition, students may achieve deeper understandings of science concepts if they are encouraged to use mathematics to quantify and explain science relationships (McBride & Silverman, 1991).

Scientific knowledge is advanced through observation and manipulation of observable phenomena whereas mathematicatics often involves patterns and relationships that are not so bound (Berlin and White, 1995). Processes fundamental to both disciplines are induction, drawing general conclusions from specific instances, and deduction, conclusions or inferences obtained by arguing from the general to the particular. These authors suggest that all students should be provided with opportunities to acquire understandings of these two processes through "the increased use of mathematical modelling in science classes and the use of student-generated scientific data in mathematics classes" (p. 25).

A number of mathematics and science educators consider the constructivist approach to learning as being beneficial (Malone & Taylor,1993; Harlen 1987; Osborne & Freyberg, 1986). This approach emphasises that knowledge is built on prior understandings and experiences and is organised around concepts involving processes and situations constructed over time. Processes basic to science include classifying, measuring, inferring, analysing and interpreting data, communicating, and formulating and interpreting models. As Berlin and White (1995) point out, these processes are also emphasised in mathematical problem solving. They suggest that integration allows students to engage in challenging, real life problem-solving tasks that enable higher order thinking skills to be utilised. Co-operative learning strategies may also reinforce the collaborative nature of the two disciplines with the teacher in the role of facilitator.

Lehman (1994) surveyed practicing elementary and student teachers and found that although they perceived integration of mathematics and science to be viable and important, they did not feel it was common classroom practice. Perceived problems were a lack of background knowledge in both disciplines, the additional time required to integrate teaching, classroom management, and a lack of commercially developed resources that teachers could utilise.

To achieve integration of mathematics and science, McBride & Silverman (1991) advocated that teacher educators should implement ideas of integration in preservice training. Stuessy (1993) describes the development of an integrated course for elementary preservice teachers which she claimed resulted in "significant positive gains in students' beliefs about themselves as successful teachers of mathematics and science" as well as higher levels of analysis and synthesis skills on tasks specifically related to integrated teaching (p. 61). There seems to be little research, however, in which teachers' knowledge of links between science and mathematics concepts has been investigated.

In N.S.W. neither the primary science nor the primary mathematics curriculum documents suggest ways in which teachers might integrate specific concepts in these disciplines and teacher education programs differ in their approaches to developing student teachers' skills in integrating Key Learning Area content. The aims of this study were to investigate a method of measuring primary preservice teachers' recognition of links between mathematics and science; their identification of mathematical topics related to activities outlined in units of work from the N.S.W. K-6 Science and Technology syllabus; and factors that seemed to affect their identification of such topics. This information could be used to assist teacher educators to design approaches to integrated lesson planning for science and mathematics at the primary school level.

Method

The sample

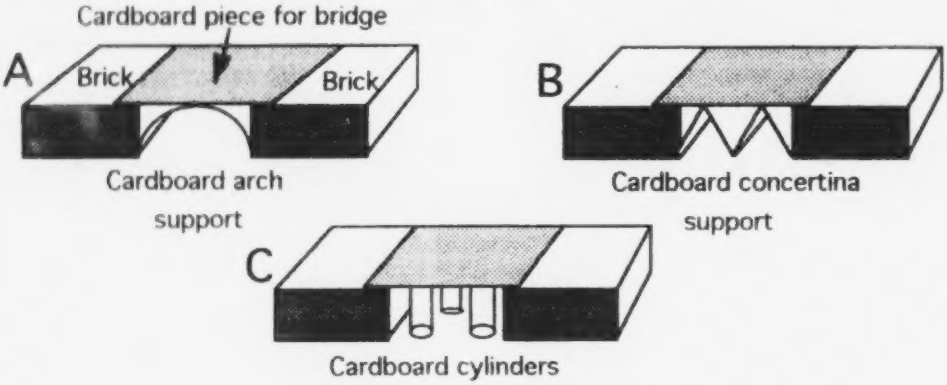
The sample consisted of 65 student teachers from teacher education programs at two N.S.W. tertiary institutions, one a concurrent Diploma in Education (Dip. Ed.) program (N=24), the other a Bachelor of Education (B.Ed.) program (N=41). The students were in the third and second years of their respective courses. The questionnaires were given to the students during workshop sessions at the end of the year.

The questionnaire

The student teachers were asked to complete a questionnaire which comprised three science design-and-make tasks (bridges, weather and semaphores) drawn from the Support Materials based on the N.S.W. K-6 Science and Technology Syllabus. The bridge task is shown in Figure 1. For each task a sequence of six teaching activities was given and students were asked to list mathematical concepts that could be linked to the activities. The students were asked to give sufficient details to show how the mathematics concepts linked to the activities. A completed example of an additional task ("Sinking and Floating") for which mathematical concepts were shown linked to the sequence of teaching activities was provided to illustrate the required responses. An indication of age, level of science and mathematics completed at the Higher School Certificate (HSC) level was also requested.

Task: Test three different supports for a bridge to determine which is the strongest.

The teacher gave each group of children six bricks (all the same size), three small sheets of cardboard, pencils, a ruler, a protractor, grid paper and scissors. They were also given a set of scales, a container and a small bucket of pebbles. The teacher then asked the children to make the bridges shown in the diagram and test which design was the strongest. The activities she planned are shown in the table below.



Activity	List maths concepts that might be necessary for or developed in each activity?
Children cut out the parts they need to make each bridge.	
Children make the three bridges shown in the diagram.	
Children test each bridge to determine which is the strongest	
Children record and then compare their results.	
Children check if the amount of cardboard used to make each bridge was the same.	

Figure 1 The bridge task from Science and technology unit Stage 2: *Indoor, Outdoor*

Scoring the questionnaires

For each task and activity the mathematical linking concepts listed by students were scored using the following criteria. "Good" linking concepts were considered to be those which related to the activity and were sufficiently detailed to give an idea of the mathematics involved (e.g. "measurement of length" rather than "measurement"). A score of four was given for one concept, five for two concepts and six for more than two concepts of this type. Connections which appeared reasonable but lacked specificity were allocated a score of either two or three, for one or more responses respectively. Such connections usually consisted of single word answers (e.g. "graphs"). Responses for which connections to the activities could only tenuously be made were scored as one. Responses which listed general processes (e.g. "investigating" or "problem solving") or no response were given a zero score and an incorrect mathematical response received a score of negative one. The maximum numerical score for each task was 30.

The background information for each student was also coded. Age was given a score from 1-6 for the following age ranges: less than 20, 20-24, 25-29, 30-34, 35-39, 40 years or over. Mathematics level (or equivalent) was scored 0-4: 0 being no mathematics studied, to 4 being 4 Unit Mathematics. Science courses were scored as follows: zero, no Science studied; one, Science for Life, two for each 2 Unit Science subject studied (Geology, Biology, Chemistry, Physics, General Science); three, 3 Unit Science; four, 4 Unit Science.

The relationship between the levels of mathematics and science studied and the total score for the tasks was investigated. An analysis of the most common mathematics concepts given for each task was also undertaken.

Results

Total scores for the three tasks ranged from 0 to 72 (mean 30, standard deviation 18) and were skewed towards lower scores. The distributions of scores for each of the tasks are shown in Table 1.

Table 1 Distributions of total scores for the three tasks (percentages)

Score range	Bridge	Weather	Semaphore
0-5	20	23	43
6-10	32	23	28
11-15	25	31	20
16-19	9	11	7
20-24	12	9	2
25-29	2	3	0
Mean (SD)	11.2 (6.3)	11.4 (7.3)	7.4 (5.9)

Since 85% of the sample were less than 25 years of age, there was not a sufficient spread of ages to be confident that there was any relationship between age and total score. However, none of the ten older students (25 years of age or older) scored less than 24 and 40% of them scored above 50 whereas 42% and 13% of the younger students fell into these categories respectively.

Table 2 A regression analysis of age, school level mathematics and science as predictors of total scores for mathematical linking concepts for the three tasks

Source	Degrees of freedom	MS	F	Probability
Regression	3	2854	15.4	0.000
Error	61	186		

Predictor	Coefficient	Standard deviation	t- ratio	Probability
Age group	5.0	1.8	2.8	.006
School maths	-.9	2.3	-.4	.709
School science	8.3	1.5	5.7	.000

A regression analysis was carried out to investigate the effect of background variables on the total score. The results of this analysis are presented in Table 2 and suggest that the most significant predictors of total score are previous school science experience ($P<0.001$) and age ($P<0.01$). These two variables accounted for 43% of the variance in the total score obtained for mathematical linking concepts. The results for the three oldest students were unusual; one student with no background in science scored far higher than would be expected while the other two scored lower than expected. None of these students had studied mathematics in senior secondary school.

An examination of the most frequent responses to each task suggested that many students were dependent on links suggested by the wording of the teaching activities. The most common mathematical concepts listed for the weather task were "measurement", "scale", "map", "speed" and "direction", all terms used to describe the teaching activities. For each task, measurement concepts were more frequently listed than number or space concepts. However, students often did not indicate what would be measured, and only one student mentioned the concept of units. For the bridge task while about half the sample mentioned area, only about half of these students also mentioned length, even though a ruler was listed among the materials provided to children.

Discussion

The questionnaire: The results of this study suggest that the questionnaire devised to assess primary preservice teachers' knowledge of links between mathematics and science was feasible for use in a group situation. Its main disadvantage was that the open-ended responses were difficult to categorise because insufficient details were given. Use of the questionnaire in an individual interview situation would overcome this problem but would increase the time required.

Identification of mathematical topics: The results are not reassuring in terms of student teachers' recognition of contexts for mathematical topics which should have been covered at primary and secondary school. Some students gave an impression of grasping at straws with suggestions such as "calculus", "trigonometry", "force". "Number" was frequently listed, presumably because measurement was suggested in all the teaching activities and measuring involves "numbers". However, the operations of addition, subtraction, multiplication, and division were mentioned only infrequently. Informal discussions with some of the students revealed that in their practice teaching experiences they were more likely to integrate science lessons with English or craft activities, rather than with mathematics.

One possible reason for not integrating mathematics with other curriculum areas may be that it is perceived to be a hierarchical and sequential subject that is difficult to program when planning thematic work. Another reason may be student teachers' lack of mathematical knowledge; the results of this study suggest that student teachers may not recognise applications of school mathematics when these are presented in unfamiliar contexts. Many student teachers may need guidance, not only in selecting worthwhile mathematical tasks, but in recognising opportunities to point out and consolidate mathematical concepts in other curriculum areas (Lappan and Theule-Lubienski, 1994).

Importance of prior knowledge: The factors that seemed to affect student teachers' identification of mathematical linking concepts appeared to be level of senior secondary science and age. The level of mathematics studied in the final years of secondary school did not seem to relate to the number or quality of the mathematical links made. Since this study was primarily exploratory, this finding needs replication, as well as explanation. It may be that science teachers build on mathematics knowledge, especially measurement, as part of their teaching, whereas the converse is less likely. However, in this study the science subjects taken at school were not considered separately; many students had taken biology in which mathematical connections are not commonly made. The effects of different science subjects also requires investigation.

There were several important limitations to this study. The first was the sample; students were at different stages in their courses and the two programs (BA Dip. Ed. and B.Ed.) have quite different approaches. The students in the BA Dip.Ed., although in their third year, were only in the first year of professional development units. A sample that included students from a wider range of tertiary institutions, and perhaps focussed on final year students, might give a clearer picture of factors that may influence effective integration.

A second limitation was that task order was the same for all students. Student teachers' responses to the last task (semaphores) were found to be least adequate. Students may have simply lost interest in this task or they may have been unfamiliar with semaphores as a system of communication. In any future study this limitation could be overcome by including only two tasks, counter balanced for order.

If teachers are expected to integrate science and mathematics instruction, integration must be included in preservice training (Lehman, 1994). The results of the present study show that preservice primary teachers did not find it easy to identify and articulate mathematical ideas in science contexts. Studies are needed to determine effective methods of assisting primary classroom teachers to integrate these two Key Learning Areas for concepts where there is a natural connection. Syllabus documents for both science and mathematics should also highlight specific links for key concepts.

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Problems with Probability

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This research examines misconceptions in probability held by a sample of pre-service primary teacher education students. Questions were selected and modified from those reported in the research literature in order to examine; the misuse of heuristics, a false assumption of equal likelihood, misunderstanding of independence, awareness of counter-intuitive probabilities, and belief in other fallacies. Questions were accompanied with a request to explain the reasoning employed. Explanations were partitioned into disjoint categories according to the type of misconception and cognitive level of response. Results of selected questions and a summary of the response levels are presented here for discussion.

Rationale

Recent curriculum developments in primary school mathematics have seen a much greater emphasis on the role of probability in the classroom. The importance of understanding probabilistic concepts in modern technological societies is now well established but the inclusion of probability into the mathematics curriculum is a relatively recent development. Furthermore the teaching of the topic is "a very difficult task, fraught with ambiguity and illusion" (Garfield & Ahlgren, 1988, p. 57) and studies have shown that the content knowledge of topics in probability and statistics for both primary and secondary teachers is often deficient. (See for example, Peard, 1987; Shaughnessy, 1992).

Thus it is contended that pedagogical problems in teaching probability are compounded by lack of teacher knowledge of content and the presence of misconceptions. In order for the effective use the limited time available for the topic of probability in the content area of pre-service primary teacher education courses, research is required to determine what conceptions and misconceptions our pre-service primary teacher education students bring with them. Watson (1992, p. 560) has suggested that "it may be possible to use the results of research to make more precise and confident statements to help teachers and students understand and apply the concepts in the Chance and Data statement." She expressed concern that recent initiatives in curriculum development "have been taken without the benefit of previous educational research in Australia on the learning of probability" (p. 556).

Aims

This research aims to determine the nature and extent of misconceptions in probability held by second year pre-service primary teacher education students enrolled in the B.Ed course at the Queensland University of Technology, Semester 2, 1995. These misconceptions include:

- the use of the heuristics of representativeness and availability;
- the false assumption of equal likelihood;

- independence of events;
- awareness of counter-intuitive probabilities; and
- belief in other fallacies.

The study examines the types of reasonings associated with use of these misconceptions in order to "search for alternatives ... to explain peoples' flawed probability estimates" (Shaughnessy, 1992, p. 471).

Background to the Study

The difficulties pupils encounter with probabilistic concepts have been confirmed by the results of a number of recent studies. For example, in the Fourth National Assessment Education Program, (NAEP), Brown, Carpenter, Kouba, Lindquist, Silver and Swafford (1988) reported widespread "difficulty with items involving probability, except those involving very simple concepts" (p. 242) and that "only 5% of (Year 11) students correctly answered questions involving compound probabilities" (p. 243). In examining areas of difficulty in probabilistic reasoning, Garfield and Ahlgren (1988) reported that some concepts that are difficult because they are unlike anything the student has thought of before and other concepts cause difficulties because they run counter to intuitive ideas that the students already have. Fischbein, Nello, & Marino, (1991) commented, there appears to be "no natural intuition for evaluating the probability of a compound event" (p. 534). The present study examined some of the students' beliefs about probability prior to instruction in their second year mathematics subject.

Some Misconceptions in Stochastics

Tversky and Khaneman (1982) claimed that most misconceptions in probability concepts among adults could be attributed to one of two heuristics; *representativeness* and *availability*. Shaughnessy (1981) also reported that misconceptions used in estimating probabilities by college entry students could often be attributed to these heuristics. However, not all misconceptions can be attributed to either of these heuristics, nor can the use of heuristics explain some of the types of reasonings associated with misconceptions. The assumption of equal likelihood in situations where none exists is one example. Shaughnessy (1992) summarised research in stochastics. In addition to discussing the use of the heuristics of *representativeness* (p. 470) and *availability* (p. 472), he examined research into:

The base-rate fallacy. Shaughnessy (1992) reports that the phenomenon is often attributed to representativeness, but claims that other factors such as "fundamental attribution error .. or some sort of causal reasoning" (p. 472) may be the real cause of the errors of probability estimates.

The failure to recognise independence. This is sometimes attributable to representativeness in, for example, the gambler's fallacy. In other instances it may be attributable to availability such as, for example, the selection of Lotto numbers. However some occurrences may not be attributable to either of these heuristics and the present research explores this possibility.

The assumption of equal likelihood when none exists. Again, this is often attributed to some other factor or heuristic. However, the present research examines instances where this may not be possible and looks for alternative explanations.

The confusion between a conditional and its inverse. Shaughnessy (1992, p.474) cites the example "the difference between the probability that I have measles given that I have a rash, and the probability that I have a rash given that I have measles."

The conjunction fallacy. Shaughnessy (1992, p. 473) cited research studies in which students were reported to confuse situations of conditional probability P (A I B) with the conjunction P (A n B) and reported that "the problem seems to occur primarily with the students' translation of conditional probability tasks, which then affects their understanding of the problem." He reports that many of the studies rely on a "forced-choice task methodology ... and were unable to obtain detailed insight into why the students were confusing conditionals with conjunctions."

The awareness of counter-intuitive probabilities. Instances of this too are often attributed to some other factor or heuristic, such as representativeness in the case of the "birthday problem".

Methodology

A set of 12 data questions was selected and modified from those reported in the research literature in order to examine the above misconceptions. This set of data questions formed the *major research instrument* of the study. Many previous research studies into misconceptions in probability have used a multiple choice format. (See, for example, Brown et al., 1988; Fishbein et al., 1991). Shaughnessy (1992, p. 479) comments that information from multiple choice items is often sketchy, incomplete and "not very helpful in clarifying students' thinking." Thus in the present research, all questions, whether multiple choice or open response, are open-ended and are accompanied with a request to explain the reasoning employed.

The questions were given to a sample of 50 students enrolled in Year 2 Mathematics Curriculum. Most of the students had completed a Year 12 mathematics subject which included some studies in probability. The analysis of the responses in the present study identified four levels according to level of sophistication in a similar manner to that of Watson & Collis (1993).

Level 1. In interpreting probability situations no analysis or evidence of use of probability principles is demonstrated. Features may include: the use of irrelevent information, subjective judgements, disregarding quantitative information, guessing at random, belief in control of probability and absence of any reason. Responses that use recent experiences to predict or estimate probabilities, availabilty, are included in this level.

Level 2. Some evidence of use of the use of probability principles and appropriate quantitative information is evident, but they may be incomplete or are incorrectly used. Probabilistic reasoning based on the assumption of equal liklihood when none exists and The use of the representativeness hueristic is considered to be illustrative of this level.

Level 3. Probability principles are used correctly used and an awareness of the role of quantification is evident. However, such quantification is precise or numerical.

Level 4. Probability principles are used correctly and relationships are explained quantitatively.

[Note, using this classification it is possible that "correct" answers can be accompanied with low-level or incorrect reasoning while "incorrect" responses may still show some high level reasoning.]

Selected Questions, Results and Discussion

Question 1

An unbiased coin is tossed five times in a row and lands *heads* each time.
On the next toss:

(a) *tails* is the more likely outcome
(b) *heads* is the more likely outcome
(c) *heads* and *tails* are equally likely.

Explain you choice: ______________________________

Results for Question 1. Only two students demonstrated the use of representativeness. Thirty stated that the coin was unbiased or that the probability remained the same (*L3*). Eighteen quantified the probabilities as 1/2 of 50% (*L4*).

Question 2

On a roulette wheel there are 18 red numbers, 18 black numbers and one green (zero). On any throw, each number is equally likely to occur. Suppose that the ball lands on "red" six times in a row. On the next roll is it now more likely to land on the "red", the "black", or are they both still equally likely? Explain your response: ______________________________

Results for Question 2. Four students used the representativeness heuristic to conclude that black was now more likely.

Discussion of Question 2. The question explores what is referred to in the literature as "the gambler's fallacy." (del Mas & Bart, 1989, p. 45) This misconception is well documented as an example of "local representativeness" (Tversky & Kahneman, 1982, p. 5). Research by Shaughnessy (1981) of its use by college students in his study showed a much higher incidence than that of the present study. However, of the remainder who recognised equal probabilities only three quantified the probability as 18/37, (*L4*).

Question 3

In a six-from-40 LOTTO,
Jim chose: 1, 2, 3, 4, 5, 6
Mary chose: 32, 33, 34, 35, 36, 37
and Bill chose: 9, 12, 27, 31, 35, 38
Does any one have a better chance of winning than the others? If so who? Why?

Results for Question 3. Eleven students said Bill had the better chance. Of these six gave reasons that showed a clear use of availability while the other five gave no specific reason, although they may nevertheless have been using the heuristic.

Discussion of Question 3. There is a common misconception that apparently random combinations such as Bill's are more likely than sequences or patterns. This misconception has been widely reported. Its use by up to 22% of respondents in the present study is consistent with that reported in the literature.

Question 6

If I throw a pair of fair dice is it harder to get a pair of sixes than it is to get a five and a six? Why or why not?

Results for Question 6. Only 17 recognised that a pair was less likely. Of these only three quantified the probabilities (L4). Five cited experience with dice as the reason.
Discussion of Questions 6. Fischbein et al. (1991) and Peard (1994) used these questions in researching pupils understanding of compound events. Fischbein et al. concluded that "there is no natural intuition for evaluating the probability of a compound event" (p. 534). The results in the present study of 17/50 or 34% correct would at first appear to be much better than those reported in the literature. However only 3 or 6% demonstrated quantitative reasoning. Peard (1994) reported that 10/ 40 or 25% of Year 11 students in his study answered correctly, six of these citing experience and four using computation.

Question 7

The spinner shown (Figure 1) is spun once.

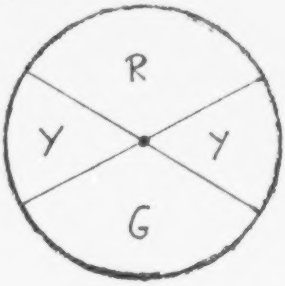

Figure 1

What is the probability that *yellow* will show? Explain your reasoning.

Results for Question 7. Thirty six responded 1/2 or 2/4. Thirteen answered 1/3, and one answered 2/3.
Discussion of Question 7. This question examines the misconceptions relating to the assumption of equal likelihood when none exists. Its use by such a high proportion of students is significant.

Question 9

A group of 30 people are selected randomly from the population.
The probability that at least two of them will have the same birthday is:
- (a) zero
- (b) very low
- (c) more than 50%
- (d) more than 90%

Explain your choice ________________________________

Results for Question 9. Forty five students selected (b). Of these, 23 gave an incorrect quantification (*L2*); 19 gave no real reason (*L1*); and three reasoned that the ratio of the number of people to the number of days in the year was low (*L3*).

Discussion of Question 10. Shaughnessy (1992) reported that most people are surprised to learn that the probability is high (70%) since this is counter-intuitive. Tversky and Khaneman (1982) cite it as an example of the use of the representativeness heuristic.

Question 11

In a pick-a-box quiz. there are three boxes. One box contains the prize and the other two are empty. You are a contestant who gets to select a box. After making your selection, the host, Monty, opens one of the two boxes to show that it is empty. (Monty does this with all contestants and Monty always knows which box holds the prize). Monty asks you whether you want to stick with your choice or change your selection.

Which of the following do you do now?

(a) Stick with your original choice
(b) Change you selection to the other box
(c) Toss a coin to decide
(d) Something else.

Explain your choice ______________________________

Results for Question 11. Forty students selected (a). Of these 23 reasoned new probabilities of 1/2 (*L2*), while 17 were unable to formulate any reason (*L1*).

Discussion of Question 12. Shaughnessy (1992, p. 475) has used this problem extensively with students, including pre-service teachers, to point out some of the difficulties with conditional information. He reported similar results. Various forms of this misconception have been recorded as a paradox in probabilistic reasoning. The probability of guessing the right box is clearly 1/3. Thus if you decide to stick with your original choice, this probability remains 1/3. However, if you use the strategy of changing your choice *after* Monty has opened one of the empty boxes (conditional probability), your probability of winning improves, not to 1/2 but to 2/3. This result is clearly counter-intuitive, but can readily be explained by the fact that using the strategy there is only one out of three ways the contestant can lose; that is by choosing the winning box initially.

Question 12

A deck of 21 green and yellow cards are shuffled and dealt twice.
The following sequences are observed:

A: Y Y Y G G Y Y G G Y Y Y G G G G G G Y Y Y
B: G Y G Y G Y G Y Y G Y G Y Y G Y G G Y G Y

Which sequence would represent the better shuffled (more random) deck?
A, B, or are both equally random? Why?

Results for Question 12. Twenty three students assumed equal likelihood. Of these 14 gave a Level 2 reason, while nine gave either no reason or used irrelevant information such as "the number of cards is the same" (L1). Six correctly recognised that A is "too regular".

Discussion of Question 12. It can be shown that if a coin is tossed *n* times, then there occurs a run of heads of length log2 *n* with a probability converging to 1 as *n* approaches infinity. Applying this theorem to the deal of 21 cards, a run of 4 or 5 yellow or green cards is to be expected. (log2 21 = 4.4). Thus sequence A is more likely than B. Falk (1981) asked a similar question to secondary students and reported that students tended to pick the sequences where more "switches" occurred. The responses of the tertiary students in the present study are similar to those reported by Falk (1981).

Summary

It would appear that the level of misconception in the use of the representativeness and availability heuristics, independence of events, compound probability and the lack of awareness of counter-intuitive probabilities are widespread and exhibited by the students in much the same proportion as is reported in the literature. The assumption of equal likelihood when none exists was much higher than expected. This misconception was evident in Question 6, 36%; Question 8, 72%; Question 12, 46%; and Question 13, 46%. Furthermore an analysis of the level of responses shows that on relatively few occasions did the students use quantitative or relational reasoning with only 17% of all responses categorised as Level 4. Probability principles were correctly used in another 23% of all responses categorised as Level 3. A full 60% of responses were made without evidence of analysis or with an incorrect use of probability principles.

Bearing in mind that the majority of students have had some formal education in probability further education as a part of their pre- service teacher education is clearly needed. It cannot be assumed that they are necessarily free of the confusion and misconceptions that permeate this difficult topic.

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Attitudes towards mathematics: What about NESB students?

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In this paper we draw on data from the second stage of a three year study conducted in eight Victorian, state co-educational secondary schools. In 1995, one Year 7 and one year 9 class, from each of four schools with high numbers of non-Anglo cultural background students, participated in the project. Four sources of data were collected. This paper focuses on the analysis of questionnaire responses and where possible snippets of student interviews have been included in an attempt to clarify issues raised.

Introduction

Are students from non-English speaking backgrounds [NESB] advantaged or disadvantaged in mathematics? Should we focus on newspaper reports which highlight the high achievements of some groups, often students who have come to Australia from Asian countries? Or should we concentrate on data which indicate that mathematics is often a difficult and alienating subject for students for whom English is a second language? In this paper we focus on students who may be experiencing conflicts in mathematics in terms of gender, ethnicity or class. We report on the second part of a three year study, undertaken in 1995 with students from four schools with a predominantly non-English speaking background. (In 1994 the four schools involved had high percentages of predominantly English speaking students). (See Brew, Pearn, Bishop and Leder, 1995).

Explanations and interpretations of underachievement in mathematics have tended to move away from the cognitive domain and to focus more on student attitudes (McLeod, 1992; Schoenfeld, 1992). Moreover as the social dimension (Bishop, 1985) has come to be recognised as a highly significant factor in mathematics education, so the need has arisen to carry out studies which examine the roles of particular individuals and groups in influencing students' attitudes towards mathematics and mathematics learning. In fact, "The unique culture of each mathematics classroom is the product of what the teacher and pupils bring to it in terms of knowledge, beliefs and values and how these affect the social interactions within that context" (Nickson, 1994, p. 28).

According to a report released about the study of young people's attitudes to secondary education (Dua, 1994), 31% of girls felt they were pressured to underachieve at school but only 20.9% of males believed there was such pressure on girls.

> Girls felt pressure from all quarters - not only from male students at school but also from their own peer groups and also from their parents ... There's a widespread feeling among educators that there are no longer these pressures on girls to underachieve but this survey shows that's just not the case and they are still subjected to stereotyped roles. (The Age, 12/9/94, p.5. In *Schoolgirls tell of pressure to fail*, quoting Andre Dua).

In this study we explored the perceptions and relationships involving individual students, their peers, their mathematics teacher and their parents. These were assumed to be the most significant 'others' likely to influence students' attitudes. As well as providing summary data for this cohort of mainly non-Anglo cultural background students, we include snippets of interviews which clarify student responses.

The Sample

The four 1995 schools were selected on the basis of their high percentage of NESB students (determined from data from the Victorian Department of School Education) with two schools from a low/medium socio-economic status [LSES] area and two from a medium/high socio-economic [HSES] area. Socio-economic status was based on Census data of the school's catchment area, participation in the Disadvantaged Schools Program, and proportion of students receiving the Educational Maintenance Allowance [EMA]. Two schools were located in the North Western Region; two in the South Eastern Region of Victoria. One Year 7 and one Year 9 class from each of the four co-educational schools participated in the project. The sample comprised 188 students, 95 girls and 93 boys.

Data Sources

Data were collected from four sources. Questionnaires were administered to the students in each of the four classes studied. Each class was videotaped for a minimum of three lessons. These videos were used to observe students who were perceived by their teachers as being either successful or unsuccessful with their mathematics learning. The targeted students and, where possible, their parents were interviewed.

Questionnaires

To ascertain the students' attitudes towards mathematics, a multi-dimensional questionnaire was developed, using items from the Fennema-Sherman Attitude Scales (1976), the Mathematics Attribution Scale (Fennema, Wolleat & Pedro, 1979), the Individualised Classroom Environment Questionnaire (Fraser, 1990) and items developed by the researchers based on statements from *The National Statement* (Australian Education Council, 1991). This questionnaire was administered to all students from the sixteen classrooms participating in the project in 1994 and 1995.

The questionnaire had four distinct sections. *Your Views about Mathematics* tapped students' perceptions of mathematics. *More Views about Mathematics* assessed students' attributions for success and failure in mathematics: to ability, effort, task and environment. *How Good Are You?* determined the students' perceptions of their own ability in mathematics and how they thought their parents, teachers, and peers would rate them as learners of mathematics. The *Individualised Classroom Environment* questionnaire was used to obtain each student's perception of the learning environment within the mathematics classroom. Most items required students to respond on a five point scale from Strongly Agree to Strongly Disagree. The number of items in each section and one example from each section are shown in Table 1.

Table 1. Questionnaire items.

Questionnaire Section	Number of items	Representative Item.
Your Views about Mathematics	42	Maths is one of the most worthwhile and necessary subjects to study at school.
More Views about Mathematics	9	Imagine you have not been able to keep up with the rest of the class in maths this term. This happened because: Students sitting near you wouldn't work.
How Good Are You?	8	How good are you at maths?
Individualised Classroom Environment	50	The teacher helps each student as much as possible.

Questionnaire Results

Students' General Perceptions

Your Views About Mathematics: Students generally thought mathematics was useful, believed they were reasonably persistent in their mathematical studies, were not particularly anxious about mathematics, felt supported by their teacher, believed mathematics was logical, did not believe mathematics was a male domain and were relatively confident about studying mathematics.

Differences emerged on the basis of students' perceived performance in mathematics. When compared with their general views about mathematics, students who rated themselves as 'low achievers' were uncertain about the usefulness of mathematics, believed themselves to be less persistent and perceived themselves to be far less confident about mathematics than other students. They were also more anxious about doing mathematics, believed they received less support from their teacher, and were less able to relate to mathematics as a logical subject. How these factors converge and are experienced by low achievers emerged in interview. Two examples follow,

I: What is it like being in your class?

E (Year 9 boy): It just gets really boring working by yourself for 50 minutes doing nothing. It gets really boring. It is the most boringest part of the day, maths is. I hate going to maths.

I: Do you ever ask your teacher for help?

E: Sometimes, when I do work.

I: What happens when you do?

E: She comes over and practically tells you the answer to the questions and doesn't help you do it.

I: Why do you think she does it like that?

E: I don't know, type of person she is, she knows it. It is difficult when you know something and try and describe it to some-one. You have to tell them the answer and then they will understand it.

I: Do you ask any friends around you for help?

E: I am not allowed to sit with my friends. I would say that I work better around friends but I don't behave as well. I work well but I don't behave. Sitting by myself I behave better but I don't work. I find it easier to work with friends, they know how to describe it to you. It is easier for a friend describing it than a teacher, and I work better around them.

A year 7 boy who believed that he was 'about average' at mathematics perceived that his teacher would rate him as 'under average' because "I always muck around. Or she always thinks I muck around". He reported that he found it difficult to get help from the teacher.

I: Do you ever ask your teacher for help?

A: No, because then she will always get angry at me.

I: Why does she get angry with you if you need help?

A: I don't know. It is just ..., she never helps me , because I asked her for help once, and she didn't. She said all this stuff and then said, 'Oh, try it yourself', which I had already done.

Classroom observations revealed this student was disruptive in class, that the teacher worked with him but he was not always receptive to her assistance. The incident described above was probably one that occurred when the teacher lost her patience but unfortunately this is the incident the student chose to remember.

How Good Are You At Maths: Students indicated overwhelmingly that they wanted to do better mathematically than they were doing (83%) and they perceived that their parents also wished them to do better (70%). The low achievers generally had lower performance aspirations. In interview, differences between perceived parental and child expectations were explored. In the example below, a low achieving Year 7 boy wanted to be rated as 5 (excellent) but perceived his parents would be pleased with a rating of 4 (very good). This may restrict the student's attempt to strive for a better performance in mathematics. It would appear that this student's perceptions have been influenced by his father.

I: What maths do you like the most?

M: Times, divide, plus and that. Arithmetic.

I: You would like to be a 5, but you think your Mum and Dad would like you to be a 4. They would like you to be better but you don't have to be excellent?

M: Dad said I want you to learn times, plus and a bit more about maths than that.

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More Views about Mathematics: Students generally attributed their success in mathematics to the external factors of classroom environment and task while failure was most strongly attributed to the task. Students who perceived themselves as low achievers attributed their success to the task while students who perceived themselves as high achievers were more likely to attribute their success to their ability. Low achievers were also more likely than high achievers to attribute their failure to lack of effort and ability.

Classroom Environment: This part of the questionnaire measured students' perceptions of their classroom and included the five factors: participation, personalisation, independence, differentiation and investigation. Results indicated that although students felt they could interact with their teacher and participate in classroom activities, they did not believe they had control over decisions made about their own learning. While they indicated they were sometimes involved in tasks that developed their problem solving skills, they perceived there was a lack of provision for individual learning preferences, abilities, interest or rate of working.

The data suggest that students who perceive they are doing well in mathematics believe that students in general receive more personalised attention in the classroom. This is consistent with the results of the first part of the questionnaire in which low achievers indicated they received less support from their teacher than high achievers.

Gender Differences

Your Views About Mathematics: It was particularly interesting to find that student attitudes concerning 'Maths as Male Domain' and 'Anxiety about Maths' were similar to those found by previous studies in Victorian schools (Forgasz and Leder, 1995). While both boys and girls in general rejected the notion that mathematics is a male domain, girls rejected this idea more strongly than boys ($F_{1,160} = 22.99$, $p<0.001$). Evidence was obtained, from both boys' and girls' interviews, of a perception that boys were naturally better at mathematics than girls. This view was rationalised at times by boys in terms of differing levels of interest portrayed by boys and girls. For example, "Basically more boys like maths than girls, and they like to be engineers." In comparison, at least one Year 9 girl suggested that girls are more creative than boys and that this has implications for how well they perform.

I: Tell me why girls have to work harder than boys at maths.
Y: Because I think that boys are naturally born with the ability. I have always felt they are more bred into that subject and girls like the creative area.

Although not statistically significant a relationship was found between students' perceived performance and their perception of mathematics as a male domain. This trend was most apparent at Year 7 where low achieving girls and boys indicated they are less convinced that mathematics is not a male domain.

Boys perceived themselves to be more confident at mathematics than girls (boys: average of 4.0, meaning they are confident and girls an average of 3.6, meaning they are more uncertain about their level of confidence, $F_{1,163} = 12.16$, $p< 0.001$). At Year 7, this gender pattern was only evident for students who rated themselves as average.

Gender differences emerged for the perception of mathematics as a subject (Table 2). Girls believed more strongly than boys that *to learn maths you need to explain what you are doing*. This pattern was more evident at Year 7. Girls were less convinced that *mathematics made sense*. Boys were more likely to disagree with *In maths there should always be one right answer*. There was no gender difference for a similar question: *In maths it is possible to have more than one right answer*, suggesting that girls are as aware as boys that it is possible to have more than one right answer when working mathematically but are more in favour of there *being only one* right answer. This perception was illustrated in interviews. For example, K (Year 9 girl): With problem solving it is really hard and I like working with things where they have a basis. With problem solving you have to think of everything and everyone has different ideas so it is hard to come up with a solution. And there is lots of different answers you can have. I like it when you have only one right answer.

Table 2: 'Perception of mathematics as a subject' by gender.

'Perception of mathematics as a subject' (1= strongly disagree and 5 = strongly agree)	Girls	Boys	t-Test ‡ denotes $p < 0.05$
Most of the time maths makes sense.	3.2	3.6	$t_{182} = 2.31$ ‡
To learn mathematics you need to explain what you are doing.	4.0	3.6	$t_{176} = 1.97$ ‡
*In maths there should always be one right answer.	2.8	3.3	$t_{182} = 2.39$ ‡
In maths it is possible to have more than one right answer.	3.8	3.9	-

* For this statement 1= strongly agree and 5= strongly disagree.

How Good Are You At Maths?

Self Rating: Boys were more satisfied than girls with their level of performance in mathematics (Table 3).

Table 3: Satisfaction with current performance in mathematics.

	I want to do better	I am happy with my performance	I don't want to do as well as I am.
girls	91%	8%	1%
boys	71%	29%	0%

Furthermore, boys considered themselves better at mathematics than did girls (mean for boys = 3.7; for girls = 3.3, $F_{1,172} = 12.42$, $p < 0.001$). This pattern was consistent for perceived ratings of significant others except for peers: teachers (mean for boys = 3.6; for girls = 3.3, $F_{1,169} = 5.33$, $p < 0.05$), mothers (mean for boys = 3.9; for girls = 3.3, $F_{1,168} = 15.8$, $p < 0.001$), and fathers (mean for boys = 3.8; for girls = 3.2, $F_{1,158} = 14.14$, $p < 0.001$). Boys were more likely than girls to believe their peers under-rated their performance ($F_{1,167} = 13.31$, $p < 0.001$) but the majority of boys and girls perceived no difference between self and peer rating (Table 4).

Table 4: Peer Influence.

	'Peers over-rate my performance'	'Peers equal-rate my performance'	'Peers under-rate my performance'
Girls (n = 91)	35%	55%	10%
Boys (n = 92)	14%	65%	21%
Overall (n=183)	25%	60%	15%

Our results may help to explain the findings of the National Youth Survey (Dua, 1994) which found in response to the question asked of 15-25 year olds "Do you think there are undue social pressures on girls to underachieve at secondary school?" that 26%

agreed, 59% disagreed and 15% were undecided. This survey found that females and males differed significantly in their views, with 31% of girls feeling undue pressure to underachieve but only 20.9% of males believed that there was such pressure on girls.

Classroom Environment: For boys, lower achievers perceived their classrooms to be less personalised. This pattern was not evident for the girls who, across perceived performance levels, consistently rated the personalisation of the classroom as high. These results are reflected in students' responses to levels of teacher support. Low achieving boys clearly felt less cared for by their teachers. A sense of being punished by teachers for misbehaving, also evident in earlier comments quoted from low achieving boys, probably contributed to this perception of less personalised positive attention.
I: Did you ask your teacher for help?
J: No I didn't because if I put up my hand for help he wouldn't come over to me.
I: Why do you think he didn't come over to help you?
J: I don't know, because he used to think I was a trouble maker.

Year 7 boys also perceived that classroom tasks were more differentiated than did the Year 7 girls, where differentiation relates to students' perceptions that tasks are set with an emphasis on the individual's ability, learning style, interest and rate of working. This was more evident in the classrooms of the two schools listed on the Disadvantaged Schools Program. As there was no relationship between personalisation and perceived performance levels, it is not clear how to interpret this result. Further analyses of the interview and video data may shed some light on this finding.

School Differences

The four schools were chosen based on their high percentage of NESB students with two LSES schools and two HSES schools. Differences between schools could be due to their socio-economic status, NESB enrolment or teacher-related issues. Explanations for school differences need to be explored more fully through an analysis of interviews and videotape data.

Your Views About Maths: A school difference emerged for the 'Maths as a male domain' construct and the order of means suggests that socio-economic status was a contributing factor. LSES schools had means of 3.7 and 4.0 while HSES schools had means of 4.2 and 4.3 ($F_{3, 160} = 7.02, p < 0.001$). From interview data, poor language comprehension may have been a compounding factor in this result. Two examples follow,

I: Girls are logical enough to do well at maths, you disagreed with that. What would be the reason for that?
B (Year 7 boy): I don't know what logic means.
I: Do you think that girls can think step by step?
B: Yes.
I: So you didn't understand the question.
B: Yes.

I: Girls often have to work harder than boys to do well in maths, you are unsure, why are you unsure about that?
V: 'Cause sometimes girls are better than boys and boys can be better than girls, so it is mixed, I am not sure about that.
I: It says that girls often have to work harder, do you think that they often have to work harder?
V: Yes, 'cause boys muck around more than girls.
I: So you are saying that you see girls working harder than boys?
V: Yes.

While there appears to be some problems in the interpretation of questions, the girls' average score for this construct was still higher than the boys' at both DSP schools.

This pattern was consistent across all the data. A wish not to reveal sexist beliefs in interview may have been a factor but for some boys this was not apparent. For example:

I: You have agreed that girls have to work harder than boys to do well at maths. Tell me about that experience, is that what you have noticed?
M: Some girls they get marks better than the boys but sometimes boys do, depends. They don't listen and boys, they listen all the time.

Compared to other schools, students at one school indicated they were more supported by their teachers. This particular school had implemented a proactive program aimed at improving the rapport between staff and students, for example, having a staff-student common room.

Success/Failure Attribution. Students at DSP schools were less likely to attribute their failure in mathematics to lack of effort. The attribution of success to the environment was also significantly higher for Year 7 students at DSP schools.

Classroom Environment. In one school students generally perceived their classroom to be the least personalised, the least encouraging of student participation and that they were less likely to be given investigative tasks. This class was continually changing during the time of the study. New migrant students were frequently being placed in the class and then subsequently moved to another. Some students were frequently absent. Hence the opportunity for the teacher to create a cohesive and supportive group environment and a supportive relationship with the students was continually being challenged. Interestingly, this was reflected in the students' responses on the paper-and-pencil instrument.

Students at another school felt they had the least independence. As this was consistent across both year levels it may be a school rather than a particular teacher effect. Questions in the independence scale were challenged by the staff from this school for not actually measuring student independence. Rather, the Year 9 teacher perceived it was perhaps measuring teacher control. The Year 9 teacher did take considerable care in deciding with whom students would work during group activities, and this action would have diminished the measured level of student independence in the classroom.

I: Your teacher chooses who you work with sometimes. What is that like?
K. That is alright as you don't get along with people and you have a top group of people and they always go together and other people are left out. And when the teacher chooses they get different people.

These results suggest how more detailed interview data can illuminate responses made to paper-and-pencil instruments.

Conclusion

The results obtained from the questionnaire have corroborated the findings of previous studies concerning gender differences in attitudes towards mathematics. This consistency makes us more confident about the validity of our other findings. Students' perceived performance ratings have provided new information concerning students' attitudes towards mathematics, particularly in relation to the impact of peers and parents. Further analysis of the student interview data and interviews with parents (still to be collected), may help to clarify the emerging patterns.

Because students' attitudes towards mathematics differed significantly on the basis of school, year level, gender and students' own performance level, our results have implications for the teaching of mathematics. Lower achievers perceive their classroom differently from high achievers and express considerable conflict with the subject as it is taught. For example, they attribute their failure more to the nature of the task and do not believe they have the ability to do mathematics.

Year differences revealed substantial variation in students' perceptions about the nature of mathematics classrooms. Year 7 students reported that their mathematics classrooms were more differentiated than Year 9 students. Year 7 students were found to be less anxious and more confident than Year 9 students. Year 9 students were more likely to attribute their success to the environment than were Year 7 students and the Year 9 low achievers most strongly agreed that failure was due to effort. In addition, the Year 9 students perceived mathematics to be less about following rules and obtaining one right answer. Clearly the Year 9 experience is different from the Year 7 experience for these students, and in some way the higher confidence levels for Year 7 students has not been maintained. Year 7 students also perceived their mothers and fathers would like higher performance ratings in mathematics than did Year 9 students. More detail on this apparent family influence may be revealed from our analysis of interviews with parents.

The outcomes of one school's strategy to improve teacher-student rapport were confirmed by our results. School practices that took account of the differing cultures experienced at home and at school included time allocation during school hours for homework with special support from peers, older students and teachers. Students at this school felt more supported by teachers and felt their classrooms were more personalised. The importance of a caring and supportive school environment, especially for students from other cultural backgrounds, needs further investigation.

The issues for NESB students appear to confirm what we know about students in general and their attitudes to mathematics. Further analysis will involve a comparison of the 1995 schools with their 1994 counterparts to investigate the compounding variables that apply to these special groups.

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K-6 Teacher Beliefs About the Learning and Teaching of Mathematics

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This paper has been developed as part of a continuing investigation of the use of manipulatives in mathematics learning and teaching by K - 6 teachers. The focus is the beliefs held by teachers about mathematics, mathematics learning and mathematics teaching. The grouping of particular beliefs which allows the development of a profile of these teachers is considered.

Introduction

In the past decade, there has been considerable comment related to the importance of teacher beliefs in the mathematics classroom (Barnett & Sather, 1992; McLeod, 1992; Pajares, 1992; Weissglass. 1992). This investigation of the beliefs held by 252 teachers, working during 1995 in government primary schools in the south western suburbs of Sydney, adds to this growing literature and provides some basis from which decisions concerning mathematics curriculum and pedagogy might be made. In addition, it provides a basis for future study of the actual practices of such teachers.

Teacher beliefs and mathematics

There appears to be more than the knowledge of facts and formulas involved in one's ability to do mathematics (Garofalo, 1989). In recognising this, the complex interaction of the affective and the cognitive domains in the learning and teaching of mathematics continues as an important area of investigation for mathematics educators, both nationally and internationally (Howard, 1994; McLeod, 1992; Southwell, 1995; Southwell & Khamis, 1992). In such investigations a belief can be defined as "*any simple proposition, conscious or unconscious, inferred from what a person says or does, capable of being preceded by the phrase: 'I believe that...'*" (Rokebach, 1968, p.2). Pajares (1992, p.327) suggests that beliefs are "*the single most important construct in educational research*".

All teachers have beliefs about mathematics and its learning and teaching. Baroody (1987, p.5) notes that

> *... everyone has a set of beliefs about how mathematics is learned. These beliefs have an influence on all aspects of teaching. They govern what is considered appropriate to include in a curriculum and when topics should be taught; they determine the importance an educator attributes to gauging readiness skills or exploiting children's curiosity and interests; and they effect how educators teach skills and concepts, evaluate progress and remedy difficulties. In brief, whether conscious or not, beliefs about mathematics learning guide decision making and, in the end, influence our effectiveness as mathematics educators.*

Beliefs about mathematics, and one's ability to do mathematics, influence how people study and perform mathematics (Garofalo, 1989). Personal beliefs incorporate one's world view of mathematics and influence the context in which mathematics is done (Schoenfeld, 1985). One way of examining teachers' espoused beliefs about mathematics has been to categorize them into those related to the nature of mathematics, the learning of mathematics and the teaching of mathematics.

One study which investigated the mathematical beliefs of students and teachers (Southwell & Khamis, 1992) reported that teachers firstly need to identify their held beliefs about the nature of mathematics and the learning and teaching of mathematics, and, secondly, to consider the influence of these beliefs on the students' learning environment (Barnett & Sather, 1992). The present investigation of teacher beliefs links them to the teachers' major task of facilitating the mathematics education of their students.

Methodology

The data upon which this paper is based were collected using a specifically designed questionnaire consisting of both multiple choice and open-ended questions covering the following areas:

1. respondent demographics such as gender, age, position in school, nature of initial teacher education, length of teaching experience, class(es) currently taught, class size, classes taught over the last ten years;
2. use of concrete materials in mathematics learning and teaching such as which materials are used, why, how often and how they are used, and the areas of mathematics in which they are used;
3. beliefs about mathematics, mathematics learning and mathematics teaching.

Section 3 of the questionnaire sought responses to a set of teacher belief statements which were derived from Australian Education Council (1991), Barnett & Sather (1992), Mumme & Weissglass (1991), Weissglass (1992) and Wood, Cobb & Yackel (1992). During its preparation, the questionnaire went through a number of iterations as a result of input from several mathematics educators.

In August, 1995, the questionnaire was posted, with reply paid envelopes, to 25 government primary schools in the Metropolitan South West Region of the NSW Department of School Education. The schools were randomly chosen from the 150 government primary schools in the Region. Sufficient numbers of the questionnaire were posted to cover all teachers at these schools. Two hundred and fifty-two completed questionnaires were received.

Results

Demographic data

Of the 252 respondents, 218 (87%) were female. The (former) Metropolitan South West Region of the New South Wales Department of School Education was often referred to as the 'nursery' for the State in terms of the relative inexperience of its teachers. Hence, it is somewhat surprising to note that 49 (19%) of the respondents had in excess of 20 years teaching experience, while 66 (26%), 62 (25%), 67 (27%) had from 11 to 20 years, 6 to 10 years, 1 to 5 years teaching experience respectively. Only 8 respondents (3%) had less than one year of teaching experience. The school positions held by the respondents are reported in Table 1.

Table 1 School positions held n=252

Current position	Number	Percentage
Principal	11	4
Deputy Principal	10	4
Assistant Principal	19	8
Executive Teacher / AST	23	9
Classroom Teacher	155	62
ESL Teacher	31	12
Other Teacher	3	1

Fifteen (6%) of the respondents described themselves as two year trained teachers, 78 (31%) as three year trained with a Diploma in Teaching, 14 (6%) as three year trained with a Bachelor of Teaching, 110 (44%) as four year trained with a Bachelor of Education and 11 (4%) as teachers with more than four years of training. Another 24 (9%) respondents described their teacher education status in other ways such as Diploma / Graduate Diploma.

Respondents were asked to indicate the class or classes in which they were currently teaching mathematics. These responses were grouped into the four categories K - 2, 3 - 4, 5 - 6 and Nil Response. There were 33 (13%) Nil Responses to this question, perhaps indicating that these respondents were not teaching mathematics during 1995. Details are given in Table 2.

Table 2 Mathematics teaching during 1995 n=252

Range of Years	Number	Percentage
Kindergarten to Year 2	98	39
Years 3 and 4	61	24
Years 5 and 6	60	24
Nil response	33	13

Statements of beliefs - Frequency of teacher responses

Table 3 reports the overall frequency of teacher responses to the 20 statements of beliefs concerning the nature of mathematics and mathematics learning and teaching. Teachers were asked to respond to each belief statement by indicating whether they disagreed with the statement, were undecided about it, or agreed.

Table 3

Statements of belief - Frequency of responses (Percentage of respondents) n=252

Statement of belief	Disagree	Undecided	Agree
Mathematics			
1. Mathematics is computation	87 (36)	41 (17)	116 (48)
2. Mathematics problems given to children should be quickly solvable in a few steps	138 (57)	45 (18)	61 (25)
3. Mathematics is the dynamic searching for order and pattern in the learner's environment	11 (4)	62 (25)	178 (71)
4. Mathematics is no more sequential a subject than any other	142 (57)	40 (16)	66 (27)
5. Mathematics is a beautiful, creative and useful human endeavour that is both a way of knowing and a way of thinking	24 (10)	75 (31)	147 (60)
6. Right answers are much more important in mathematics than the ways in which you get them	214 (86)	20 (8)	14 (6)
Mathematics learning			
7. Mathematics knowledge is the result of the learner interpreting and organising the information gained from experiences	5 (2)	12 (5)	234 (93)
8. Children are rational decision makers capable of determining for themselves what is right and wrong	90 (36)	76 (30)	85 (34)
9. Mathematics learning is being able to get the right answers quickly	212 (85)	19 (8)	20 (8)
10. Periods of uncertainty, conflict, confusion, surprise are a significant part of the mathematics learning process	20 (8)	26 (10)	206 (82)
11. Young children are capable of much higher levels of mathematical thought than has been suggested traditionally	17 (7)	83 (33)	151 (60)
12. Being able to memorise facts is critical in mathematics learning	71 (29)	57 (23)	120 (48)
13. Mathematics learning is enhanced by activities which build upon and respect students' experiences	1 (0)	7 (3)	244 (97)
14. Mathematics learning is enhanced by challenge within a supportive environment	1 (0)	6 (2)	244 (97)
Mathematics teaching			
15. Teachers should provide instructional activities which result in problematic situations for learners	9 (4)	23 (9)	215 (87)
16. Teachers or the textbook - not the student - are the authorities for what is right or wrong	176 (71)	56 (23)	16 (6)
17. The role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge	94 (38)	44 (18)	109 (44)
18. Teachers should recognise that what seem like errors and confusions from an adult point of view are children's expressions of their current understanding	8 (3)	36 (14)	206 (82)
19. Teachers should negotiate social norms with the children in order to develop a cooperative learning environment in which children can construct their knowledge	8 (3)	56 (23)	184 (74)
20 It is unnecessary, even damaging, for teachers to tell students if their answers are correct or incorrect	207 (83)	32 (13)	12 (5)

As with questionnaire research in general, a limitation to this study is that all results must be considered in the context that responses to the belief statements are dependent on the interpretations given to them by the respondents.

Beliefs about mathematics

Forty-eight percent of the respondents indicated a belief that *mathematics is computation*. In spite of this, 86% disagreed with *right answers are much more important in mathematics than the ways in which you get them*. As well, 71% believe that *mathematics is the dynamic searching for order and pattern in the learner's environment* and 60% believe that *mathematics is a beautiful, creative and useful human endeavour*. Interestingly, 57% of respondents disagree with the statement that *mathematics is no more sequential a subject than any other*.

Beliefs about mathematics learning

Ninety three per cent of respondents believe that *mathematics knowledge is the result of the learner interpreting and organising the information gained from experiences*. This is supported by 97% of respondents who believe that *mathematics learning is enhanced by activities which build upon and respect children's experiences* and *by challenge within a supportive environment*. Further, 82% of the respondents saw that *periods of uncertainty, conflict, confusion, surprise are a significant part of the mathematics learning process*.

Almost half (48%) of the respondents agree that *being able to memorise facts is critical in mathematics learning*. Sixty percent agreed that *young children are capable of much higher levels of mathematical thought than has been suggested traditionally* while 85% disagreed that *mathematics learning is being able to get the right answers quickly*. This can be contrasted with the 57% of respondents who disagreed that *mathematics problems given to children should be quickly solvable in a few steps*.

Of particular interest is the belief that *children are rational decision makers capable of determining for themselves what is right and wrong* where 36% of respondents disagreed and 34% agreed. This item will be the subject of further analysis.

Beliefs about mathematics teaching

The belief that *teachers should provide instructional activities which result in problematic situations for learners* was agreed to by 87% of the respondents with 71% disagreeing that *teachers or textbooks - not the student - are the authorities for what is right or wrong*. The majority (74%) of respondents agreed that *teachers should negotiate social norms with the children in order to develop a cooperative learning environment in which children can construct their knowledge*. However, this finding must be tempered by consideration about *whose* social norms are to be negotiated.

The belief that *the role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge* polarised the respondents with 44% agreeing with the statement and 38% disagreeing. On the other hand, 83% of the respondents disagreed with the statement that *it is unnecessary, even damaging, for teachers to tell students if their answers are correct or incorrect*. This would seem to contrast with the response to the statement concerning authorities for what is right and wrong.

Eighty-two percent of respondents agree that *teachers should recognise that what seem like errors and confusions from an adult point of view are children's expressions of their current understanding*. This would appear to be quite a shift from a traditional view of errors as **wrong** answers which indicate children's ability levels.

The interconnectedness of the belief statements

Through a chi-square analysis of the responses to these belief statements, some statistically significant relationships ($p<0.05$) between the statements have been identified.

Statistically significant results

The belief that *mathematics is computation* (1) is significantly related to three beliefs:

- *mathematics problems given to children should be quickly solvable in a few steps* (2) ($p<0.001$)
- *being able to memorise facts is critical in mathematics learning* (12) ($p<0.0005$)
- *the role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge* (17) ($p<0.00005$).

That these four beliefs are related is not surprising in that all four represent facets of what might be called a transmission or absorption approach to mathematics learning and teaching (Baroody, 1987; Wright, 1991). Relationships between pairs of these four belief statements are also statistically significant: 2 and 12 are related ($p<0.02$); 2 and 17 ($p<0.002$) and 12 and 17 ($p<0.0001$).

Another group of belief statements which display statistically significant relationships are *children are rational decision-makers capable of determining for themselves what is right and wrong* (8) with:

- *mathematics is no more sequential a subject than any other* (4) ($p<0.05$)
- *being able to memorise facts is critical in mathematics learning* (12) ($p<0.02$).

However, 4 and 12 are not related significantly to each other.

Profiling teachers through espoused beliefs

The above statistical analysis suggests that the set of beliefs: 1, 2, 12 and 17 when held by teachers, tend to be held strongly as a group. These beliefs can be characterised as *transmission* beliefs. On the other hand, there is a set of beliefs which are held by the majority of the respondents but which are not held as strongly, as a group of beliefs, as the transmission set. Nonetheless, it is possible to pair these beliefs if 60% of the respondents (an arbitrary limit chosen by the researchers) agree with both belief statements, disagree with both or agree with one and disagree with the other. This enables a profile of the group of teachers to be inferred.

For example, 86% of all respondents disagreed with the belief statement that *right answers are much more important in mathematics than the ways in which you get them* (6). While this result itself is spectacular enough, of further interest is the way in which this result is linked to the response rates to other belief statements. Of all the respondents, the percentage which disagree with statement 6 and:

- agree with *mathematics is the dynamic searching for order and pattern in the learner's environment* (3) is 63%;
- agree with *mathematics knowledge is the result of the learner interpreting and organising the information gained from experiences* (7) is 82%;
- disagree with *mathematics learning is being able to get the right answers quickly* (9) is 77%;
- agree with *periods of uncertainty, conflict, confusion, surprise are a significant part of the mathematics learning process* (10) is 72%;
- agree with *mathematics learning is enhanced by activities which build upon and respect students' experiences* (13) is 84%;

- agree with *mathematics learning is enhanced by challenge within a supportive environment* (14) is 84%;
- agree with *teachers should provide instructional activities which result in problematic situations for learners* (15) is 75%;
- disagree with *teachers or the textbook - not the student - are the authorities for what is right or wrong* (16) is 66%;
- agree with *teachers should recognise that what seem like errors and confusions from an adult point of view are children's expressions of their current understandings* (18) is 73%;
- agree with *teachers should negotiate social norms with the children in order to develop a cooperative learning environment in which children can construct their knowledge* (19) is 66%;
- disagree with *it is unnecessary, even damaging, for teachers to tell students if their answers are correct or incorrect* (20) is 70%.

All of these beliefs, perhaps with the exception of 20, display a child-centred approach to teaching and learning mathematics. That so many of the respondents adhere to these beliefs both individually and as a group is heartening to mathematics educators who have worked for a shift towards a child-centred approach over the last 20 years.

This analysis indicates that grouping belief statements can help lead to the development of profiles for the respondents. It is anticipated that further analysis of the data, including the investigation of the relationship between these espoused beliefs and the classroom practice of the teachers, will add to our knowledge in this area.

Conclusion

This paper has investigated the espoused beliefs of teachers and pointed to the need for further investigation. A natural extension of this investigation is to include the beliefs of secondary mathematics teachers in the broader research agenda announced by Perry & Howard (1994).

The results of this study suggest that there is a need for mathematics educators to identify more clearly the place of memorising facts in the learning of mathematics, to continue discussion about the role of computation in mathematics and to seek more balance between students obtaining the right answers in mathematics and the importance of the mathematical processes that they go through.

It is not only the mathematics learning and teaching which needs to be investigated further. It is also necessary for mathematics educators to consider the complex web of social interactions which occur in the mathematics classroom within the framework of negotiated social norms and to identify the constraints and supports within this social context which determine the degree to which the teacher is able to enact his / her beliefs about the learning and teaching of mathematics.

Given the overwhelming support for mathematics learning which is built upon students' experiences and takes place in a challenging and supportive environment characterised by periods of uncertainty, conflict, confusion and surprise, it is pertinent to ask how one establishes and maintains such an environment. Many of these characteristics have been associated with what have become known as *constructivist classrooms* (Lo, Wheatley & Smith, 1994; Simons, 1995; Wood, Cobb, Yackel & Dillon, 1993). A further follow up to this study would be to undertake observations in the mathematics classes of K - 6 teachers to ascertain the degree to which their espoused beliefs are evident in practice.

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Focussing on the Self as a Learner of Mathematics and Statistics

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A comparison of three consecutive implementations of a Mathematics and Statistics subject suggests that a focus on the self as a learner of statistics in addition to one on learning the course content may result in preferred outcomes. Modification of students' perceptions of themselves as learners can alter the acceptability of the experience of learning Mathematics and Statistics, whilst maintaining high performance levels. The body of the paper explores the exercises and manner through which the focus, on learning how to learn statistics, has been created and suggests ways in which the experiences of one class of students may be used to improve the experiences of successive generations of students.

Introduction

A constructivist epistemology of knowledge and the associated teacher-as-researcher research paradigm, have led to what this writer has experienced as a profound change in teaching practice and to the theme of this paper which is how teachers may incorporate a focus on learning how to learn Mathematics and Statistics into their teaching of the discipline.

Epistemology of knowledge

The key epistemological idea underlying the teaching and learning discussed in this paper, and the research process into it has been that knowledge is actively constructed (Von Glaserfeld, 1988) irrespective of the teaching pedagogy (Klein, 1995). What follows from this for the teacher is a need to assess what the student knows and how they have constructed that knowledge. This includes understanding the social context within which knowledge is constructed. The drive to know what knowledge has been constructed and how it may have been constructed has provided the impetus to finding and developing techniques which allow me, as the teacher, and the students to examine the processes involved in learning, and in learning Mathematics and Statistics. These techniques that will form the focus of this paper.

The reflective practitioner

How the techniques have been developed, as distinct from the impetus to have them, has been through the usage of a teacher-as-researcher or reflective practitioner research paradigm. Throughout the development and evaluation of successive implementations of a tertiary preparation course in Mathematics and Statistics, part of a program called Gateway, I have been involved in deliberate reflection upon what has transpired in the classroom. Qualitative and quantitative evidence to confirm or refute explanations as to what has been happening or experienced both inside and outside of the classroom has been gathered throughout the practice of teaching. The reflective process has been sometimes akin to ethnographic work relying on meticulous observation and description of events and interactions in the classroom. At other times the evidence to confirm or deny conjecture has been through questionnaires, the analysis of critical incidents, thematic analysis of written material gathered through student exercises or even independent evaluations. It has been this recent approach of being a teacher-researcher rather than a traditional researcher that has made a profound impact upon, and insight into, the teaching undertaken by the author of this paper.

Comparing Outcomes

To demonstrate that a focus on learning may result in comparable outcomes to those from courses which focus on technical content and processes a comparison of three implementations of a tertiary preparation mathematics and statistics subject is presented (and reported elsewhere in greater detail (Porter *et al.*, 1994)). The implementations could be characterised as:

Base implementation: This, the first implementation provided base level outcomes to enable later comparisons. This implementation used the same pedagogical techniques as subsequent implementations and was consistent with what various authors would consider as constructivist approaches to teaching. Pedagogical techniques included experiential learning, small group work, activity based learning, classroom discussion and other language oriented activities and video materials. The focus was on implementing a curriculum developed from the ongoing work of unpacking statistical experts' knowledge (Porter *et al.*, 1993).

Curriculum focus: The second implementation modified the curriculum of the first, adding units and deleting units, sequencing them in order to provide students with better explanations and/or experience of data which could highlight the theme that, Statistics is about variability. The prime focus was on the mastery of technical content and processes.

Learning focus: The explicit focus of this implementation required that students examine the learning experience in addition to identifying and mastering the statistical ideas. All three implementations involved an examination of the process of learning but in the third implementation this was made explicit to the students. Whilst in early implementations students were placed in situations which would demand that they be responsible for extracting from an activity the important statistical ideas, by the third implementation students were made explicitly aware that the process of taking responsibility was one to be learned as well as the Statistical material itself. They were to reflect upon themselves as a learner in the process.

In Table 1 it is evident that base Mathematics and Statistics performance improved greatly in the curriculum focussed implementation, despite students' poorer pre-test scores. This was maintained when the orientation of the course shifted to include an explicit learning focus. The improvement in submission rate is also of interest with students seemingly more prepared to submit work. The decline in completion rate in the learning focussed course was explained in terms of outside influences eg eviction, illness interrupting students' study.

Table 1: Mathematics and Statistics Outcomes

		Implementation		
		Base	Curriculum focus	Learning focus
Number commencing		64	79	61
Number completing		47	62	37
Mathematics	mean	82.4	64.0	62.8
Pre-test	std	14.7	20.3	16.2
Course Completion %		73	78	61
Assignment submission rate %		76	95	100
Assignment	mean	7.4	8.8	10.7
mark /15	std	3.8	2.9	1.8
Statistics part	mean	10.8	8.6	9.2
of exam mark /15	std	5.1	2.6	2.4
Overall mark	mean	71.3	78	81.5
/100	std	13.6	10.1	6.3

The three implementations were sequential and did not follow an experimental research paradigm with subjects being randomly allocated to different conditions. The teacher-as researcher paradigm used the reflective process and associated evidence to direct changes to teaching. As such, outcomes following the change are not interpreted as causative rather as viable reasons for the successful change in teaching. The evidence suggests that it is viable to take time in the mathematics and statistics classroom to focus upon issues related to learning as this does not necessarily detract from the outcomes of learning the discipline. This is important as far too often in teaching there appears to be too little time to cover the content let alone explore other issues such as learning.

As can be seen from Table 2, the curriculum focussed implementation whilst resulting in improved grades was accompanied by a decline is students' evaluations of the course. For the subject, which was part of a tertiary preparation program, to be considered a success it was seen as necessary for students to be more inclined to undertake further study rather than less inclined, mastery of content alone was considered inadequate. One of the advantages of including a focus on the learning process was that there was an improvement in student's attitudes toward the learning experience.

Table 2: Percentage of students responding to two most positive rating categories in formal evaluation.

		Implementation	
	Base	**Curriculum**	**Learning**
The lecturer sequences material in the unit very illogically (1) to very logically (5)	76	47	97
The lecturer stimulates me to think about the subject never (1) to always (5)	70	51	76
The lecturer has demonstrated understanding this subject very poorly (1) to very well(5)	86	85	91
I have felt enthusiastic about attending lectures in this subject never(1) to always (5)	63	45	88
The subject material has usually been very uninteresting (1) to very interesting (5)	55	40	82
The lecturer has presented the material very unclearly (1) to very clearly (5)	60	38	79
This lecturers interest in assisting me to learn is non existent (1) to very high(5)	70	66	88

Focussing on Learning

Garfield (1995) in listing the implications of constructivist theory for the learning of statistics concludes "students should be encouraged to assess their own learning as well as their notions of how they learn, by giving them opportunities to reflect on the teaching/learning process." This has certainly resulted in more positive experiences for these and subsequent classes of students.

The techniques for drawing the focus to the process of learning may be subtle or obvious, take moments or be part of an extended exercise, they may be indirect as a by product of a statistical exercise or deliberately induced. The focus may be on how student knowledge itself is constructed or on the affective nature of the process.

Whatever techniques are utilised however it is important for students to be aware that the learning process is considered as important as learning the material itself, as learning should continue when the teaching instruction stops. Without this being made explicit, students may, as with the 'curriculum" implementation of the course, fail to realise the tremendous advances that they have made, not because of the statistical concepts they had mastered, but because their approach to learning would allow them to master many more.

Techniques for providing the focus on learning how to learn

Setting the scene

In week one the students were asked the simple question 'What is the name of the course that you are about to undertake?' the unanimous response from all three classes of students was 'Mathematics and Statistics' rather than 'Learning Mathematics and Statistics'. This perception was rectified through an initial lecture explaining the approach to teaching and learning that was to be adopted. Usually a couple of students will indicate at the end of class that they 'feel better or more relaxed' after this explanation, their Mathematics anxiety having diminished a little.

During the introduction to the subject students were informed of the approach to learning that had been adopted.

> My teaching is premised on the assumption that each of you as learners will take what I have to say and put it together to form different constructions of knowledge, different perspectives on that knowledge. This premise is a basic tenet of constructivist theory and it contrasts with the notion that each of you as learners are passive vessels into which we pour knowledge, store it until it is again released in its uncontaminated form. The communication of what it is you understand is crucial if I am to help you develop appropriate representations of that knowledge. To aid that communication I have lots of class discussion and minor writing assignments...Approaching teaching from the perspective of constructivist theory, I require you as students to participate in activities and to make sense out of the ideas that those activities engender. In our work we will attach labels to ideas and repeat them week in week out so that you come to understand the language and ideas of statistics. I will not give you a lecture, sets of notes, readings...I want you to think about what you observe in class. I want you to piece together a jigsaw puzzle of ideas, to develop an understanding of what statistics entails. What are the most central ideas, what ideas are of lesser importance. You will need to learn to describe what it is we do and what we have observed, talk about it and write about it. (Introductory lecture)

The students were also introduced to the issues that confronted students in the previous two implementations. These teacher identified issues involved students: trusting their own ability to think about and do mathematical and statistical work; ascertaining the relevance of the material to their chosen discipline; ascertaining the personal relevance of the material to their lives; modifying expectations about the discipline when it was not as expected; dealing with uncertainty as characterised by 'tell me what it is I have to know, give me a rule or a formulae to follow'; being able to reflect upon what they had experienced in an appropriate manner, that is to be able to focus on the most important concepts or ideas rather than focussing on the easy concepts or the hard concepts, or upon the specific example rather than the generalisation or vice versa; in being able to recreate ideas rather than simply remembering them; and, being able to write about statistical ideas.

In the first class students are also given, in groups, a completed Maths test and asked to identify the strategies they could use for checking the answers were correct. This was a useful way of introducing students to the notion that there is more than one way of completing a task correctly although sometimes some methods may be easier or more precise than others. Many verbalise how they had always thought that they were wrong if their strategy was different to that being taught.

Exercises for focussing on learning

The snake: In the first or second class, students are also asked to draw a winding snake or stream (Pope, 1993) and on it to chart the various incidents that previously influenced their learning culminating in the experience that led to them applying to enter the Gateway program. This exercise was to focus them upon those incidents that had helped or hindered them in previous studies. At the end of the subject this was repeated with students being asked to focus on the journey through the subject, see for example Figure 1.

Personal challenges: Students are also asked to anticipate "What are the personal challenges that you think you will need to face and conquer in order to learn (1) your Mathematics?; (2) your Statistics? and, (3) the material in the other units in Gateway?" This exercise repeated at the end of the subject provided material which is then used at the commencement of the subsequent implementation.

Sharing student histories

The snake exercise and the personal challenges exercise have both generated data that has been taken back to subsequent classes. For the tertiary preparation groups anxiety appears to be the highest when the first piece of assessment is due. Students, feeling uncertain, want to know what they have to learn and what they are to give back to you. Using the stories gathered from previous students the new class is alerted to the commonality of their experiences, the paths that students follow, the range of emotion they may encounter and the fact that they, with perseverance, emerge at the other end of the course. A number of examples such as that in Figure 1 are shared.

Reflecting on one's current learning

During tests: It is the middle of a Maths test and the tension can be felt. Students are asked to stop, close their eyes and note their anxiety level. They reminded that the tests are to help them assess how much they have learned, why are they anxious?

Mid Course: Similarly, midway through the course at the start of class the instruction is as follows 'Close your eyes, breathe deeply for a few moments...now think about the challenges you have to face in learning your Mathematics and Statistics...pause...What are they...pause...Take a few minutes to write them down. Would anyone care to share what your challenges are and the strategies you are using to overcome them?' Strategies to deal with problems such as 'My partner is complaining that I spend to much time studying' are identified. Strategies such as 'I promised my partner that I would take a week holiday when I finished'. Other strategies to deal with the difficulty in understanding the Mathematics may include 'finding another textbook' or 'doing some examples everyday'.

Morning Tea: A scheduled 'morning tea'/ social gathering to focus on learning issues was included in one implementation.

Assessment for Learning

What students learn is often influenced by the nature of the assessment tasks. The assessment tasks in this course are an integral to the learning process. Full marks are given for three or four pieces of homework, if submitted. They are to provide students with feedback as to the quality of their work, clarification of ideas, development of writing in order that they have developed the appropriate quality and depth and breadth of ideas that are required for their major assessment work. In recent classes many students have chosen to complete more than the required pieces of assessment. The accompanying activity sheets will often suggest that students reflect not only upon the activities conducted in class but the experience of doing them, for example, "what did you learn about yourself whilst working in a group situation?"

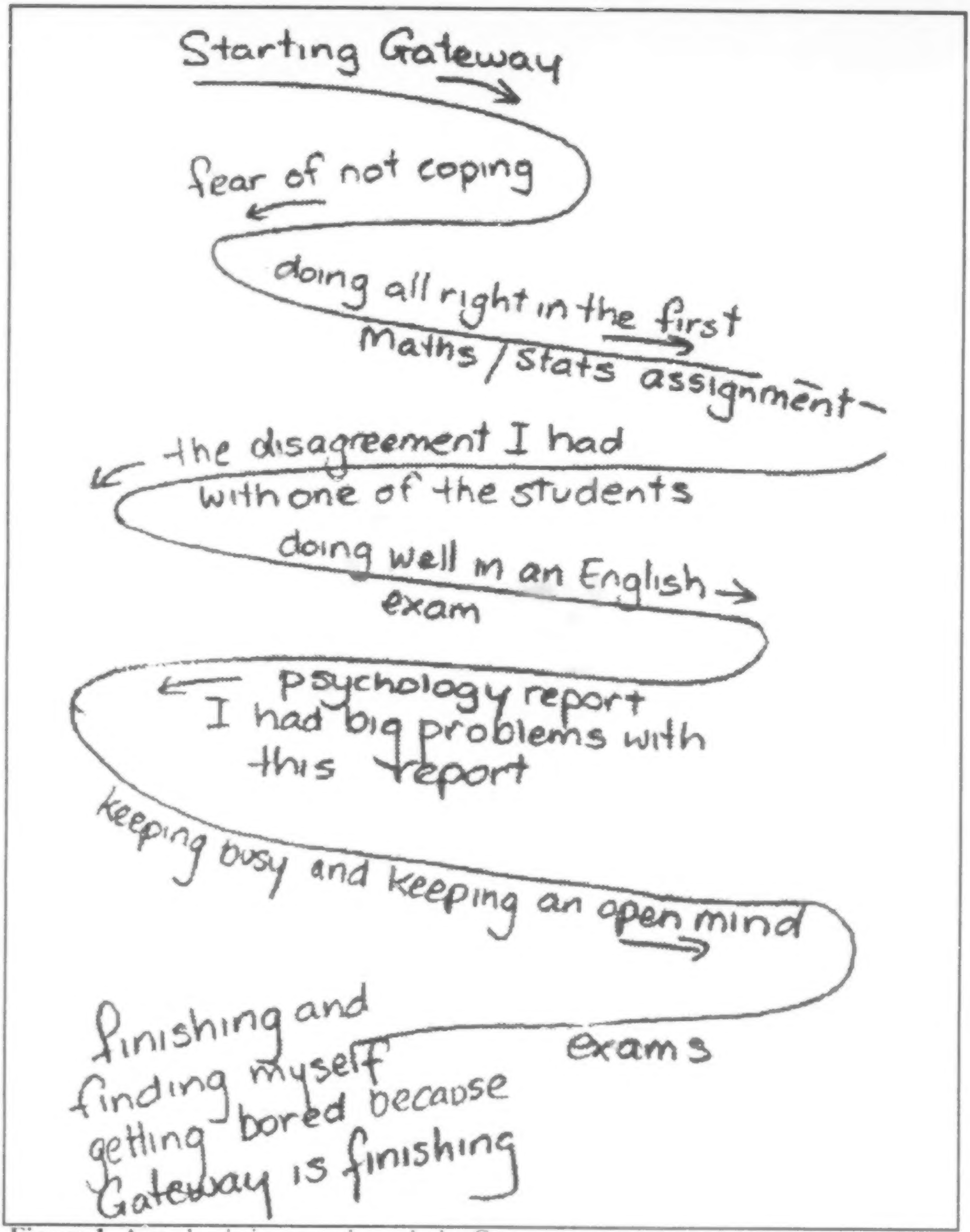
Starting Gateway
fear of not coping
doing all right in the first Maths / stats assignment -
the disagreement I had with one of the students
doing well in an English exam
psychology report I had big problems with this report
keeping busy and keeping an open mind
exams
finishing and finding myself getting bored because Gateway is finishing

Figure 1. A student's journey through the Gateway program

Class Discussion

How are you going? Student homework in these courses has required that students reflect upon what has transpired in the week's statistical activity (the statistical exercises chosen provoke a wealth of statistical ideas and data), what ideas were generated, what procedures they used and why they did so? How are you going? What was the purpose of the work last week? are good starter questions to get students to enter the debate or provide information regarding the state of their own learning. A recent example of a discussion, after a peer assessment session involved a student asking the question, 'Will you be giving us written feedback on our assignments?" Not satisfied with peer assessment, following a teacher generated set of criteria, observing other assignments which could provide ideas as to both content and layout, many students wanted to be told what was required by the teacher. This question was re-directed to illustrate a introductory week's discussion regarding the need for many students to be told precisely what to learn rather than to learn to select for themselves what is important in a situation, for the latter may invoke feelings of uncertainty and lack of trust in oneself. In the ensuing discussion some cited their awareness of the requirement that as part of the homework and peer assessment they were to identify for themselves what they believed was appropriate and to learn from the situation. Others become aware that they had self censored in their writing and learning "I didn't know you would want graphs or that we could put in our own data" . They had attempted to give the teacher what they thought the teacher wanted rather than responding to their own sense of appropriateness of material.

Providing a learning commentary during class exercises

Evidence of thinking statistically: Several of the class exercises provide the teacher with an opportunity to challenge students perceptions of themselves as mathematical or statistical thinkers. One exercise, involves students working through a worksheet developing ideas of location and spread, using both given criteria and self developed criteria to assess which they consider to be a good measure. When the answers are obtained, for example a zero for the sum of the deviation scores, or a large number for the sums of squares or an incomprehensible variance, students think they are wrong or stupid because they do not understand. Drawing attention to their thinking, indicating how it is that they are right and how they perceived the need for a better measure such as the standard deviation which they could understand, can be used to affirm the students as statistical thinkers.

Teacher talk:: Teacher talk can help. My own experience was of saying to the Professor 'now why do you use this technique instead of that' only to be told what I already understood 'well the first is easier to understand'. That teachers can be waylaid by the same thoughts, the same self doubts as the novices can be instructive.

Developing the teaching role

Using knowledge of the learning experience

Knowing that students experience anxiety, self doubt or uncertainty during the process of learning has not necessarily been used as an excuse to remove the experience. Teachers in this program consider that one of the most important aspects of learning is learning how to make decisions about selecting information from the volumes of information available. Those decisions take place in an atmosphere of uncertainty, unlike the task of rote learning material that has been ordained as needing to be learned. An important aspect of learning occurs when one is not told what to learn, but asked to identify what is important in a situation and justify why it is important. One of the classes in this subject involves students in sorting two sets of seashells in order to determine if there was a difference caused by a maritime accident? In exercises such as this students learn about statistical issues such as measurement, the need to operationalise definitions and how comparisons may be made. In addition students may learn about how their learning is motivated. Student involvement in this task generally mirrors students' preparedness to make decisions about data. They start hesitantly, then have period of active sorting and then ask 'what does one do with the broken shells?' or 'what do we do with these shells when there are only a couple?' Decisions are required on the part of the teams, they are decisions which are seemingly too hard involving too many choices and it is often at this point that some members will feel suddenly bored, unable to continue, in conflict with others. The parallel with students undertaking thesis work and the decision to quit or continue is striking, the question as to whether or not to continue the search for solution is one often asked by students when they experience self doubt or a lack of confidence. These are learning issues to be discussed.

Developing learning theory

For a teacher the process of examining what students have constructed, how they have learned and asking 'how could they have come to that construction?' is an illuminative process. It enables the development of personal theories of teaching and learning. For example, using the constructivist notion that reflection is an important part of constructing knowledge, students in these classes were asked to reflect upon what has transpired in class and to write about it. Little was I aware that for some the process of reflecting was one that in the classroom, at least, needed to be learned. Students needed to learn to focus on a coverage of important ideas rather than easy or hard ones, they needed to know how to remain focussed on the statistical ideas rather than be diverted to for example the enviromental theme embedded in a given topic. They needed to learn to focus not only on the detail of their own data but to recognise the generalisations possible from all data gathered. They needed to identify that the inability to write was not necessarily the inability to have good thoughts about the topic. For this teacher asking 'why did this student write this as homework?' and attempting to find the cause, rather than dismissing it as the student's problem or lack has led to many changes in teaching practice in order to accomplish certain ends. Developments in the teacher's personal theory of teaching and learning have occurred. An increased vitality in teaching has been experienced as each piece of assessment (once drudgery to mark) has become an new piece of data affirming or denying conjectures about how learning is taking place and suggesting further changes to practice.

Conclusions: Which exercises and when?

Since the first three implementations were completed several other groups of students have been taught statistics in a learning focussed manner. These groups include: more tertiary preparation students, Arts students, Law students; and staff and students in non credit introductory statistics workshops. Reflective practice, observing the impact of activities upon students, getting to know the interests of students groups, constantly listening for students' voices, asking them for their reactions, observing body language will all direct the extent to which the learning focus should be maintained and which exercises are best undertaken. Student commentary suggests that it is no one exercise that is instrumental in achieving that connecting experience, different students have insights with different exercises. A variety of interactions is warranted. What does appear important is that throughout the course there is a safe and accepting environment where there is ample opportunity for discussion of the subject matter and the experience of learning it. Many comments by students start "It fell into place when I overheard...".

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Patterns, Language and Algebra: A Longitudinal Study

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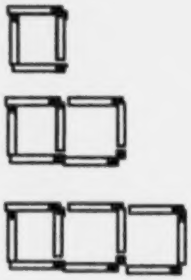
This paper reports on the some findings of a longitudinal study into children's understanding of number patterns by considering the changing use of natural language and symbolic notation over time. The study had its roots in an earlier study that provided a classification system for the responses. The two findings reported here suggest that responses to pattern stimulus items change over time and that the use of natural language in the highest category seems to be a necessary precursor to the emergence of algebraic notation.

Introduction

Following the popularisation of using number patterns as a contextual vehicle for the initial development of algebraic ideas (Mason 1985; Pegg and Redden 1990a; NSW Syllabus 1989; Romburg 1989; Australian Education Council, 1991) some research output has begun to report the efficacy of the approach (Pegg and Redden 1990b; Arzarello 1991; MacGregor and Stacey 1993). This research has used a variety of instruments and strategies to identify response categories to pattern stimulus items. The next phase of this research might investigate the dynamics of children's development in this area of cognition. This paper is based on some findings from a survey of 1435 children (Redden 1994a; Redden 1994b) and investigates some further issues in the context of a longitudinal study.

An analysis of the survey data (Redden 1994a) has identified an hypothesised sequence of development in the natural language used by children to describe number patterns in response to both geometrical and numeric stimulus items. An example of the items used is represented in Figure 1.

Here are some chains
of Matches



(a) What is the next term in the pattern?

(b) Describe a general rule for the pattern in natural language

(c) Calculate the value of an uncountable term (eg. n=80)

(d) Write their rule in the symbolic notation of mathematics

Figure 1
A number pattern Stimulus Item

In brief, there were four major response categories identified for question (b) above which are listed here with a brief descriptor of the category and a sample response.

1. **Inappropriate** response. This reflected no attempt to answer the question or a failure to understand any aspect of the question. e.g., "It depends."
2. **One example**. This class of response gave the value for a specific example rather than a general description. e.g., "10 squares needs 31 matches."
3. **Successive** description. Respondents only made use of the dependent variable. e.g., "You start at four add three every time."

4. **Function**. This group of respondents described a relation between a dependent and independent variable. e.g., "The number of squares times three plus one gives the number of matches."

This sequence of development was validated by a number of procedures which included investigating the modal response category for each age cohort, comparing the response categories with categories identified in other studies (Stacey 1989; Ursini 1990; Meira 1990), and by using the SOLO Taxonomy (Biggs and Collis 1982) as a theoretical framework to analyse the complexity of the response. Further, the categories of symbolic notation were found to be strongly associated with the categories of natural language. The categories for classifying the symbolic notation were reduced to four for the purposes of further analysis. They were:

1 **No attempt**. No attempt was made to use symbolic notation.
2. **Operation Symbols**. Attempts at symbol use were restricted to those commonly used in arithmetic. e.g., "3 x 4+1".
3. **Arbitrary** use of letters. This category included a range of responses including replacing every number with a letter (a+b=c) or trying to incorporate some form of repetitive notation e.g., "a+3+3+3+3.......... forever".
4. **Algebra**. Successful use of algebraic notation. e.g., "y=4+3(x-1)".

In fact, the strength of the association between natural language and symbolic notation, and its existence across a number of pattern items, suggested that the ability to use natural language to describe number patterns may be a necessary precursor to being able to provide adequate algebraic descriptions of the relationships inherent in the number patterns.

To further investigate these and other issues a longitudinal study was undertaken over a two year period. Two of several research questions of the longitudinal study are discussed in this paper.

1. Does children's ability to express generality using natural language change over time?
2. Does the symbolic language of algebra follow the natural language development?

Following a brief outline of the research method, an analysis in relation to each of the research questions is presented. The paper concludes with a short discussion of some implications of the findings.

Method

Twenty six children from years five six and seven were selected as a stratified sample of the 1435 children included in an earlier survey. The stratification reflected both age and ability. The children of the sample were interviewed five times at six monthly intervals over the two year period. These are referred to as studies 1 to 5. During the thirty minute interviews they were asked a number of questions about three pattern stimulus items of varying complexity. The discussion here will focus on the responses to the items at the middle level of complexity. A number of items at this level of complexity were constructed of which the item in Figure 1 is typical.

The interviews were audio-taped and later transcribed by the researcher for later analysis in conjunction with the written responses of the children. The interview structure was developed from that used by Booth (1984). The planned questions were supplemented by additional probes to clarify meanings and understandings when ambiguity existed.

The suitability of a longitudinal design for this study is detailed elsewhere (Menard 1991, Cohen and Manion 1989). However, there are a number of difficulties associated with such a design such as institutional change (some children changed schools), sample mortality (five children left the study) and the measurement effect of repeated interviews. Cohen and Manion (1989) argued that repeated interviewing often induces the subjects to respond in a manner different from that which would be provided in a more natural setting. In this context the researcher may be providing cues as to which type of responses are considered desirable. However, in this study it is not what induces the change that is of interest but rather the nature of the change. There are a number of influences that may induce the change, including maturation of the subjects, experiences in mathematics classes at school, and participation in this study. The way in which these contribute to the change in children's responses is beyond the scope of this discussion.

Analysis

Overview of Change

This part addresses the question of:

> Does children's ability to express generality using natural language change over time?

In determining whether or not there was a general change in children's ability to describe number patterns, attention was to be focused on the natural language responses. The Rasch model has been used here to assist in the identification of changes over time in students' ability to respond to the stimulus items. This was facilitated by an analysis of the item difficulty delta values reported by the item analysis statistic, "Stat-Tau", option of Quest (Adams &Khoo 1993, p.34).

Traditionally the Rasch model has been used to analyse students' responses at a single point in time on a number of items of variable difficulty. The Rasch model is capable of reporting δ_i values for each item. The δ_i values indicate the difficulty of that item in Study *i* compared to the other items in the set (Adams and Khoo 1993). If a set of identical items is presented to children over a period of time then the item difficulty (δ_i) values reflect changes in the children's ability to respond to the items. Hence, it is hypothesised that an item in Study 1 will have a higher difficulty value than the corresponding stimulus item in Study 2 and so on until Study 5. That is,

$$\delta_1 > \delta_2 > \delta_3 > \delta_4 > \delta_5$$

Once the values of δ_i are established their sequence can be simply observed; however, some test of significance of the differences needs to be applied. A number of alternatives in structuring such a test are available. The existence of a difference between the five delta values could be investigated, (i) by considering the null hypothesis that:

$$H_0\colon \delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5,$$

(ii) by investigating stepwise differences using the null hypotheses:

$$H_0\colon \delta_1 = \delta_2, H_0\colon \delta_2 = \delta_3, H_0\colon \delta_3 = \delta_4, H_0\colon \delta_4 = \delta_5$$

or (iii) by investigating the significance or otherwise of the total change over the two-year period. The null hypothesis for this investigation would be:

$$H_0\colon \delta_1 = \delta_5$$

It was decided to use the first and third methods for investigating significant differences since they would provide answers to the questions:

> Does any significant difference exist? and,
>
> Was there a significant difference over the two-year period of the study?

Method (ii) was seen to have less potential since the repeated testing would result in increasing the chance of a Type I error, and since the SOLO Taxonomy predicts some instability in responses during development from one level to the next it would be possible for reversals in delta values to appear. These reversals would be most likely in adjacent studies with little development time between them.

The test for homogeneity of effect sizes (Hedges and Olkin 1985) was used for investigating the significance or otherwise of differences among a set of item difficulties. It is clear that there is a general shift from the lower levels of response to the higher levels over the two-year period (see Table 1). There is also a ceiling effect on many of the children's responses as they were already responding at the highest level at the beginning of the study or they achieved responses at the highest level during the study. Quest only uses cases and items that do not have a perfect score. Some of the infit statistics indicated a number of reversals, which were to be expected for two reasons. The first is the volatility of responses predicted by SOLO as transition between cognitive structures occurs. Secondly, the items lose power to discriminate as the ceiling is approached and a reversal is therefore more likely to occur. The fit statistics are reported elsewhere (Redden 1995).

Table 1 reports the frequency data for the natural language responses. A consistent pattern of growth is evident across the two years of the study. The ranks of the item difficulty values (δ) are consistent with the predicted pattern, i.e.,

$$\delta_1 > \delta_2 > \delta_3 > \delta_4 > \delta_5 .$$

Further, on investigating the homogeneity of effect sizes, using the Q value described by Hedges and Olkin (1985), it was found that the differences are significant at the 0.01 confidence level. (Q=20.599, df =4). Hence the null hypothesis that

$$H_0 : \delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5$$

is rejected. Additionally, the null hypothesis of equality of delta values between Study 1 and Study 5 was rejected. (Q=15.738, df=1 and P<0.01).

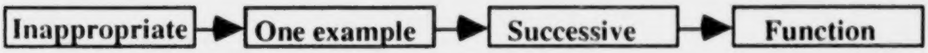
Table 1
Frequency of Responses and Item Difficulty Values

(n=21)	IA	IEG	SUCC	FUNC	Difficulty (δ) SD	Rank of Difficulty δ
Study 1	8	5	3	5	1.14 .29	1
Study 2	3	4	5	9	.09 .31	2
Study 3	2	0	11	8	.08 .37	3
Study 4	2	0	6	13	-.49 .37	4
Study 5	1	0	6	14	-.82 .40	5

From the discussion in this part there is strong evidence in support of the proposition that:

> An improvement in the children's ability to express generality over the two years of the study can be detected?

While there is clear evidence of general growth, what can be said about the actual path taken by the children in the sample? The suggested hierarchy of responses reported earlier was:



Of the 84 transitions made between categories ten were classified as reversals since they failed to follow the predicted sequence by responding at a level below that of a previous response. However, all reversals with the exception of two were later compensated for. Thus it is argued that the predicted hierarchy reflected the general growth pattern of the children in the study and that the reversals reflected the instability of response patterns as the cognitive structures are changing.

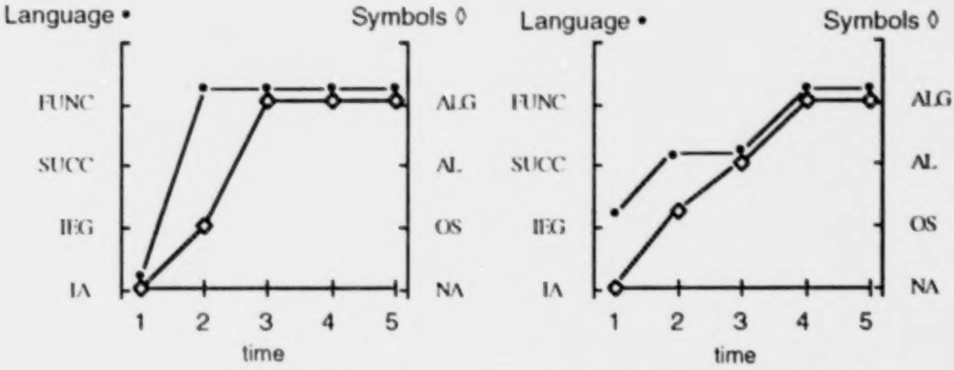
Relationship between Natural and Symbolic Language.

Associated with each natural language response the children were asked to provide a symbolic language response. It has been reported elsewhere (Redden 1994b) that there was significant association between the categories of natural language and the categories of symbolic language. In particular these pairs of associated categories are: **(Inappropriate, No attempt), (One example, Operation symbols), (Successive, Arbitrary use of Letters)** and **(Function, Algebra).**

In order to investigate the question:

Does the symbolic language of algebra follow the natural language development

individual profiles for each child were developed which mapped their responses to the request for both natural and symbolic language against time. Two such profiles are presented here as Figure 2.



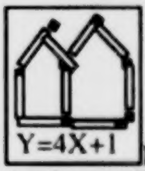
Response Profile of Brett

Response Profile of Anthony

Figure 2

In both the profiles in Figure 2 the higher level of natural language appeared before or at the same time as the associated level of symbolic language. Of the 315 pairs of responses recorded in the study all but six followed this pattern. In 92 cases the level of symbolic response was below the level of natural language response, while in 217 pairs the levels matched as the earlier study predicted

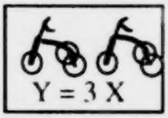
While the proposition that the symbolic language of algebra follows the natural language development is overwhelmingly supported by the frequency data the protocols provided insight into two issues related to this issue. The first is the issue of how do children provide accurate algebraic descriptions of patterns without functional relationships being first expressed in natural language. It will be recalled (see Figure 1) that between asking for a pattern description in natural language and symbolic language the children were asked to apply their rule to an uncountable example. It was this question that forced one child to reconceptualise his rule as the following protocol shows.



Researcher: How did you do Part (A)?
Anthony: For every house starting off with 1 house you need 5 matches and to add another house you need 4 for every one then I counted up those there and added 4 more
Researcher: Added 4 more? (Those there are the number for an extra house) So you have counted matched for 3 houses and added on 4 to get 17.
Anthony: Yes.
Researcher: Part (B). You have used a different rule?
Anthony: Starting with one house add 4 each time......... (goes to Part (C))............ 37 x4+1=149.
(then returns to Part (B) and changes description).......... Times the number of houses by 4 and add 1.
Researcher: Why are you doing that?
Anthony: I did Part (C) first then on a scrap of paper I got the same answer that I had (to Part (A))
Researcher: You got 17?
Anthony: Yes I had a go then I thought up another way and got a new rule
Researcher: So you did Part (C) then went back and did Part (B)?
Anthony: Yes.
Researcher: Part (C) forced you to think about it in another way?
Anthony: Yes.

It is clear that Anthony was forced to reconceptualise his pattern description by the need to calculate an uncountable example. If children are to be taught to express generality it may not be enough to merely provide the opportunity to express generality. They need to be given the opportunity to see the value of expressing generality.

The second issue that arose from the protocols was that of the permanence or otherwise of the relationship between natural language and symbolic language. There was limited evidence of children becoming so competent in their use of algebraic symbols that their search strategies for pattern descriptions took place using that symbolic language. Two children, Peter and Anthony, began to use algebraic notation instead of natural language. However, this practice appeared only towards the end of the longitudinal study. Anthony appeared to use algebraic language because of its convenience, while Peter seemed to conduct his search for the rule in symbolic language. In Study 5, Anthony initially asked for permission to use symbolic language.



Researcher: Part (A)?
Anthony: 3 x 4 = 12
Researcher: Part (B)?
Anthony: Can I just use Symbols?.... 3 x T = N
Researcher: What does that mean?
Anthony: Three times T equals N.....T is the number of triangles and N is the number of wheels.
Researcher: Part (C)?
Anthony: 3 x 55 = 165
Researcher: Part (D)?
Anthony: (as for Part (B)).

Researcher: If you had to write it in English what would you say?
Anthony: The number of tricycles times three gives you the number of wheels.

As Peter's confidence grew more reliance was placed on the symbolic notation.

□	1	2
△	2	5

Y=3X-1

Researcher: Part (B)?
Peter: There it is times 2 , then times 2 + 1, times 2+ 2, Times 2+ 3......Times 2+ 10 +4...Can I write it out....hey....2 times 4 is 8..+4 no plus 3....oh....so 2x2 +1...Yes I've got it....(writes (5x2) + (5-1)=14)....so its Xx2 +(X-1)=y.....yes that's it now to do it in an English sentence. The computer multiplies the top number by two and adds one less than the top number to its answer.

By Study 5 he was writing the pattern descriptions in symbolic notation and could not see the point in repeating the description in natural language. In other words, algebraic notation had become his preferred means of communication.

Conclusion

This paper has reported the results of a longitudinal study that attempted to probe the nature of change in children's response patterns using both natural language and symbolic notation. It was also found that the natural language descriptions of number patterns seem to be a necessary prerequisite for the emergence of algebraic notation as a means of describing the generality of number patterns. It may well be that the necessity for articulating accurate natural language descriptions is a transitory phenomenon.

While some children were describing patterns at the highest level at the beginning of the study, others did not achieve the highest level of response during the two years of investigation. It can be seen that the concept of expressing generality in the form that facilitates algebraic notation is not automatically available to all children who may be exposed to the Year 7 and 8 mathematics syllabus in N.S.W.

The other focus of this longitudinal study was to consider the anomalies identified within the data when it is compared to the theoretical relationships identified. These anomalies included children being able to provide correct answers to uncountable examples, and provide algebraic notation without providing functional descriptions in natural language. It was found that in most cases children are forced to reconceptualise the pattern description when confronted with a problem they cannot answer with their initial response. It would seem that in some cases it was not enough to ask for a generalisation, but the need for, and power of a generalisation needs to be demonstrated.

The study pointed to the elusive nature of children's cognition. In particular the potential for a lack of congruence between children's externalised responses and their 'hypothetical cognitive structure'(Biggs and Collis 1982, p. 22). The variability in the response patterns indicated a number of cases of mismatch to be identified and clarified. This process was facilitated by two design features. The first was the multiple data components collected for each stimulus item, which allowed apparent inconsistencies among responses to each component to be analysed more deeply. The second feature was the interview environment, where a variety of clues as to the child's true intention could be gleaned. This interview environment can be seen as a dynamic interaction between subject, the researcher and the stimulus item. This is contrasted with the more static environment (from the researcher's perspective) of survey data.

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Goal Transformation In Professional Development Programs

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One strand of curriculum implementation research concerns itself with the faithful implementation of an innovative curriculum. From this perspective, it is reasonable to suppose that the successful implementation of a curriculum change will be significantly affected by the congruency between the goals of the curriculum change initiative and the goals of the professional development program designed to achieve that change. There are at least five stages in the development and implementation of a professional development program designed to support curricular reform. These can be identified with the actions of specific individuals: Program initiators; program developers; trainers of workshop presenters; workshop presenters; and, teachers. The goals of the curriculum and consequently of the professional development program must be reinterpreted at each stage. This study documents the ways in which the goals of a professional development program were transformed in the course of its development and implementation.

The introduction of the *Curriculum Standards Framework: Mathematics* (CSF) (Board of Studies, 1995a) into the Victorian education scene has lead to a number programs and publications whose aim it has been to explain the CSF and smooth its introduction. The professional development program *Linking Mathematical Instruction and Assessment through the CSF*, a program put in place by the Victorian Board of Studies in 1995, and its related publication *Mathematics assessment activities using number tools and procedures* (Board of Studies, 1995b) are examples. This professional development program has provided the opportunity to research some key concepts to do with curriculum implementation and professional development.

Since we are exploring the transformation of goals, which exist only as statements, the careful use of terms is imperative. In this study,*Curriculum* has been taken to be a plan of the content, the instructional actions and the learning experiences intended. *Implementation* of curriculum involves the putting into place or into action of that curriculum. This term assumes that the curriculum has some concrete nature, so that a direct check is possible to ascertain whether or not it is in place or in action. The distinction between the *intended* curriculum, the *implemented* curriculum and the *assessed* curriculum is at the heart of the CSF implementation, since the CSF aims to effect curricular prescription through the specification of learning outcomes. *Professional development* has been defined as the processes that develop or improve employment related knowledge; in this case the attitudes or skills of teachers. *Implementation* of a professional development program, is taken to involve the stages of *conception, development* and *practical implementation. Conception,* in this case, involves identifying a need to be met by the intended program, *development* involves forming a plan of actions to meet the identified need, and *practical implementation* involves the putting into place or action of that plan. *Assessment* has been defined as "the process of gathering evidence about a student's knowledge of, ability to use, and disposition toward mathematics and of making inferences from that evidence for a variety of purposes" (NCTM, 1995, p. 87). This definition is broader than definitions of the past which have been analogues of terms such as testing, measurement and evaluation.

The structure and the focus of *Linking Mathematical Instruction and Assessment through the CSF*, and the current climate of change in assessment practices, made it eminently suitable for an investigation into the transforming of goals in the delivery of a professional development program. It is clear from the literature (for example Boomer, 1989; Fullan and Pomfret, 1977), that the maintenance or transforming of goals in a professional development program affect the extent to which a given curriculum change is implemented. In particular, the number of stages involved in this program's implementation, allowed for ample examination of the goals at each stage, and for changes both of an overt and of a subtle nature in the way these goals were implemented and understood. The five stages of development and implementation can be identified with the actions of persons engaged in the following roles: program initiators, program developers, trainer of presenters, presenters and teachers.

Theoretical Context

Curriculum implementation

Much research has taken place regarding the conditions that allow or restrict the implementation of curriculum. There has been considerable discussion regarding whether the term "implementation" is completely appropriate. It has been suggested that the word might better be replaced by "enactment", given the acknowledgment that wholesale insertion of curriculum is impossible, and that there will always be adaption by the players in the process of change. As Doyle and Ponder (1978, p. 4) state: "Teachers adapt, rather than adopt innovative practices". Generally, however, the term "implementation" continues to be used, but with the aforementioned qualification kept in mind.

Planned changes in curriculum usually involve the altering of one or more of the following aspects: goals, organisation, role of the teacher, content, instructional strategies, classroom management, materials or evaluations (Snyder, Bolin and Zumwalt, in Jackson, 1992, p. 402). A number of factors can impact on the extent of the fidelity of implementation of a curriculum change. Such factors are considered significant from both a fidelity perspective and from the mutual adaption view (Snyder, Bolin and Zumwalt, 1992, p. 416). Fullan (1982) relates several of the factors capable of affecting implementation to the *characteristics of the change*, specifically: the need and relevance of the change; clarity; complexity and the quality and practicality of the program. He identifies the following factors as *characteristics at the school district level*: the district's history of innovative attempts; the adoption process; district administrative support; staff development and participation and time-line and information systems (evaluation). *School level* factors affecting curriculum implementation are listed as: the role of the principal; teacher-teacher relationships and teacher characteristics and orientations. Factors of an *external* nature are to do with government agencies and external assistance.

Clarity is one of these factors that is said to influence the implementation of curriculum. According to Snyder, Bolin and Zumwalt (1992, p. 416), clarity, in this sense refers to "the understanding of goals and means of innovation by users". It has been found that the greater the understanding of the goals of an intended curriculum change and what is to be gained from its being taken up, the greater is the magnitude of implementation. The importance of the clarity of goals has been acknowledged elsewhere as a factor of influence when curriculum is being implemented. Fullan and Pomfret (1977) and Hall and Loucks (1981, pp. 46-47) when referring to different studies into the implementation of curriculum change, refer to the lack of clear goals by program facilitators as mitigating factors retarding the success of their implementation.

Furthermore, it has been found by Bird (1986) and acknowledged by Snyder, Bolin and Zumwalt, in Jackson (1992, p. 417) that the process of adaption actually enables participants to clarify their understanding of the philosophy and goals of the intended curriculum change, and that of their own goals. The analysis which follows was motivated by our acknowledgement of the centrality of goal recognition and goal formation in the process of professional development.

The need for a clear understanding of the goals of a proposed implementation of curriculum or professional development program was specifically referred to by Boomer (1989). He refers to the difficulty in maintaining a clear understanding of the resons underlying professional development programs, and hence the goals, in the "controlled process approach - system initiated" model: "No matter how well the ELIC tutors control, monitor, and insist on the grasping of principle, the process eventually loses its grip as it gets further from the enlightened source. It is a problem at the heart of the central working of any curriculum document, textbook, or innovative program"(p.167).

Boomer's use of the term "enlightened source" appears to challenge the legitimacy of practitioners re-interpreting program goals. We see this interpretive process as inevitable. Individuals have a professional responsibility to reconstruct program goals in forms that reflect local issues and priorities. The essential point is that the interpretation of the goals of a curriculum change initiative, and the congruent goals of the professional development program designed to achieve that change, play a significant part in its chance of successful implementation.

Changes in mathematics curricula and assessment in Victoria:

In Victoria, the *Curriculum and Standards Framework (*CSF*)* (Board of Studies, 1995a) has been developed with the National Mathematics Profile as its basis. The CSF is also an example of the world wide trend towards "accountability of schooling", including an attempt to measure what is learned using outcomes (Stacey, 1994). In accord with the world wide change to the use of a variety of assessment practices, the Victorian Board of Studies (BOS), the body charged with the implementation of the CSF in Victorian schools, is encouraging teachers consider the use of alternative assessment techniques to measure student learning using the CSF.

The Victorian Board of Studies responded to the concerns of teachers and schools with a pilot professional development activity, publication of a related book, and a state-wide professional development program. The pilot program was conducted in seventeen schools in 1994. Teachers, assisted by tertiary mathematics educators, developed assessment activities linked to the CSF. The book, entitled *Mathematics assessment activities using number tools and procedures* (Board of Studies, 1995c), detailed in an annotated form, a selection of the activities that were developed during the pilot program. The book has now been used as a resource for a professional development program entitled *Linking Mathematical Instruction and Assessment through the CSF* (henceforth to be referred to as The Program), which was conducted at the district network level for teachers of Preparatory grades to Year 8 across the state of Victoria. The Program has been used as a basis for this investigation into the transforming of goals in the delivery of professional development programs.

Methodology

Aims

The aims of the investigation were twofold; firstly to describe the goals as understood by the initiators, developers, trainers of presenters, presenters and teachers involved in The Program; and secondly to describe the ways the key terms and phrases, which make up the goals, change from stage to stage of The Program's implementation. The investigation involved description and analysis of the language used by participants in The Program and how it communicated the nature of their goals, given that a participant's understanding of the goals impacts upon the way a program is implemented. Here, *participant* refers to an individual involved at any of the stages of implementation.

Data collection

Data was collected during interviews, questionnaires and observations of training sessions. It was also collected from a range of documents used in The Program. A summary of the data sources is in Table 1. In the development and implementation of this program, the developers and trainers of presenters were the same individuals. Accordingly, the two stages of implementation - developers and trainers of presenters - were combined reducing the five stages of implementation to four.

Table 1
Summary of data collection

STAGE OF IMPLEMENT -ATION	PARTICIPANT	WRITTEN MATERIAL	INTERVIEW	OBSERVATION
1	**INITIATORS/ POLICY MAKERS**	DOCUMENTS	INTERVIEW	OBSERVATION TRAINING SESSION
2	**DEVELOPERS/ TRAINERS OF PRESENTERS** *See note below*	DOCUMENTS	INTERVIEWS WITH TWO PRESENTERS	OBSERVATION TRAINING SESSION
3	**PRESENTERS**	Q'NNAIRES	INTERVIEWS WITH TWO PRESENTERS	OBSERVATION PROFESSIONAL DEVELOPMENT PROGRAM
4	**TEACHERS**	Q'NNAIRE	INTERVIEWS WITH FOUR TEACHERS	

A range in data sources satisfied the need in qualitative research to provide corroboration and comprehensiveness. In addition, the dangers of the exclusive use of self reporting techniques, such as questionnaires and interviews in data collection was acknowledged through the use of multiple sources to cross validate the data. Content analysis was undertaken for all data sources. Data from the interview transcripts, questionnaires, observation records, and documents were examined. All references to program goals and purposes, both explicit and implicit were collated.

In many cases the goals addressed a number of key points. In order to accurately and succinctly describe and summarise the goals, they were annotated with code letters according to the key points implicit in each. Table 2 shows a selection of the key points and letter codes. Table 3 shows an example of the annotation of goals with the key point code letters. The first code letter represents to the most dominant key point for that goal.

Table 2
Letter codes for key points - selection

KEY POINTS	CODE LETTER
assessment - in general	A
the CSF - in general	C
rich assessment tasks - development	T
relevance to teachers' classroom practice	R
linking	L
instruction	I
Reporting	r

Data collation and categorisation: A frequency analysis was conducted on appropriate data from the presenter and teacher questionnaires. Collated key words and phrases from completed questionnaires were entered into the Statview statistical analysis computer program or analysis. Results were used in conjunction with the descriptive data to compile accurate descriptions of the goals. Consideration of the ways the goals of the program are transformed was made by investigating the changes in meaning of key words and phrases, with respect to several key areas into which the goals have been categorised. Each of the four stages of implementation of the program are included: program initiators, program developers, presenters and teachers.

Table 3
Examples of annotating goals with key point codes

GOAL	CODED KEY POINTS
To assist schools to link assessment in mathematics to instructional decision making	L A I
To bring the CSF closer to teachers' classroom practice	C R
To enable teachers to link the results of assessment to the schools' current mechanisms for reporting	L A r
Assisting teachers to use the CSF to develop assessment tasks which address the questions: ⇒ What do these tasks tell me about what my students understand and can confidently do? ⇒ How can I build upon what I've learned about my students? ⇒ Do some students need additional help as we move on?	L C T A I

The key areas have been chosen because of the extent to which each has been addressed by the players in each of the stages of implementation. Alternatively an area of goal has been chosen because of the fact that it has *not been addressed* at one or more stage, and the possible impact that lack of being addressed might have on how a goal has been transformed. In the second case the identified goal arose from the research literature and the document analysis undertaken in this study. The key areas of goals investigated were:

- the linking of instruction and assessment;
- relevance or usefulness of the CSF;
- provision or development of rich assessment tasks;
- teachers sharing and discussing;
- a variety of assessment methods;
- outcome-by-outcome approach to assessment;
- link between assessment and reporting.

Choice of examples of goals: As this section of the investigation was concerned with how the understanding of words and phrases changes from stage to stage of the implementation process, the selection of the examples by which to characterize the goals at a particular stage was undertaken with the specific purpose of capturing diversity. The choice of extreme examples, in terms of their variation from the meaning of particular words and phrases from that employed by other individuals at other stages of implementation, allows the *extent* of the changes to be more readily considered. The additional selection of a small number of other less extreme examples provides opportunity for consideration of the more subtle nuances of meaning.

Results and Discussion

Key groupings of goals were chosen to investigate the changes in meaning of the terms and phrases in the goals at each of the four stages of implementation of the professional development program. Each table below conveys a description of the transformation of a particular key grouping of content expressed in the goals of the professional development program. An example of a goal from each of the four stages of its implementation is provided along with annotations to assist description of the changes in which are referred to. A discussion of the transformations is presented under each table. A further more general discussion follows. The key groupings chosen for inclusion in this paper were *the linking of instruction and assessment; relevance or usefulness of the CSF* and *teachers* *sharing and discussing*. The goal transformations associated with each grouping are set out in Tables 4, 5, and 6.

Table 4: *The linking of Instruction and Assessment*

EXAMPLES OF GOALS AS EXPRESSED AT EACH STAGE OF IMPLEMENTATION	ANNOTATIONS
INITIATOR *To improve teachers' understanding of important milestones in students' mathematical development , and their ability to move students towards and then beyond these milestones*	A goal directed at *all* teachers, showing an understanding of the terms *assessment* and *instruction* and the link between them
DEVELOPER *Putting a state-wide focus on the process (of) generating good assessment activities, which may be very well related to instructional activity; or stated another way: "linking their instruction and assessment .."*	A developers' concern is shown for the process of encouraging teachers to develop rich assessment tasks - the assessment vehicle being introduced.
PRESENTER *To provide(teachers with) exemplary materials, and show how they link in with assessment.*	The emphasis is moved from 'instruction' in general as described in the first two stages, to the *elements* of the instruction process - in this case, *exemplary materials*.
TEACHER *To relate the CSF to what we are doing in our classes, particularly in reference to assessment and recording*	The term 'instruction' is changed to the broader phrase of 'what we are doing in our classes'.

Discussion: Differences in goals in this category were differences of focus and of scale. The progression in scale goes from general decontextualised statements (Initiator), through statewide practice (Developer) to the provision of exemplary materials (Presenter), and finally to the level of "what we are doing in our classes" (Teacher). The change in focus appeared to be a shift from seeking to improve teachers' practice to relating the CSF to "what we are doing."

Table 5: *Relevance or usefulness of the CSF*

EXAMPLES OF GOALS AS EXPRESSED AT EACH STAGE OF IMPLEMENTATION	ANNOTATIONS
INITIATOR *To show teachers in the field how other teachers have used the CSF in creative and sensible ways to inform their teaching and assessment.*	• Inference that teachers *are not using* the CSF or are *not using it in correctly*. • Inference that teachers will see relevance and usefulness of a teacher practice *if other teachers are involved*.
DEVELOPER - PERSONAL GOAL *To demystify some of the things about the CSF....to make teachers feel ' this is not going to be something that is going to burden me so much, that I don't do the job I'm supposed to do which is teaching my kids'*	Recognition that • teachers currently find the CSF *removed from their experience* • teachers need reassurance that use of the CSF may *assist* their teaching *rather than take away from it*
PRESENTER *To make the teachers see that the CSF is not far from what they are actually doing now*	Acknowledgment of the need to connect the CSF with current classroom practice
TEACHER *Making the CSF relevant to our teaching and our needs*	Inference that the CSF will have to change - not the teachers' practice.

Discussion. The initiator's goal can be contrued as an attempt to inform and reform teachers' practice through the application of the CSF. The developer describes the need to mediate between teachers and the CSF. The presenter's goal is to show the relevance of the CSF to current classroom practice. In this, the Presenter appears to take the Developer's more general goal and reinterpret it in practical terms. In framing his goal, the teacher adopts the position that the CSF *is* removed from classroom practice, but that its content can be adapted to be made more relevant to that practice.

Table 6

Teachers sharing and discussing

EXAMPLES OF GOALS AS EXPRESSED AT EACH STAGE OF IMPLEMENTATION	ANNOTATION
INITIATOR *Workshops will enable teachers to: analyse students' work gathered by teachers in the schools themselves; make links between teaching and assessment in topics currently being taught."*	• The term *workshop* and *the nature of the activities* infers that teachers will be working together. • Sharing or discussing are *not directly* stated.
DEVELOPER *This is a vehicle for getting the teachers together to discuss things... getting them together to share ideas*	Strong emphasis on sharing and discussion.
PRESENTER *To get people to talk about what they are doing, and share...*	Strong emphasis on sharing and discussion.
TEACHER *The actual sharing of information with other teachers: what works and what doesn't .*	• Strong emphasis on sharing and discussion . • Acknowledging the *value teachers place on learning from other teachers' practice*

Discussion: Developer, presenter and teacher all acknowledge the value of sharing and discussion in a professional development program. The initiator's goal assumes that sharing and discussion will take place, but implies (by omission) that sharing and discussion represent the means to the achievement of other goals, rather than a goal in themselves. It appears the *developer* has either initiated a goal independently of the initiator or inferred one from information provided by the initiators. Here, the goal of sharing and discussing, as constructed by the developers was endorsed by the presenters and the teachers in increasingly specific and practical terms.

General Discussion regarding the Transformation of Goals: The initiator's goals were generally concerned with *altering teachers' practice*. Consider for example the use of phrases such as "... to improve ...their ability to ..." and "Assisting teachers to ... develop ...". The developer's goals, were also concerned with changing teacher practice but were *less prescriptive*. An example of this is shown in the use of terms such as "...teachers will explore...". The presenter's goals were to do with promoting a *change in attitude* or an *awareness* in teachers rather than changing their practice. Consider for example the expressions "To develop the ability of teachers to see..." and "...to make the teachers more aware of". The presenters' goals were perhaps a closer match to the teachers' goals and to the realities of professional development outcomes than were those of the initiator or the developers, as evidenced by the teachers' goals which often did not entertain a change in practice. Rather, the teachers' goals sought "to relate" the proposed change to their *existing teaching practice*. This is demonstrated when a teacher's goal is stated as "Making the CSF relevant to our teaching..." So the goal is to change the CSF, rather than the teaching. In general, at each stage, the goals were reinterpreted, so that they met three criteria specific to the needs and perspectives of the particular role of the individual and the demands of that stage in the implementation of the professional development program. These three criteria were that the particular goal: made sense; was achievable; and, conformed to the nature of the individual's perceived role in the professional development program.

Conclusions

When Doyle and Ponder (1978) say that "Teachers adapt rather than adopt innovative practices" they could well be referring to ***innovative goals*** as far as this study is concerned. For example, from stages 1 through to 4, a particular goal was variously interpreted as certain terms were replaced with "synonyms" and key linkages appeared and disappeared. For example, ***Instruction*** broadened to become *Classroom Practice*, and the *CSF* was only sometimes linked with ***assessment*** .

The notion of adaption concurs with the curriculum enactment perspective to curriculum implementation (Snyder, Bolin and Zumwalt, in Jackson, 1992, p. 404), which applies to goals also, as far as this study is concerned. A feature of some of the goals as understood by the initiators was that of altering teachers' practice. The general thrust for teachers is not to alter their practice but to ***adapt*** or ***interpret*** the proposed innovation ***to suit*** their practice.

The link between curriculum implementation and professional development is clear in this study. For example, the ***relevance*** of The Program to teachers' practice was one of the *characteristics of the change* as described by Fullan (1982). And, indeed, relevance was a goal characteristic emphasised by developers, trainers and teachers in The Program . The relative significance of the other factors in this category of Fullan's varied across the four respondent role types. The ***need*** for the change, its ***complexity*** and its ***practicality*** were all addressed in the goals of the developers. The goals of presenters related to complexity and practicality (for example: manageability of the recording assessment information) and aim to demonstrate a need (for example: encouragement of the use of a range of strategies). The teachers' goals also referred to the need for manageable recording practices, and acknowledged the need for the curriculum implementation.

It is clear that the key terms and phrases can alter greatly in meaning in many ways, from stage to stage of the process of implementation of a professional development program. At times the change appears slight, a mere juxtaposition of words, but the change in meaning is great. At other times the words of a goal might be completely different and yet the meaning remains essentially as intended, the choice of words merely reflecting the task of the person at a particular stage of implementation.

Many of the goals as understood by the participants in the process of implementation of the professional development program ***Linking Mathematical Instruction and Assessment through the CSF*** were found to change, at times subtly and at other times profoundly. The fundamental character of this change can be seen in the comparison of the goals of the program initiator and the teachers, and this transformation exemplifies the difference between a policy perspective and a practice perspective. The first takes policy as central and immutable and seeks to inform and reform practice; the second takes practice as central and seeks to interpret or reconstruct policy in order to relate it to practice. This difference of perspective drives the process of goal transformation and plays a major part in determining the outcomes of any professional development program.

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What Does An Expert Teacher in Instructional Computing Look Like?

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The aim of this study was two-fold: first, to describe exemplary practice in teaching with computers; and secondly, to isolate criteria appropriate for the identification of expertise in instructional computing that might be useful for future research in this area. Specific criteria for identifying expertise in instructional computing were derived from the literature and from the findings of a case study. It is suggested that documenting expert teaching procedures will provide educators with examples of and ideas for improving practice in mathematics education.

The proliferation and flexibility of the computer as an information processing tool makes its use in the classroom almost limitless. However, instruction involving computers is not as well integrated into our mathematics curriculum and practice as the increased access and curriculum rhetoric might indicate (Moursand, 1994; Zorn, 1987).

Instructional computing in mathematics can be defined as teaching using the applications of computing which serve "...the goals and functions of mathematics instruction..." (Hatfield, 1982, p. 43). Many studies emphasise how little research there is in instructional computing (Moursand, 1994; Wiburg and Carter, 1994; Zorn, 1987). The research from which this report emanates adopted a proactive approach to researching instructional computing in mathematics by investigating the practices of a teacher who, based on criteria derived from the literature on expertise and instructional computing, was considered to be an 'expert' teacher in the area. Case studies of this nature can provide teachers, teacher educators and aspiring teachers with guidance and ideas for improving their own practice. Research into an effective instance of instructional computing in mathematics is essential in the light that knowledge of a variety of cases is an important part of teacher knowledge (Shulman, 1986, 1987).

Shulman (1986, p31.) states that the "wisdom of practice" comprises teacher knowledge of rules; teacher knowledge of particular cases; and teacher knowledge of ways to apply appropriate cases. A difficulty for the researcher and the teacher in developing their knowledge of 'a variety of cases' in the area of instructional computing stems from the fact that such an environment is always being redeveloped with new hardware and software being introduced constantly (Johnson and Leonard, 1993). A further problem also exists in that it is difficult to isolate and refine criteria and behaviours which indicate expertise in an area of teaching that has, as yet, received little attention from researchers (Berliner, 1988). Documenting the practices of an expert teacher requires that we identify a broad range of criteria or behaviours and observe how teacher knowledge manifests itself within these criteria or behaviours (Berliner, 1988). These criteria can then be used to identify and guide observation of an expert teacher in instructional computing.

Hence, an initial aim of the study reported here was to isolate criteria that could be used to identify an expert teacher in instructional computing. A suggested list of criteria was derived for the purposes of this study through a synthesis of three areas of literature: general expertise; teaching expertise; and current research in instructional computing. Data gathered from a case study of a teacher recommended for his success in teaching mathematics using instructional computing was then analyses to help refine and validate the criteria derived from the literature.

A second aim of the study was to contribute to the development of a body of case study literature. From this body of knowledge, it is anticipated that teachers will be able to draw information on the most effective methods for teaching mathematics involving computers.

Isolating criteria to help identify expertise

An expert can be defined as "someone who has attained a high level of performance in the domain, as a result of years of experience" (Foley & Hart, 1992, p. 235-6). However, this definition does little to describe how we can distinguish the expert from the experienced. Experts in a particular field possess "an orgainsed base of procedural knowledge that can be accessed readily and used with superior modelling and self regulation skills" (Chi and Glaser, 1988, p. xxi). Primary school educators work across a number of topic areas and need not only the knowledge of that field but procedural knolwedge of the best ways in which to teach in these various fields. It is the teaching strategies within topic areas as well as the pedagogical content knowledge that distinguishes the expert teacher (Fennema & Frank, 1992).

Expertise is often studied by comparing expert and novice performance in a field. Chen (1993) compared a series of case studies of teachers at various stages in their professional development to derive a model of reflective teaching that can be critiqued and applied by educationists. This model shows the diversity of factors which affect expert teaching, by illustrating that the integrated thought processes that result in teaching expertise are the result of knowledge, experience, and cultural and societal influences.

Leinhardt (1988) believes that expert teachers develop lessons that can be broken down into segments and routines. He suggests that the value of these routines is their predictability and security for the students. Leinhardt (1988, p. 50-51) identified three types of routines: management routines, involving placement and movement in the classroom; support routines, involving specific instructional strategies; and exchange routines, relating to communication in the classroom. In addition, the expert teacher exercises and makes blatant their agenda, and can predict student responses providing consistent, set responses or "lesson scripts" for their lessons (p. 52). Leinhardt stresses the need for more studies in specific topic areas of teaching to further our understanding of expertise.

The incorporation of instructional computing into a class program leads to diversification of the types of knowedge, skills and understandings required by teachers to teach effectively (Zorn, 1897). The role of the teacher is ambiguous in the instructional computing classroom, because the roles they assume are distinct from and more diverse than the roles required in the traditional classroom in terms of knowledge, skills and understandings to teach effectively. Teacher role is seen as the key component to changes that instructional computing has brought to mathematics teaching (Hatfield, 1982; Kay, 1995).

By identifying, defining and critiquing the roles required to teach mathematics through instructional computing, it is possible to isolate and describe attributes we might expect to characterise an expert teacher in this field. The following criteria were isolated following a synthesis of the literature pertaining to teacher expertise and instructional computing. They include:

- The establishment of procedures for presenting, working through, revising and evaluating instructional computing in mathmatics (Berliner, 1988; Leinhardt, 1988);
- Have specific strategies for linking on-screen and off-screen computer work (Hoyles and Noss, 1992).
- Exhibit flexibility in these strategies to cater for individual differences in student understanding (Hoyles and Noss, 1992);
- Have established guidelines for communication, providing opportunities for the use of the vocabulary of instructional computing in mathematics (Berliner, 1988; Leinhardt, 1988);
- Have set lesson scripts that format vocabulary and discourse in instructional computing lessons (Berliner, 1988; Leinhardt, 1988);
- Understands the relationship between educational goals and teacher roles in instructional computing in mathematics and adopts the appropriate role for specific educational purposes (Cox and Cumming, 1990; Heid et. al 1990; Laridon, 1990).

This list was later refined to include attributes emerging from a case study of a primary school teacher recommended for his success in teaching mathematics using instructional computing. Before presenting the extended list, it is necessary to provide a summary of the case study analysed to isolate additional criteria and validate the existing list. Reporting of the study's outcomes will be limited to the presentation of the criteria.

An Expert Teacher in Instructional Computing : A Case Study

The Setting

The independent school involved in this case study is situated on the northern beaches of Sydney and contains approximately 750 boys ranging from Year 5 to Year 12. The main subject of the study was a teacher (Jack) who had been identified within his school as 'experiencing success' in teaching mathematics with computers.

Jack is a primary school teacher with fifteen years experience and is currently teaching twenty-four Year 6 boys. Besides having access to two computer rooms, one with 17 computers and another with 15, Jack had secured 7 other computers for his classroom through personal means. Being familiar with word processing programs, Jack received initial instruction in the computer program, LOGO, in 1991. Since that time, he has instructed his students in its use and has furthered his own knowledge of the program through personal exploration.

Data Collection

The study took place over a period of one school term (approximately 10 weeks) with 8 separate observational visits occurring. The flexible nature of the instructional computing environment demanded flexible methods of data collection, such as participant observation, observational note-taking, collection of children's work samples and interviewing. Such a variety of data gathering techniques assisted in providing the most comprehensive description of the observed phenomena.

Observation sessions involved participant observation during instructional computing lessons, where audio taping of discourse was used to assist in the review of observational notes. Observational field notes were taken as suggested by Taylor (1995), and later added to by the researcher, and again after transcription of the audio-tape.

Interview sessions involved informal, non-directive interviews during class time with both the teacher and his students. A structured interview was used initially to gain an overview of Jack and his philosophy, questions being based on the assumption that this teacher would exhibit the characteristics of an expert. Structured interviews were also undertaken at the conclusion of the study, one using questions based on developing an overview of the Jack's perception of his students' performances in instructional computing over the term, the other with a colleague of the targeted teacher.

Collection of student work samples was vital, because when students are programing on the computer, their work sample is an indication of their reasoning. Interviews based on student explanation of their work samples yielded data concerning the depth of student understanding of the issues involved in programing.

Results and Discussion

Interviewing the teacher in class yielded data which illustrated clearly the expertise of the teacher concerned. Through in-class interviews, the teacher and researcher had the students on-going work as a reference, making the informal in-class interview the most important method of data collection in this case study. Jack exhibited a vast knowledge of the level of mathematical understanding of his students, describing succinctly through reference to the students' current and past work, and how each work sample exhibited different levels of understanding for each student. Through this he was able to comment on the types of thinking and understanding student programing encourages and the problems some students were having. This illustrates, as the literature suggests, Jack's broad knowledge base in relation to content, his in-depth understanding of the educational purpose of the particular program in use, and his knowledge of student thinking processes, all of which suggest his expertise in instructional computing (Berliner, 1988; Fennema and Franke, 1992l; Shulman, 1986).

Audio-taping of interviews yielded data that illustrated aspects of Jack's teaching that was indicative of his expertise. It also assisted in the revision of field notes, while transcription allowed the researcher to analyse closely and compare lesson scripts. Lessons followed a set script, as Leinhardt (1988) indicated they would, for expertise in teaching. Students were first reminded of past lessons, and encouraged to ask questions of the teacher about their problems with specific commands. Often these problems were pre-empted by Jack. In these cases, commands were listed and revised and students were asked to predict and plot the movements of the LOGO turtles. Literature indicates the importance of lesson scripts in providing students with a sense of predicability and routine (Leinhardt, 1988). Jack's pre-empting student thinking in programing, and his planning of strategies to address the problems they might encounter, indicates not only his knowledge of student thinking in instructional computing, but also his commitment and enthusiasm for investigating the program himself, and hence identifies him further with the criteria for expertise.

Observing both direct on-screen instructional computing lessons and off-screen instructional computing lessons, reinforced data gathered on lesson scripts. Both types of lessons involving instructional computing followed the same type of script observed in related work in Jack's classroom. These included revision of problems either hypothetical, based on Jack's prediction of student difficulty with a particular command or deriving from student work, followed by focussing students' attentions to tasks and on-screen work, or by providing a demonstration conducted by Jack or a student under his direction. Lessons concluded with discussion of what was achieved, revision and solution of the difficulties that some students had, and finally a monitoring of student progress and goals for the next lesson.

In short, the study highlighted a number of successful teaching strategies and teacher characteristics employed by Jack during instructional computing sessions. These included a broad base of content knowledge that related specifically to instructional computing in mathematics, planning open-ended lessons that provided both scaffolding and opportunities for student extension, and analaysing difficult programs so that they could be easily explained to students experiencing difficulty. In addition, the use of peer tutoring, computer demonstrations, off-computer instruction, set lesson scripts and setting behavioural expectations were key strategies contributing to Jack's perceived expertise in instructional computing.

Refining Criteria to Identify Expertise in Instructional Computing

Not only was the list of criteria identifying expertise in instructional computing previously presented validated from the findings of the case study, but additional criteria were identified. It must be made clear that this list is in no way exhaustive or comprehensive in nature. It is presented here to provide a frame of reference for future research in this area. While Leinhart's (1988) expert teacher routine labels were used to organise the criteria according to related attributes, additional labels were included to incorporate attributes isolated in the present study.

Management Routines

An expert teacher in instructional computing will have:

- established procedures for movement and placement of students in instructional computing lessons (Leinhardt, 1988).
- established behavioural expectations.

Support Routines

An expert teacher in instructional computing will have:

- established procedures for presenting, working through, revising and evaluating instructional computing in mathmatics (Berliner, 1988; Leinhardt, 1988).
- specific strategies for linking on-screen and off-screen computer work (Hoyles and Noss, 1992).

- exhibit flexibility in these strategies to cater for individual differences in student understanding (Hoyles and Noss, 1992).
- specific strategies for linking on-screen mathematics to the curriculum.
- planned open-ended lessons that provide scaffolding and opportunities for student extension.

Exchange Routines

An expert teacher in instructional computing will have:

- established guidelines for communication, providing opportunities for the use of the vocabulary of instructional computing in mathematics (Berliner, 1988; Leinhardt, 1988).
- set lesson scripts that format vocabulary and discourse in instructional computing lessons (Berliner, 1988; Leinhardt, 1988).

Teacher Role

An expert teacher:

- understands the relationship between educational goals and teacher roles in instructional computing in mathematics and adopts the appropriate role for specific educational purposes (Cox and Cumming, 1990; Heid et. al 1990; Laridon, 1990).

Teacher Knowledge and Understanding

An expert teacher:

- exhibits knowledge and understanding of the content of a specific computer program and the mathematical implications of this knowledge for the students.
- understands the ways in which students work to attain valuable mathematical learning through a specific instructional computing program.

Values and Attitudes

An expert teacher:

- Shows enthusiasm and genuine interest in instructional computing.

Conclusion

It is difficult to isolate and refine criteria and behaviours which indicate expertise in an area of teaching that has, as yet, received little attention from researchers. A synthesis of the different areas in the literature pertaining to expertise, coupled with an analysis of a case study, has allowed us to generate a set of criteria that can be used to identify an expert teacher in instructional computing and provide educators with examples of and ideas for improving practice in instructional computing.

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Statistical Concepts of High School Students: Some Findings from Fiji

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Growing interest in statistics and probability in schools has resulted in a change to developing concepts rather than carrying out calculations. However, statistical reasoning is not easily acquired by all students and common errors (misconceptions) reflecting difficulties have been identified in several studies. This paper presents the results of a study which explored form five (14 to 16-year-old) Fijian students' ideas of statistics and probability and how these related to their previous school and cultural experiences.

Introduction

Despite the growing use of statistics and probability in our society, there is substantial documentation of the systematic errors and biases that people display when attempting to interpret statistics in their daily lives. Kahneman and Tversky (1983) state that even highly educated professionals who use statistics and probability in their daily lives display biases when attempting to interpret the statistics they produce.

Research on students' understanding of probability and statistics provides evidence that students appear to have difficulties developing appropriate intuitions about fundamental statistical ideas (Brown et al., 1988; Bright and Hoeffner, 1993; Shaughnessy and Bergman, 1993). In a review of the literature, Garfield and Ahlgren (1988), give three reasons for some of the difficulties which have arisen in teaching statistics at the college level. First, students are generally weak in rational number concepts and proportional reasoning used in calculating, reporting and interpreting probabilities. For instance, the concepts of combinatorics and relative frequency both assume a grasp of proportionality. Secondly, many students have a distaste for probability and statistics through having been exposed to its study in a *highly abstract and formal way* (p. 47). Thirdly, statistical ideas often appear to conflict with students' experiences and how they view the world.

Shaughnessy (1992) reports that most of the research in probability and statistics has been done with elementary school children or with college students, resulting in gap in our knowledge about students' conceptions of probability and statistics at the secondary level. In order to help inform teachers and curriculum designers, it is crucial to carry out investigations at this level. He adds that to obtain a clearer picture of student conceptions, beliefs and misconceptions, it is essential to have teaching experiments and more long-term interventions with a strong clinical methodological component.

The purposes of my research were to see how students related their views of the world and prior experiences to statistics in the classroom, and to determine whether Fijian students were different from other students. This paper reports the results.

Method

Sample

The secondary school selected for the research was a typical Fijian Indian high school. The sample consisted of a class of 29 students aged 14 to 16 years of which 19 were girls and 10 were boys. According to the teacher, none of the students in the sample had received any in-depth instruction on statistics prior to the first interviews. The whole class participated in the first phase of interviews, and 14 students participated in the later phases during which the class teacher taught a unit on statistics and probability. This group of 14 was representative of the larger group in terms of abilities and gender.

Interview Process

Each student was interviewed individually by the researcher in a room away from the rest of the class. The interviews were tape recorded for analysis, and notes were made of student non-verbal behaviours observed during the interview. Each interview lasted about 40 to 50 minutes. Paper, pencil and a calculator were provided for the student if he or she needed it.

Interview Schedule

Open-ended interview questions and tasks were selected and adapted from those used by other researchers. The appropriateness of these interview tasks for the Fijian children was established by checking the tasks with the Fijian Ministry of Education Mathematics Syllabus (Ministry of Education, 1988).

Results and Discussion

Analysis of the interviews and children's written records indicates that the students used a variety of models for solving the statistical problems. The data also revealed that many of the students held beliefs and used strategies based on prior knowledge which, would tend to inhibit their development of statistical ideas. This paper focuses on two types of experiences— school and cultural ones. The aspects of cultural experiences discussed are religious and everyday experiences. Extracts from typical individual interviews are used for illustrative purposes. Throughout the discussion, I is used for the interviewer and Sn for the nth student.

School Experiences

Aspects of prior school experiences used inappropriately by the students related to sample, median, range, event, proportional reasoning, compound events, normal distribution and drawing graphs.

Sample: A key factor in selecting a sample is that it is a fair representation of the wider population. Representation is often achieved by picking a random sample or selecting a reasonably larger sample. The results of this study indicate that when students were asked to define the word sample or to select samples, they drew upon their prior knowledge. Criteria such as smartness, intelligence and ability to talk were used as the basis of selection, as the following interchange shows.

> I: *How would you pick this sample so that it is a fair representation of the whole class?*
> S3: *Those students who have been to a meeting.*
> I: *The students haven't been to the meeting yet. You are to select the students.*
> S3: *Select the students who are talkative. They can talk and have some confidence in them.*
> I: *Isn't this sample biased towards the students who are talkative in class?*
> S3: *Eh ... They might go in the meeting and start shivering.*
> *I:* *Would you say that it is a fair sample?*
> S3: *It is a fair sample because some students in our form hardly talk and when they go and talk in the front the voice is so slow that we can't hear. We are sitting in front, still we can't hear and they hardly talk.*

The students did not seem to realise that their samples were biased towards bright and talkative students and not typical of the population. The idea of representation had no meaning for them. Instead they appeared to be deeply convinced that, in selecting a sample, it is impossible to disregard the personal ability of the individuals. They did not realise that the names could have been drawn out of a hat, or random number tables could have been used.

Median: While 12 students could calculate the median from a set of data, only five students could calculate it when data were presented as a frequency distribution. Often, prior school mathematics experiences distorted students' thinking. For example, when asked to find the median from a frequency distribution, one student drew upon prior knowledge of sets and decided that identical scores should not be listed more than once!

Range: The range is the difference between the highest and lowest value of the variable but when asked to find the range for the task in Example 1, two students used their prior experience of relations and functions to conclude that the range was the first element in the data set.

Example 1: Weight problem

A small object was weighed on the same set of scales separately by nine students in a science class. The weights (in grams) recorded by each student are:
6.3 6.0 6.0 15.3 6.1 6.4 6.2 6.15 6.3.
What is the range?

This is evident in the following interchange:

I: *What is the range?*
S14: *6.3.*
I: *Why do you think that 6.3 is the range?*
S14: *The range I know of is the domain and the range. If there is 2.5 and 3.5, range is the first number and domain is the second number.*

Event: An event is a subset of the sample space. It is the outcome that is of our interest, or the desired outcome which is being investigated in the experiment. For example, tossing two coins and checking the possibility of getting two heads. The event is {HH}. A number of students in this study did not have a clear understanding of this term. While constructing their meanings for the term event, four students drew upon their prior sports experiences as indicated in the following interview.

I: *What does the word event mean to you?*
S22: *It is a happening that has to take place, say a match that has to take place tomorrow, that is organised.*
I: *Where have you used this word before?*
S22 *In athletics where there are track events and field events.*
I: *What does the word event mean in statistics?*
S22: *Event that takes place.*
I: *Can you give an example?*
S22: *When we toss a coin and find the probability.*
I: *So what is the event, tossing the coin or calculating the probability?*
S22: *Tossing the coin.*

Proportional Reasoning: The ratio concept is crucial to a conceptual understanding of probability. The results of the present study indicate that this concept is not well understood by the students. Rather than attending to proportionality information given on a marble task, (10 white and 20 black marbles versus 20 white and 60 black marbles) some students based their reasoning on previous sports experience. The following discussion illustrates this misunderstanding:

> I: *Now Meena says that the game is not fair because in Ronit's box there are more white marbles than in her box. What is your opinion about this?*
>
> S20: *Eh ... So if you play a game it should be equally in number.*
>
> I: *What do you mean by equally?*
>
> S20: *Like soccer if you are playing there should be 11 players aside. Then you will be able to have a win eh. So here it is like marble it is 10 white and 20 black marbles. So the game should be equally .*

Compound Events: In some situations, successive outcomes may be involved. For example, tossing two sixes in a row with a dice. If two dice are rolled, the probability of getting the same number on both is one-sixth and the probability of getting different numbers is five-sixths. The answers would be the same if one dice was rolled a second time, provided no particular order of numbers was required. In addition, theoretical probabilities of compound events can be analysed by using a table or a tree diagram.

For the dice problem (see Example 2), student 21 said that the probability of getting a 5 and 6 was more than the probability of getting a 6 on both because a six is hard to get.

Example 2: Dice Problem

If I roll two dice, is it more likely that a 5 and 6 will turn up, or that there will be a 6 on both dice? Or is the probability the same in both cases?

The student did not work out the sample space but based his reasoning on previous experience with board games.

For the two dice problem, students were asked whether rolling the two dice together or separately would make a difference to the results. The aim of this task was to check whether the students could extract the identical mathematical structure from the two practically different tasks. Two main types of unequal probabilities were mentioned by students. Some students considered that by successively throwing the die they had a higher chance of obtaining different numbers, while others thought that by throwing two die simultaneously they had a higher chance of obtaining different results. The explanations provided by the students indicate that they based their reasoning on their physics experiences. As one student explained,

> S22: *If you roll once it might stop here because of the friction force the other one if the surface is smooth, it might stop somewhere here.*

These findings concur with the findings of Fischbein et al (1992). They used similar problems and found that only half the children in their study could see that the two procedures lead to the same results. The researchers concluded that the children in their study thought that outcomes can be controlled by the individual because the *mathematical probabilistic structure has not been detached from the concrete circumstances and considered in its abstract generality* (p. 530). In this study, this confusion seems due to the same difficulty.

Normal distribution: The total area under the normal curve is 1. Using mean and standard deviation, this area can be approximately sub-divided. For instance, approximately 68% of scores fall within one standard deviation from the mean. Some students completely ignored this information and based their reasoning on the activities they had done in other domains. When asked to find the values for x_1, x_2, ..., x_6 on a normal curve, given a mean of 50 and a standard a deviation of 5; two students based their calculations (wrongly) on directed numbers graphs which they learnt in Form Three. The students used positive numbers above the mean and negative numbers below the mean. For instance student 22 substituted 1 for x_1, 2 for x_2, 3 for x_3, -1 for x_4, -2 for x_5 and -3 for x_6. When asked to explain, he provided the following justification:

> *If the standard deviation is five, then we have to divide this part* [points to area above the mean] *into five parts. So x_1 will be 1. In line graphs., think of this* [indicates the mean] *as zero. Then you have positive numbers on this side* [gestures right of the mean] *and negative numbers on this side* [motions to left of the mean].

Another example of students bringing other school experiences inappropriately to bear on the normal distribution idea occurred in relation to a pulse rate item. Rather than considering the item statistically, nine students based their reasoning on activities they had done in science classes. For example, they wanted to talk about various pulse rates, which they had recently learnt in a biology class, not the number of adults (out of 400) who might have a pulse rate between 68 and 76, given a mean of 72 and a standard deviation of 4.

In their biology classes, the students had been investigating the effect of exercise on the heart beat. The conclusions from these activities were applied to the statistical problems.

Drawing graphs: The bacteria growth problem (see Example 3) was used to explore students' understanding of graphs.

Example 3: Bacteria growth problem

A bacteria population cultured in a laboratory showed the growth pattern with respect to time, t, in minutes, as shown.

(i) Show this information on a suitable graph.

(ii) Why did you choose this graph?

t Minutes	0	1	2	3	4	5	6	7	8	9	10
Number (100s)	2	2.3	3.1	4	4.8	5.7	6.5	8	8.5	8.7	8.8

Although the students could draw a graph of the information given in the table, they could not give mathematical reasons for drawing a line graph or selecting a particular graphical representation. Three students said that they drew a line graph because they drew line graphs in their science lessons.

Cultural Experiences

In addition to basing their thinking on school experiences, some students based their reasoning on their cultural experiences. The cultural experiences discussed are religious and everyday experiences. The religious aspect is evident in independence and prediction tasks and the everyday experience in sample size and interpreting tables tasks.

Independence: The students thought God controls everything that happens in the world. For the baby problem (see Example 4), some students thought that one can not make any predictions because the sex of the baby depends on God.

Example 4: Baby Problem

The Singh family is expecting the birth of their fifth child. The first four children were girls. What is the probability that the fifth child will be a boy?

As one student explained:

> *We can not say that Mrs Singh is going to give birth to a boy or a girl because whatever God gives you have to accept it.*

Prediction Problem: Strong influences of religious beliefs were also apparent when students were asked to comment on an advertisement which claimed that for a fee of $20 and a sample of the mother's hand writing the sex of the baby could be predicted (with a money-back guarantee). Even when challenged to consider how the people placing the advertisement could make money, the students could not see that roughly half the babies born would be girls and half would be boys, that anyone could expect to be right in half the number of cases just by guessing. Rather than seeing that a clear profit could be made on 50% of the all the $20 sent in, powerful religious beliefs influenced the students' thinking, for example,

> *As I have told you before that God creates all human beings. He is the one who decides whether a boy is born or a girl is born.* (S17)

These findings of students drawing upon their religious beliefs concur with those of Amir and Williams (1994) who assert that culture plays a significant role in the building of a child's probabilistic thinking. It is certainly the case that the Indian culture in Fiji has a religious orientation.

Sample Size: In general, the larger the size of the sample taken, the more confidence one has that a sample statistic is a close estimation to the population parameter. People who rely on prior experiences, however, tend to estimate the likelihood of events by neglecting the sample size and by placing undue confidence in the reliability of their experiences. Rather than attending to sample size in a consumer report, two students in this study, in responding to the item about whether Mr Singh should buy Honda or Toyota (see Example 5) said that the *life* of a car depends on how one keeps it.

Example 5: Consumer Report task

Mr Singh wants to buy a new car, either a Honda or a Toyota. He wants whichever car will break down the least. First he read in *Consumer Reports* that for 400 cars of each type, the Toyota had more break-downs than the Honda. Then he talked to three friends. Two were Toyota owners, who had no major break-downs. The other friend used to own a Honda but it had lots of break-downs, so he sold it. He said he would never buy another Honda. Which car should Mr Singh buy?

They did not apply the idea of representativeness in this instance where it is really appropriate to do so. For example, student 29 explained:

> *He should buy any of the cars Honda or Toyota; it depends on him how he keeps and uses the car ... Ah Because it depends on the person, how he follows instructions then uses it. My father used to*

> *own a car and he kept it for ten years. He sold it but it is still going and it hasn't had any major breakdowns.*

Interpreting Tables: The task comparing temperatures of Ba and Sigatoka (Example 6) was used to elicit student ideas in interpreting tables.

Example 6: Comparing temperatures of Ba and Sigatoka

> Temperatures (in degrees C) were taken from Sigatoka and Ba on six consecutive days. Look at the temperatures from both the towns and decide if Ba is warmer than Sigatoka. How do you know? What else do these figures reveal about the temperatures in Sigatoka and Ba?

Example 6: Table comparing temperatures of Ba and Sigatoka

Day	1	2	3	4	5	6
Sigatoka	25	24	21	20	23	24
Ba	28	27	26	25	29	30

Analysis of the interview data showed that five students based their reasoning on their everyday experiences. These students made unwarranted topic assumptions when trying to read the table. The students said Ba was warmer than Sigatoka. When asked to explain their answer, they talked about the everyday weather conditions of Ba and Sigatoka. For example, student 21 explained,

> *Each day the temperature for Ba is greater than the temp for Sigatoka. And normally Sigatoka is called a valley. They are producing fruits; it rains there. My one uncle lives here. He mainly plants Chinese cabbages; because of the rain it grows so well there.*

When asked to explain what else the figures revealed about the temperatures in Sigatoka and Ba, student 9 continued to talk about Sigatoka being warmer than Ba because everyday Sigatoka temperature was lower. The other four students continued to base their reasoning on everyday experiences.

Although this study provides evidence that reliance upon experience can result in biased, **non-statistical** estimates, in some cases this strategy may provide useful information. For example, the students knowledge of geography may have been reasonable.

Conclusion

The results indicate that certain misconceptions noted in the literature take a strong and common form in the thinking of the 14 and 16 year-olds' involved in the present study. There were also instances where individual reasoning was influenced by beliefs and experiences about the world and events and that this reasoning remained impervious to traditional instruction.

The strategies used by the students can be explained in terms of constructivist theory. The theory emphasises that children construct their own meaning on the basis of their existing knowledge rather passively receiving knowledge from the teacher. This implies that learning is a process of adapting to and organising one's existing structures of knowledge, not absorbing ideas imposed by others. It can be inferred that in the present study, children generated links with other domains while solving the statistical problems. Both anticipated and unanticipated learning occurred as children tried to generate links between the different domains.

It should not be taken for granted that children understand spontaneously the meaning of the terms sample, event, equally likely and range. Children have to be helped to distinguish between how these terms are used in statistics and in other domains. Trivial text book definitions such as *An event is a subset of the sample space* is not of much help.

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Evaluating Student Expository Writing in Mathematics

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A previously reported coding scheme for describing student expository writing in mathematics has been developed into a set of descriptors aimed at reflecting the level of understanding being demonstrated in the writing. The original coding scheme provided a detailed analysis of the content of the writing to enable the application of the descriptors. The descriptors are intended to give researchers and teachers indications of the content of writing tasks which may help advance students' thinking. Examination of writing examples mainly from year 8 students has shown that it demonstrates little conceptual understanding of mathematics.

The use of writing as a learning aid in mathematics classes at all levels of education has received considerable attention in the mathematics education literature over the past fifteen years. The term "writing" here refers to tasks which require the use of prose as well as the usual symbolic and diagrammatic representations. The types of tasks which have been reported in use in mathematics classes vary greatly and may include journal writing (Borasi & Rose, 1989) and expository writing (Miller, 1990). A range of benefits attributed to the use of writing tasks in mathematics learning has been reported. The improved dialogue between students and their teacher and the exposure of misconceptions that would otherwise have gone undetected for longer have been clearly demonstrated.

Borasi and Rose (1989) developed a comprehensive analysis of the benefits of journal writing in mathematics and noted an increase in the content knowledge of the students as well as some exposure of their views about mathematics and its learning. In most of the reported studies, there has been little attempt to establish whether students were actually developing their understanding of mathematics through the use of writing. Many have claimed improved learning and understanding as a benefit of writing without any compelling evidence to support the claims (Powell & Lopez, 1989). Such studies have tended to assume that increased understanding will occur based on the finding from more general "writing-to-learn" studies. One study reported in Waywood (1992) and Clarke, Waywood and Stephens (1993) has provided a useful scheme for examining the development of student thinking demonstrated in their writing and this will be discussed in detail below. The study reported here extends a previous study (Shield, 1995) in which a scheme to describe student expository mathematical writing was developed. It builds on the previous study and the work of Waywood to further refine the evaluation of student expository writing with particular emphasis on the understanding of mathematics being demonstrated in the writing.

Writing and the Development of Understanding

One of the main benefits that could be realised from the use of writing activities in mathematics learning is an increase in student understanding of the subject. Understanding in mathematics learning is generally accepted to involve knowing the concepts and principles related to the procedures being used and making meaningful connections between prior knowledge and the knowledge units being learnt (Baroody & Ginsburg, 1990). A network of internal representations is established and this is developed in some way from the external representations which the learner experiences (Hiebert & Carpenter, 1992). Skemp (1976) had earlier described such understanding as "relational" and contrasted this with what he termed "instrumental" understanding which is characterised by a knowledge of many discrete mathematical processes which can be applied in very limited situations.

As well as demonstrating the type of knowledge held by the learner, the process of writing may also stimulate the development of further connections within that learner's knowledge structure. The process of connecting new ideas with existing knowledge has been described by the term "elaboration" (Hamilton, 1989) and has been studied as part of the process of reading. Elaborative processing by learners appears to make new material more meaningful and allows for more effective integration of the ideas with existing knowledge. Studies (e.g., Reder, 1980; Hamilton, 1990) have shown how elaborative processing during the learning of new ideas enhances the ability of the learner to apply those ideas in novel problems, the resulting richer knowledge base providing more options for the generation of ideas during problem solving. The process of writing can involve some of the same elaborative thinking as used in reading for students are stimulated to make meaning of the ideas they are presenting (Rose, 1989). The study reported here made use of the concept of elaboration in developing a scheme for evaluating the thinking and understanding of students as displayed in their expository writing.

Mathematical Thinking Shown in Student Writing

In a previously reported study (Shield, 1995), a coding scheme for describing the content of student expository writing in mathematics was discussed. An application of this scheme was also reported in Shield and Swinson (1994). It enables the detailed description of a piece of writing in terms of the elaborations of each main idea (known as a "kernel") and is based on Leinhardt's (1987) features of an explanation and van Dormolen's (1985) terminology for the description of the content of a mathematics textbook. The coding scheme is outlined as follows.

The van Dormolen terminology is used to describe each statement in a presentation in terms of the "aspect of mathematics" being expressed and the "level of language" used.

Aspect of mathematics: theoretical - theorems, definitions, generalisations; algorithmic - explicit 'how to do' methods; logical - the way we are allowed to handle the theory; methodological - heuristic 'how to do rules'; communicative - conventions, how to name a diagram, write a proof.

Level of language: exemplary - demonstrative, related to a specific example; relative - generalised, not related to a specific example. Within each level, the language may be procedural or descriptive.

The method then involves partitioning the writing into separate elaborations of the kernel. Separate elaboration can often be distinguished by changes in the aspect of mathematics being expressed or the level of language being used. The elaborations which may be found in student expository writing are:

Kernel	definition or general statement of the procedure
Goal statement	identification of the concept or procedure being explained
Demonstration	a worked example of the concept or procedure which may be elaborated with: (a) symbolic representation; (b) verbal description; (c) diagrammatic representation; (d) statement of convention
Legitimisation	justification for the procedure or part of it using known principles

Link to prior knowledge	extensions of prior knowledge, reference to everyday experience
Practice exercises	set of questions to be answered by the reader by modelling on the demonstration

Not all of the elaborations are usually included in individual pieces of writing and it has been found that legitimisation was rarely present (Shield, 1995). A rich description of the writing can be developed and some indication of the level of understanding may be inferred. That is, the number of elaborations used indicates how much the student has linked the idea or procedure with various representations and ideas from prior knowledge. The level of language and aspect of mathematics in some of the statements are also indicative of understanding. In the earlier study (Shield, 1995), it was found that many students rarely expressed anything other than the algorithmic aspect of mathematics and only approximately half the students could describe a procedure in generalised (relative) language, the remainder always writing only in terms of a specific example.

Establishing the Level of Understanding Evident in Writing

The present study was designed to extend the coding scheme to provide a way of more systematically determining the level of understanding being displayed in the writing. Methods reported in two earlier papers assisted in this development. In one study (Waywood, 1992) a scheme to assess journal writing based on a set of descriptors was developed. In that study, journal writing included the tasks summarising, collecting examples, questioning and discussing. Each of these tasks was further delineated into four sub-categories which enabled a detailed description of the processes evidenced in each student's writing. In conjunction with the categories, a set of progress descriptors was described. From the pattern established, a global progress categorisation of the student's journal writing into one of three modes, namely recount, summary, or dialogue could be established. Parts of the definitions of these modes are reproduced below.

> RECOUNT When students are writing in this mode, they interpret the tasks in terms of concrete things to be done: to write a summary means *record*: . . .
> SUMMARY When students are writing in this mode they interpret the tasks as requiring involvement. The involvement is utilitarian. Describing gives way to *stating* and *organising*. . . . Journals show students trying to form an overview.
> DIALOGUE When writing in this mode, students see the task as requiring them to generate mathematics. . . . Summaries are about *integrating*; questions are about *analysing* and *directing*; examples are *paradigms*; and discussing is about *formulating* arguments. . .(p. 38)

With further analysis of the students' journals, and also analysis of questionnaires for students and teachers developed for the purpose, Clarke, Waywood and Stephens (1993) confirmed and further described these progress modes and formulated a detailed description of student thinking associated with each. They were able to establish a general trend that as experience with this type of journal writing increased, the students' writing tended to progress through the modes.

Shepard (1993) addressed the issue of the role of writing in conceptual development from a theoretical standpoint. He adapted the work of a number of cognitive psychologists to describe learning in three phases, namely initial, intermediate and terminal. Within each phase, writing categories were described at progressive levels. The descriptions are summarised as follows:

Initial	record	transcription
	report	summaries - no inferences, just information

	generalised narrative	personal examples of concepts, ideas explained in their own words
Intermediate	low-level analogic	explains how to solve problem, explains what is wrong with incorrect working
	analogic	explain how concepts are related, explain how and why procedures apply or don't apply
Terminal	analogic-tautologic	propose alternative ways and understand how they are different
	tautologic	produce new method for a certain type of problem

These descriptions in some ways parallel those developed by Waywood (1992). Shepard's aim was to provide teachers and researchers with guidance for designing writing assignments in mathematics which could assist in moving students from the rote memory of algorithms towards a more abstract and integrative type of thinking.

In the present study, examples of student mathematical writing from earlier studies (Shield & Swinson, 1994; Shield, 1995) which had been coded with the earlier detailed scheme were re-examined with the systems summarised above (Waywood, 1992; Shepard, 1993). The aim was to develop, by a synthesis of the three systems, a set of descriptions for recognisable levels of elaboration of mathematical ideas which would reflect the thinking and understanding being displayed by students in their writing. Five levels of understanding have been identified, the highest corresponding reasonably closely to Shepard's (1993) intermediate/analogic mode and Waywood's (1992) summary mode. The examples used came mainly from year 8 students and no higher levels were apparent. In the following paragraphs, the five levels are described and exemplified. The levels and their examples are discussed using the terminology from the coding scheme outlined above.

1. *Approach*. At this level, the student writer presents either an unelaborated demonstration of the algorithm or a simple algorithmic description of the method which lacks sufficient definition to be a true kernel. In the first example in the appendix, the student writer was asked to explain about arranging the numbers 4831, 5107, 4970 and 5015 in ascending order. The complete response is shown. In the second example, the task was to explain about working questions such as: "What is 35% of 80?" One student's complete response is shown.

2. *Recount*. At this level, writers generally reproduce a demonstration of the concept or procedure. The demonstration is complete and may be elaborated both symbolically and verbally and the language is exemplary (related only to the specific example). It may also be elaborated diagrammatically when appropriate. The aspect of mathematics being expressed is algorithmic. In the example in the appendix, the writer was responding to the same task as in the second "approach" example. The working is complete and the algorithmic nature is emphasised by the steps.

3. *Generalise*. Writing at this level also focuses on a demonstration of the concept or procedure which may be elaborated symbolically, verbally and diagrammatically when appropriate. The main feature which distinguishes it from level 2 is the inclusion of a definition or procedural statement in relative (generalised) language, that is, a kernel is clearly stated. The aspect of mathematics being expressed is algorithmic and the presentation may include some practice exercises for the reader. In the example in the appendix, the student was asked to write a letter to an absent friend to explain all about highest common factor. The writer has stated a goal for the presentation in lines 1 to 3. Line 5 to 11 contain the relative statement of the procedure which shows that the student

has been able to generalise the method into an algorithm. In the earlier study (Shield, 1995), it was found that while most students could produce a demonstration, only about half of them could also make a general statement of the procedure.

4. *Link.* Writing at this level contains the same elaborations as at level 3 but also includes some links with prior knowledge or everyday applications of the concept or procedure. Students writing in this way are beginning to demonstrate an understanding of the relational nature of mathematics. In the example provided, the writer responding to the same task as the writer in the level 3 example has included a discussion of factors in lines 5 to 8 in preparation for the kernel and demonstration which follow.

5. *Integrate.* At this level, the student writer integrates a number of relevant ideas to arrive at an elaborated presentation which includes some statements expressing theoretical and methodological aspects of mathematics in relative language, as well as the types of content described in the previous levels. There is a logical organisation of the ideas. In the example, a year 9 student was asked to write a letter to an absent friend to explain all about solving equations, linear equations being the only ones so far covered in class work. In the example, the writer has firstly addressed prior knowledge of what an equation is and then made a general statement (lines 5 and 6) about what it means to solve an equation, a theoretical aspect of mathematics. The following two sentences (lines 6 to 10) express methodological aspects of mathematics in relative language. The remainder of the presentation then focuses on the algorithmic aspects of the procedure except for the statement on line 12 about setting out which is a communicative aspect conveying an accepted convention.

Conclusion

The scheme described in this paper provides a method of evaluating the level of understanding of specific mathematical ideas being displayed in students' expository writing. The earlier coding scheme (Shield, 1995) enables a detailed description of the content of the writing, rather than a global description, and enables decisions to be made regarding the level to be assigned. In working with this scheme, so far there has been little difficulty in assigning writing examples to one of the levels. It is expected that in further work higher levels will be demonstrated as indicated by Waywood (1992) and Shepard (1993). From the examples collected by this author and from many of the examples quoted in the literature on writing in mathematics at the school level, it is apparent that most student writing is at level 3 or lower. If the writing in some way reflects the conceptions of the mathematical ideas and beliefs held by the students, then it indicates that students are learning mathematical procedures in an instrumental rather than relational way. The aim of this scheme is to provide researchers and teachers with a framework for further developing students' thinking through the use of writing by indicating the types of content that students might be encouraged, by the use of examples, to include in their writing.

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APPENDIX

Level 1: APPROACH

You look at the lowest number and to the
next lowest and so on until you have
the highest number last.

[illegible]
You can use a calculate and
put in 35 ÷ 100 × 80 then it will
give a aswer.

Level 2: RECOUNT

— you devide the 35% by 100 beca-
2 — use (% = 100)
3 — 35 ÷ 100 = 0·35
4 — multiply 0·35 by 80 because that
5 — is what you want to know
6 — 0·35 × 80 = 28%
7 — STEP one 35 ÷ 100 = 0·35
8 — STEP TWO - 0·35 × 80 = 28
9 — STEP THREE = Answer = 28%

Level 3: GENERALISE

Dear Jane Doe,

1 — I am writing to tell you about
2 — how to find a Highest Common Factor
3 — (HCF) and the Lowest Common Mutiple
4 — L(CM) of 2 numbers.

5 — H.C.F. To do this you must have 2
6 — numbers. E.g. 16 and 40. You must
7 — then start writing down their factors
8 — When you finish doing this you must
9 — find the highest number that occur
10 — in both numbers. That number is you
11 — H.C.F.!!

12 — e.g. 16→1, 2, 4, 8, 16.
13 — 40→1, 2, 4, 5, 8, 10, 20, 40.
14 — ∴ the H.C.F is 8 because it is the
15 — largest number that occurs in both
16 — numbers.

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Level 4: LINK

Dear Jessy,

Well I know how you've been away for a couple of weeks and Mrs gave me the job to explain about Highest Common Factor and Lowest Common Multiple. Well let me start with HCF. A factor is small Than the number that you start off with. The factors are numbers that can divide into the number evenly for eg 16=1,2,4,8,16. Do you see. HCF is highest common factor, so you take two numbers 12 and 16 and you list the factors. When you've done that you look and see the highest factor both of them have otherwise you always have one.

eg 12=1, 2, 3, 4, 6, 12
16=1, 2, 4, 8, 16

They both have 4 so you're answer is. 4. Get it.

Level 5: INTEGRATE

Solving Equasions.

In an equasion it shows that something is equal to something else: $5+6 = 7+4$
Sometimes in an equasion there is an unknown number (x):

$3x + 2 = 11$

To find the unknown quantity is called solving the equasion. When solving an equasion the idea is to get x on its own. To do this you have to cancel out all the other numbers on that particular side of the equasion. Throughout the sum it is essential that the equasion is balanced at all times. To do this you have to make sure whatever you do to one side you do to the other. The equasion has to be set out as follows.

For example $3x + 2 - 2 = 11 - 2$ (You have to take away the 2 from both sides)

$3x \div 3 = 9 \div 3$ (You have to divide both sides by 3 to leave x

$x = 3$ on its own reading the answer in the equasion)

You can check the answer by substituting you answer with x and if the equasion is equal you are right eg.

$3 \times 3 + 2 = 11$
$9 + 2 = 11$
$11 = 11$

Try these to see how you go
(Remember you can check your answer).

1. $5x - 20 = 30$
2. $x - 14 = {}^{-}2$
3. $3 - 2x = {}^{-}1$

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"This Is *Your* Life": Initial Study Of Mathematical Autobiographies

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Despite being involved in a mathematics education subject emphasising student centred methods of teaching, some trainee teachers revert to 'chalk and talk' and rote learning methods. To partially overcome this practice, second year students in a primary teacher education course were asked to write their mathematical autobiographies over a period of eleven weeks. The analysis of these revealed increased awareness of mathematics as a life skill, changes in opinions and greater understanding of the influences shaping mathematical development.

Mathematics teacher educators develop courses of study for teacher trainees in order to help them put into practice the exciting and stimulating ideas they learn in their course. It is a matter of some consternation and dismay to discover that these same trainees, when faced with the demands of the classroom situation on a permanent basis, sometimes revert to the teacher centred, 'chalk and talk' method of presentation of mathematics which they themselves experienced when they were in school. It is hard to give a definitive reason for this phenomenon. It is certain, however, that even those who were not very successful with the 'chalk and talk' method seem to find some solace in imitating their teachers. This could be because of fear they might not be able to handle the mathematics the students discover if more student centred methods are used, dread of poor test results, lack of resources and time for preparation of resources, or simply because the executive in the school have told them to teach that way.

Whatever the reason for the reversion, it is apparent that something fairly radical has to be attempted in order to develop in teacher trainees the will and motivation to implement the methods which they find so fascinating in their undergraduate work. Besides providing the environment at university in which they can explore new methods, some fundamental change is needed in their understanding of the process of learning mathematics and the relationship of their feelings and attitudes to that process. Reflection is one process that could bring about this desired change.

More than half a century ago Dewey (1944) wrote about 'reflective thought'. Skemp (1971) developed the idea of intuitive and reflective intelligence. Polya (1957) and many others working in problem solving emphasised reflection in the problem solving process. Schon (1983) put forward the idea of the teacher as the 'reflective practitioner'. Boud, Keogh and Walker (1985) developed further ideas about reflection in a number of different personal and academic contexts.

Another development which dovetails into this process of reflection is the use of biographical details as the source and object of reflection. Several studies (Cameron and Wilson, 1992) have explored the use of reflective journals in the development of teaching skills. Taylor (1992) extended this idea of reflecting on present experiences to the reflection on one's personal history. Nimmo and Smith (1994) describe a longitudinal ethnographic study of the progress of four secondary school teachers. The study used biographical data and showed the influence of such data on later ability to develop as teachers.

The developmental theory of Vygotsky has, to a certain extent, superseded the theory of Piaget in that he emphasised the social origins of mental functioning. While Piaget maintained that children *learn through* social interaction, Vygotsky believed that children *need* a social environment to learn and that they learn from a more expert partner. His idea of the Zone of Proximal Development stresses the close relationship between instruction and the child's level of development. Another critical aspect of Vygotsky's theory was that language is central to learning.

The Vygotskian Zone of Proximal Development has implications for the teacher in the classroom, in that s/he needs to provide learning experiences for the child within that zone. It is also important for teachers in their interaction with their own philosophy and practice of teaching as well as thire preparation for student activity. The teacher becomes a very powerful role model and support for the learner. This in turn has implications for the development of attitudes and aptitudes in mathematics.

If attitude is defined as a way of thinking, feeling and acting, the development of attitudes will be the result of an interrelationship of a number of different aspects of one's life. The formation of attitudes involves the family, socialising processes, schooling experiences and relationships among all three as well as with role models and mentors. Biographies can often be the catalyst which brings these elements together to form or reform attitudes to learning mathematics. They can also assist the construction of knowledge, thus connecting the process to social constructivism. An emphasis on a Vygotskian form of social constructivism stresses the interrelationship of the people in the classroom, the classroom environment and both the nature and methods of presentation of mathematics.

The basis of a life history is the language and culture of the person concerned. An examination of a person's life history or biography can reveal some of the factors impinging on that person's present attitudes, beliefs and knowledge.

A variation of the life history is the autobiography. Autobiographies have both strengths and weaknesses. Their strength is their subjectivity. Their weakness is the possibility of inadequate memories leading to false assumptions. They do, however, provide memories which have been filtered through the person's subsequent experiences and concepts.

This current study seeks to bring together the need to provide teacher trainees with experiences which will release them from previously held inhibitions and negative attitudes, the reflective process, Vygotskian theory and the results of autobiographical reflection.

Subjects and Method

The subjects were primary teacher trainees in their second year. They were requested to write their mathematical life history using a double entry page mode. For a period of four weeks, the students were asked to write about three entries each week and record incidents that occurred earlier in their lives, in chronological order, if possible, and to also record, in the second column, the implications, as they saw them, of the recorded incident for their education, particularly in mathematics. For a further six weeks, the students were asked to complete their history, if they had not already done so, and then to keep a journal of their current mathematical experiences. It was suggested to them that they include their reflections on the lectures and workshops in which they participated, any reading they did and any out-of-university experience that involved mathematics. It was also suggested to them that they look for the connections between these aspects and what they had recorded in the histories.

Each of these two stages was monitored by tutors who read the submissions. For the first phase, the tutors wrote constructive comments and pointed out ways in which the students could make better use of their work. Part of the analysis of the second phase had to do with ascertaining to what extent the students were able to perceive and implement the advice of the tutors.

Two hundred and sixty seven subjects completed the task with varying degrees of personal satisfaction. From these, several have been selected to illustrate the use of this procedure as a means of creating the reflective environment necessary for developing teachers.

Whilst students wrote entries in their histories, they were actively participating in a university mathematics education subject. They were required to attend two lectures each week and two workshops in which they manipulated materials and reflected on their activity as a way of developing an understanding of the implications of the lectures for their teaching.

Analysis

The analysis took several forms. Comment on the first phase was limited to suggestions to make the process more useful to the students. The second phase was analysed to see if suggestions had been carried out. In addition, more attention was paid to the depth of the students' comments and to the extent to which the subjects had been able to make connections between the various aspects of their history. The ability of the students to recognise the relationships involved and to predict how their learning could be used constructively in the classroom was viewed very favourably. A superficial description of an event did not appear to be as useful as an in-depth reflection on the meaning of the event and its implication for the classroom. Students who did well on this first submission were given suggestions for further possibilities for reflection.

Marks were awarded for the histories in the second phase. This was done, partly because the task was part of the assessment for their normal mathematics education subject and partly because it gave the students an indication of the relative weighting of various aspects of the task. The main analysis was through comparison of the apparent attitudes and depth of reflection between the first submission and the second and the identification of particular incidents recorded that had a significant effect on the life of the student.

Results

Many students found it difficult to remember anything before they went to school and then only the upper primary school. When encouraged, they were able to remember further back and were surprised that they were able to do so.

They were also surprised to find that incidents from their past had influenced their present thinking and attitude. The reflective process appeared to free them from preconceived ideas, fears and misconceptions concerning their motives and attributions. Many expressed their delight and appreciation of the process they had experienced as it gave them new insights and aspirations.

No student reported any awareness or evidence of mathematics anxiety prior to entering school. The students also reported that the mathematics at this stage in their life was relevant to their play and social interactions, they had greater control over and autonomy in their learning, the pace of their learning and the content of their learning. They also had the "right" to withdraw from the learning environment and this right was supported by their parents.

With very few exceptions, the students expressed varying degrees of surprise and pleasure to find that mathematics permeates so much of their everyday existence. One student wrote:

> This just goes to show how math concepts totally surround us in everyday life experiences. As teachers, it is very important that we make math a fun subject in order to keep the students interested in it. If students do not take it seriously and don't try to understand the concepts, they will be affected in several ways. We must be creative and imaginative; use a variety of methods; draw on past experiences; and demonstrate how it applies to real life situations."

This quotation also illustrates the understanding which this student has reached as a result of her multiple experiences in workshops and through the autobiographical reflections. Further results will be given as comments on selected entries from a number of life histories.

Example 1. Subject Dee

27/7/95 Birth - School I remember my very first experiences with maths were all very positive. I would be told that I was "a very smart girl" by my Dad when I built walls with Lego and played with blocks. I also used to go shopping with my Mother and sister on a regular basis. My sister and I, although we didn't fully understand the concept of money, knew only too well that 20c. would buy us a drink from the machine outside of Franklins. When we received our pocket money Dad would help us by making us think about what we would like to do with the money.	Reflection I think that my attitude towards things mathematical was originally well- nurtured at home with my family. Mum and Dad would positively reinforce me when I did something like hold fingers up to show how old I was. But along with this they also assisted me to understand my mistakes and would help me to remedy them. This type of 'attention' soon came to be a pre-requisite with me with mathematical problems. Reinforcement became a crucial element of my development. My initial introduction to money was probably like other people's - pocket money. Mum gave my sister and I $1 per week to spend or save as we pleased. I guess that we were taught a sort of mini-budget system by my Dad. ...
28/7/95 Primary School In Year 2 we were introduced to the times tables at school and had to learn them by rote. We were also asked to get our parents involved by reciting them to them.	...I believe that the continual repetition was advantageous to my mathematical learning. Not only did it generate interest - as many of the games that were played in the classroom were also intermingled with the tables - but it gave us something to focus our attention on. ... Mum and Dad also reiterated the importance of the times tables to me and my little sister by buying a wall chart and (this may sound silly - but it worked!) putting it behind the toilet door!! . . .
31/7/95 ... I got really curious as to how aeroplanes could stay up in the air when one happened to fly by. I had come across this in class in about Year 2/3 but I had never fully grasped why? Dad said that when we got home we would look up the encyclopaedia and see how planes stayed in the air. Once we had figured out this we were amazed at how heavy they were. When I went to school the next day, I spoke to my teacher (Year 4) Mr E... about it and we went about comparing the weights of different things like planes, cars, people, etc. This led on to a comparison of all of the students in the class's weights and heights. We then made up a chart from this data.	I can remember this vividly because it made my sense of being able to compare things from my environment be realised in my own mind. Because I had asked the questions people were becoming interested and using enquiry as a springboard for different things; like my teacher making an entire lesson sequence out of a single incident. This experience was as a whole positive. I was never really certain when it came to the concept however of the mass of an object or thing. I don't know whether I was sick that week or what but it seems that I have never really grasped the idea not have I felt it necessary to have to have it. However, nor that I will be dealing with it with teaching I feel it would be wise to look into gaining knowledge where I know the knowledge is severely lacking. ... The sense of ownership with

this activity meant that the learning became more meaningful.

3/8/95 Early Secondary School
When I initially went into Year 7, I found the work quite a lot harder, particularly in maths than I had experienced before. My first report card that I brought home to Mum and Dad had me listed as being in the bottom 10% of the class. I was extremely embarrassed at this result because I had been giving the work a real go. The event that finally broke the straw over the camel's back so to speak, was when in Year 8 first semester we were told that we would be sitting an exam that would be a lead up to a maths competition that we were all expected to enter. I became so frustrated and upset that I burst into tears and ran out of the room. The frustration stemmed from the lack of preparation we had been given.

.... In the case of this event that I have described, I personally felt there was not enough consolidation and practising of fundamental skills and routines that would be encountered in 'spot tests' such as these. I also felt that discussion between teacher, pupils and each other is integral to clarifying ideas and concepts that are met. Often cross-ability grouping is a strategy that can be employed for students to gain ideas into how to attack problems in different ways, this sort of informal discussion makes it less intimidating for children to put forward ideas that they may have felt uncomfortable saying in front of the entire class. Teachers must consider all of these feelings of students and should attempt to recognise where problems may occur.

4/8/95 Late Secondary School
By Year 9 (late) to early Year 10 we began to realise that I was falling further and further behind with my maths. By this time Mum had made the decision to get me a maths tutor. The event that finally clinched the tutor was when I brought home my intermediate - (how I had been graded, like streaming) Maths paper and it was obvious that the errors that I was making were mostly due to carelessness - not altogether a total disregard or misunderstanding of formulae or problem solving strategies.

Once again when looking back in hindsight it is clear that students need to practise and repractise concepts that they themselves feel weak on. I know this was the case with me, for when I began being tutored and was able to air my mathematical misunderstandings with someone who could listen I began to gain confidence and feel much more relaxed about tackling mathematical problems than I had previously been (see entry dated 31/7/95). Even though I had expressed to my Maths teachers that I was having difficulties it became a problem to catch them to assist me whenever I needed help. The one to one coaching I got with my tutor benefited me far more than any of the teachers I had at school did.

Another problem in the latter years of my High School was the teacher turnover rate. I think I had about 7 different Maths teachers in two years. ... it seemed we were continually being presented with casuals who appeared to have an interest in teaching only those who could keep up. I believe this also contributed to my poor attitude towards Mathematics as a whole.

I realised that I am not a risk taker, I prefer to be shown something concretely not in an abstract manner. This emphasised to

> me the multitude of differing learning styles that can exist in one classroom and has made me stand back and think about the role of a mathematics teacher in the learner's understanding and the impact that a teacher who is willing to listen can make on students.

Comment: Comments will be made in terms of the student's perceptions concerning mathematics, apparent beliefs concerning learning and teaching mathematics and evidence of the student's growth and development.

This student has had experiences (such as shopping with her parents) which are fairly common to a number of other students. She was fortunate in that her parents took the trouble to introduce mathematical concepts into many of the everyday activities round the home and on outings. This interest on the part of the parents obviously had a great influence on her and helped to develop an initial very positive attitude to mathematics. This positive attitude was further promoted by the teacher who was alert and secure enough to use her query in a very creative way, thus involving the whole class in the learning process and also affirming her as an integral and important part of the decision making of the classroom.

Despite these positive early experiences shared with her parents, Dee developed a view of mathematics which saw the subject as becoming increasingly harder. She also perceived it as a subject in which practice is necessary. This implies a belief that mathematics consists of routines and facts to be memorised.

For Dee, mathematics needs to be expressed in a concrete form for her to learn it and practice plays a significant part in her view of mathematics learning. Ownership of the knowledge is important, as is the possibility of negotiating decisions that relate to her learning. This possibility of negotiating was no doubt strengthened by her experiences with her parents who provided good role models for inquiry, re-inforcement and decision making.

Because of her own experience, Dee recognises the necessity for the teacher to take note of students' learning styles, to listen to students and to provide practice and concrete materials or examples. Her own experience has highlighted for her the pre-eminence of affective issues in learning/teaching situations.

Example 2. Subject Bea

1.8.95 Reflections on Week 1 Reading

> Theorists pose a huge dilemma for me. From memory I believe my past teachers must have had receptive-accrual views of learning.
> *Primary School* - Maths lessons particularly in primary school, involved primarily drill and practice. Although it sometimes became a little monotonous, I feel that this method was quite successful. But now the research suggests alternatives. These cognitive-mediational views do seem to promote much more stimulating and meaningful learning experiences but it is all so foreign to me.
> *University* - My only experience with such teaching was in Maths I last years. I just hope this course and the next will provide me with practical instructional advice. I want to be able to facilitate my students in constructing their own understandings if that is the most meaningful way of learning, as the constructivist theories argue. My concern is that once I leave university, I may feel overwhelmed by everything and revert back to what I know.

Comment: Subject Bea found mathematics lessons involving drill and practice sometimes monotonous yet felt they were successful. Mathematics seems somewhat of a mystery to Bea.

It is interesting that she is challenged by her reading to consider alternative methods of presenting mathematics and seems to have espoused the constructivist theory. Her participation in workshops would no doubt reinforce theory and this is indicated by her final entry in which she writes:

> "This journal has really exposed 2 main points that I hope to incorporate in my lessons in the future. Firstly, learning experiences should be motivating and enjoyable. Secondly, and most significantly, activities should make obvious the relevance of the concepts and skills to the individual's life."

Bea has been through a transformation in thinking about the methods she will use in teaching mathematics. The anxiety expressed in her entry on August 1, however, is a timely reminder that there is the danger of reversion to old experiences and habits if she is not encouraged and supported in her endeavours to put into practice her new-found ideas.

Example 3. Subject Kay

1/9/95 *Primary School Event Recalled*: When I was at school I always had trouble remembering my multiplication tables. I have mentioned before how I sat down with my mother and went over and over my tables. Well, as I got older, in about grade 4 and 5, I would use the multiplication table graph located on the back of text books. I had this teacher in Grade 5 who made me go and stand against a brick wall for using the back of my book in an exam. I was really embarrassed and ashamed of myself.

Reflections on the mathematical significance. I would be naive to think I am the only person who had troubles remembering my tables. I am sure to come across many students who are struggling to remember them. So what will I do? I believe, rather than take away from the students their "security blankets" we should try to build on them. I am aware that students have to learn their tables. However, we are all individual learners, whom some of us will simply have difficulty putting the facts into our heads.

A way of approaching the problem posed above, from my own history, would be to analyse the multiplication table chart. Students can find different patterns on the chart, so they are re-inforcing problem solving and getting familiar with identifying patterns, whilst generating useful cues for the memorisation of these necessary basic facts. This would also help students to feel more confident and able.

This same student was able to also write about her later achievements and attribute her success to her tutor.

25/9/95 Yr 11 and Yr 12

Whilst I was completing 2 unit mathematics for the HSC I received tutoring. I used to travel to the University of Technology, Sydney, for a 2 hour class. There was a group of about 15 people. There were three tutors available, of these one of the tutors would teach us the topic for the day.

Franca, the woman who used to teach the class. was a truly inspiring person. I believe it was not only the energy that she transmitted when she was teaching, but it was the way that she took a personal interest in every individual there. She would want to know how I was going at school, my results. She showed genuine keenness.

I went from a rather average grade student in mathematics to finishing first place in my school for the 2 unit mathematics course. It surprised my teachers, my

> parents, my brother and even me when I began getting remarkable grades, but Franca never seemed surprised, just pleased for my hard work paying off.
> The point that I am trying to illustrate here is that I was always considered average at maths by teachers, my parents and myself. Franca never put a limit on my ability and I achieved more than just average. She helped me to build confidence in myself and what I could do.

Comment: Kay exhibits a broader view of mathematics than the other two students. Despite her inability to learn tables, she sees patterns as being very important in mathematics and uses them to overcome her own particular deficiency. She also incorporates problem solving as an important aspect of mathematics.

Like Dee, Kay recognises the individuality of learners and the need for re-inforcement when teaching or being taught. The good mathematics teacher, to Kay, is one who is energetic, inspiring and takes a genuine interest in students, thus underlining the importance of affective issues in the mathematics classroom. The other characteristics of the tutor who inspired Kay was that she believed in Kay - believed that she could do the mathematics. She did not set a limit on what she could do but encouraged and supported her. As a consequence, Kay was able to make great progress in mathematics.

The confidence Kay gained through having a tutor who listened to her and believed in her overcame her previous perception that she could not succeed in mathematics. Her growth in her own mathematical understanding can be inferred from her overt commitment to more meaningful ways of teaching mathematics and from her desire to help students feel confident and able to do mathematics.

Conclusion

The use of mathematical life histories can have several benefits for all and especially for teacher trainees as they seek to overcome inadequate knowledge of and attitudes towards mathematics. As in the case of the above students, reflection through mathematical life histories can provide insights into students' beliefs concerning mathematics, a greater understanding of the levels of confidence students experience and an awareness of changes and influences for change in the understanding of the mathematics teaching/learning process. The process of reflection helps to clarify ideas, transform beliefs and strengthen resolve. This, it seems, needs to be supported and encouraged within the context of good role models and a sound understanding of the teaching-learning process.

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A Framework for Research into Students' Problem Posing in School Mathematics

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Research into the potential of problem posing as a means for developing of students' understanding of mathematics has been hindered by the absence of a framework which links problem solving, problem posing and mathematics curricula. This paper presents an overview of the frameworks used by researchers for investigating problem posing, and proposes a framework for research into students' problem posing in mathematics. Examples of problem-posing situations used in a classroom with mathematically able students are presented.

Definitions of Problem Posing

The notion of problem posing has been explored by different researchers from contrasting perspectives. For example, problem posing has been viewed as the generation of a new problem or reformulation of a given problem (Duncer, 1945); as the formulation of a sequence of mathematical problems from a given situation (Shukkwan, 1993); or as a resultant activity when a problem is inviting the generation of other problems (Mamona-Downs, 1993). Dillon (1982) conceptualised "problem finding as a process resulting in a problem to solve."

Silver (1993, 1995) referred to problem posing as involving the creation of a new problem from a situation or experience, or the reformulation of given problems. Such problem posing could occur prior to problem solving (when problems are being generated from a given contrived or naturalistic situation), during the problem solving process (one can intentionally change some of the problem's goals or conditions), or after solving a particular problem (as would be the case when problems are generated on the basis of the experience gained by solving a particular problem or a set of problems).

In this paper mathematical problem posing will be defined *as the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems.*

The definition is deliberately broad to enable problem posing to fit within the goals of mathematical instruction in the context of school mathematics. The broaden definition also means that the researcher can explore the interrelationships between problem posing and problem solving as a means of mathematical instruction, and can examine the design of a wider range of problem-posing situations.

Recognition of Problem Posing

Many prominent scientists have recognised that the ability to *pose* significant questions had an equally important role to play in their scientific work as the ability to *solve* them. Einstein and Insfeld (1938), for example, wrote: "The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skills. To raise new questions, new possibilities, to regard old questions from a new angle, requires creative imagination and marks real advance in science"(p. 92).

The significance of the solution of a specific problem depends, to very large extent, on the significance of the question asked. In his investigation on "expert" and "novice" scientists, Zuckerman (1977) found that the "expert" scientists differed from the "novices" not so much in the *answers* as in the *questions* that the two groups of scientists posed.

In mathematics education, after over a decade of studies which have focused on problem solving, researchers have slowly begun to realise that developing the ability to *pose* mathematics problems is at least as important, educationally, as developing the ability to *solve* them. Commentators such as Resnick and Klopfer (1989) have recognised that helping students become competent thinkers is a central challenge for all educators. It has also been recognised that the incorporation of problem-posing activities into regular classroom situations can be a powerful approach for developing students' mathematical thinking (Silver, Kilpatrick & Schlesinger, 1990).

The mathematics curriculum documents of several countries have acknowledged the impact which problem posing could have on mathematical instruction. In Australia, for example, *The National Statement on Mathematics for Australian Schools* offered strong support for the use of open-ended problems in mathematics classrooms with the words: "Students should engage in extended mathematical activities which encourage problem posing, divergent thinking, reflection and persistence. They should be expected to pursue alternative strategies, and to pose and attempt to answer their own mathematical questions" (Australian Education Council, 1991, p. 39).

In the United States, *The Curriculum and Evaluation Standards for School Mathematics*, (National Council of Teachers of Mathematics, 1989) acknowledged the importance of having students experience some of the problem-posing aspects involved in the work of mathematicians: "Students in grade 9-12 should also have some experience recognising and formulating their own problems, an activity that is at the heart of doing mathematics" (p. 138). Investigative mathematical projects have been used in Victoria for assessing students' knowledge and skills since the late 1980s (Stacey, 1995).

There is a growing interest towards incorporating problem posing-activities into mathematics classroom (see, for example, Brown & Walter, 1983, 1990, 1993; Kilpatrick, 1987; Silver & Cai, 1993; Silver & Mamona, 1989; Silver, Kilpatrick & Schlesinger, 1990), and researchers have tried to use different frameworks for exploring problem posing. This movement makes it all the more important for researchers to develop appropriate frameworks for exploring problem posing.

Research Framework for Exploring Students' Problem Posing in Mathematics

Research into the potential of problem posing as an important strategy for the development of students' understanding of mathematics has been hindered by the absence of a framework which links problem solving, problem posing and mathematics curricula. Before the effects of problem posing and its implication for the teaching and learning of mathematics can be adequately researched, such a framework needs to be developed and refined in the light of data gained from its application in the classroom. This paper outlines the basis of such a framework, and emphasises the potential offered by extending Krutetskii's problem-solving categories as problem-posing situations.

Central to the framework proposed in this paper is the notion that every problem-posing situation can be classified as free, semi-structured or structured. All three categories have, in effect, been used by researchers for investigating various aspects of the effects of problem posing on mathematical instruction, but a framework which places a range of problem-posing situations into one of these three categories has not been proposed before.

We will describe a problem-posing situation as *free*, when students are asked to generate a problem from a given, contrived or naturalistic situation. Some directions may be given to prompt certain specific actions. Many researchers have used free problem-posing situations in their studies. For example, Ellerton (1986a, 1986b) introduced creative writing in mathematics by asking students to make up mathematics problems. She asked Australian students to pose a problem which would be difficult for a friend to solve. She also asked students to write a letter to a friend, who had been away ill from school, describing the mathematics which the class had done during the past 3 weeks. As part of the letter, students were asked to make up mathematics questions which were typical of those which they had encountered during the same period. She used this framework as window into exploring students' perceptions of mathematics. According to Ellerton (1988), "children's expression of mathematical ideas through the creation of their own mathematics problems demonstrates not only their understanding and level of concept development, but also reflects their perception about the nature of mathematics" (p. 281).

Richardson and Williamson (1982) used another form of free writing. They asked children to make up mathematical problems for each other. In his study, Kennedy (1985) used forms of writing to his mathematics students as writing letters about what were they studying, keeping logs and devising mathematical problems about a particular topic. Problem-posing activities involving much younger children have been described by Van der Brink (1985), who asked Grade 2 children to make up problems and games for Grade 1 children.

A problem-posing situation will be referred to as *semi-structured* when students are given an open situation and are invited to explore the structure and to complete it by applying knowledge, skills, concepts and relationships from their previous mathematical experiences. Hart (1981), for example, asked children to make up mathematics problems to fit given computations. Her aim was to study how children draw on concrete situations in describing symbolic expressions. Winograd (1991) used posing and sharing story problems as a research tool and found that children generally composed problems which they themselves had difficulty understanding or solving. Writz and Kahn (1982) observed that having students make up applications helped them to bridge the gap between concrete situations and mathematical abstractions. Furthermore, it appeared to help students to learn how to generalise, as well as making mathematics more meaningful to them. Students who wrote problem stories tended to learn to integrate mathematics with other subject areas and to develop creative writing skills (Bush & Fiala, 1986).

A problem posing situation will be called *structured* when problem-posing activities are based on a specific problem. In order to reveal the structure of students' mathematical abilities Krutetskii (1976) used a research tool involving students in finishing or reconstructing a specific problem structure. In his study he used problems with unstated questions, problems with insufficient and problems with surplus information.

In several studies, researchers have asked students to pose problems similar to a given problem as a tool for exploring some aspects of their mathematical performance. Hashimoto (1987), for example, found that asking students to pose a problem *similar* to a solved problem can be a useful teaching technique for providing a mirror into students' understanding of mathematical concepts.

Stover (1982) investigated the consequences of having students make format changes to mathematics problems. In this study, sixth-grade students were asked to modify one of three structural format variables (by adding a diagram, or removing extraneous information, or reordering information) in the statement of a problem, and observed substantial improvement in students' ability to solve word problems of the type they had learned to modify. Smilansky (1984) investigated the relationship between being able to solve problems and to pose problems in the same domain. After he had collected students' responses to a mathematical test, he distributed a skeleton test page and asked the pupils to create new problem which would be particularly difficult in the future version of the test. Smilansky found a low correlation between the performance on the problem-solving task and the problem-posing task in the same domain.

An Investigation of Problem Posing in the Classroom

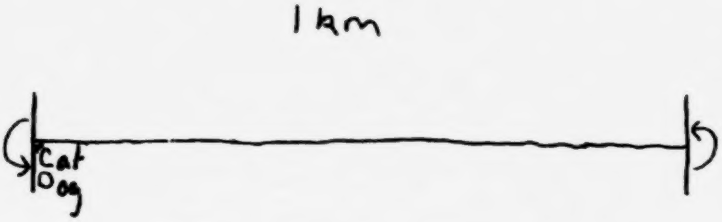
In order to investigate the range of problem-posing situations which could be used as part of problem solving environments, a one-year study with mathematically able children was undertaken. A total of 40 Years 8 and 9 students from different schools in Perth, were involved in a mathematics enrichment program. Students took part in the program for one hour per week, from the beginning of February to mid-November, 1995.

The design of the problem posing situations was based on the following basic assumptions: (a) problem-posing situations should correspond to, and arise naturally out of, pupils' classroom mathematics activities; (b) problem-posing situations could be generated from textbook problems, by modifying and reshaping the language and task characteristics; (c) problem-posing situations should be a part of pupils' problem-solving activities.

Most of the structured problem-posing categories were inspired by Krutetskii's (1976) work. In fact, although Krutetskii's major focus was problem solving, his insights into the relationship between problem solving and problem posing has prompted us to reflect on how his ideas could be extended to embrace both problem solving and problem posing. This infers that Krutetskii's problem-solving categories can be readily applied by educators wishing to develop quality structured problem-posing situations for mathematics classrooms.

Free Problem-Posing Situations

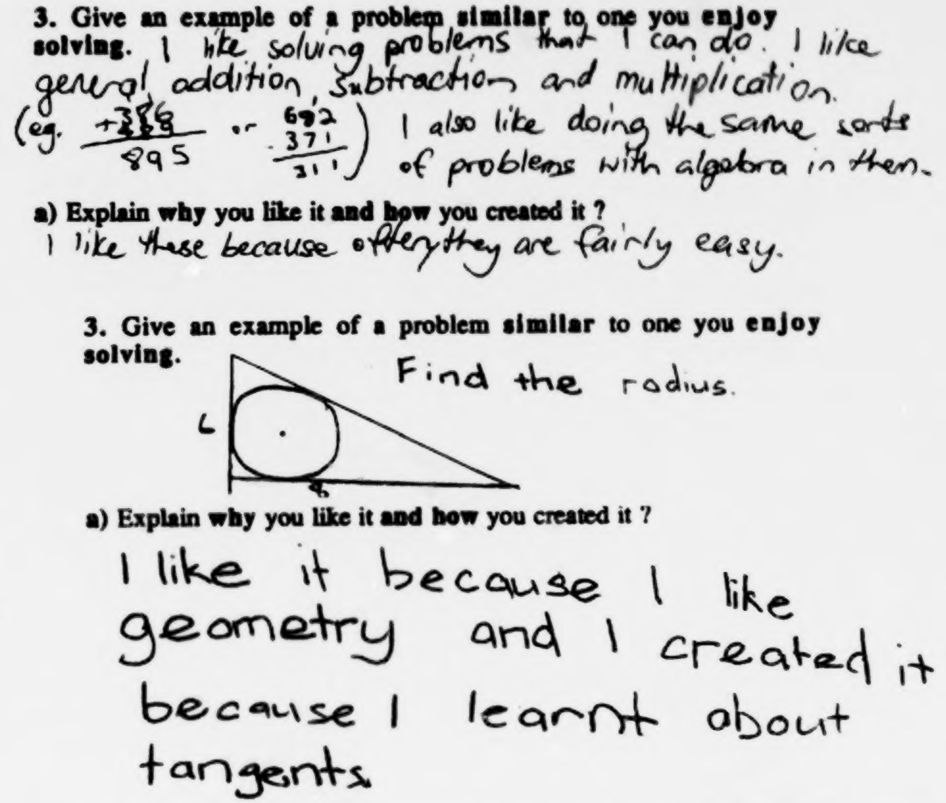
In order to encourage students to reflect on their specific previous experience, the free problem-posing situations used in this study were addressed to problem posers, or placed the problem posers in a situation where they were forced to consider the person(s) for whom they were posing the problem. For example, students were asked to pose problems for mathematics competitions (see Figure 1); problems which they like (see Figure 2); problems which they found difficult (see Figure 3); and problems which would have to be solved by their teacher.



1. If, over a distanceof 1km, the road has a big piece of concrete along the middle, and the dog and cat can't see/smell over the concrete; How long would it take for the dog to overtake the cat if once they had reached the end, they turned around and ran on the other side of the road; and the cat ran at 10kph and the dog ran at 20kph?

2. If, under the same circumstances, the dog tired after 2kms, and only ran at 15kph, how far would the dog have to run before he overtook the cat?

Figure 1. Mathematics problem posed by one student for a mathematics competition.



3. Give an example of a problem similar to one you enjoy solving. I like solving problems that I can do. I like general addition, subtraction and multiplication. (eg. $\begin{array}{r} 386 \\ +509 \\ \hline 895 \end{array}$ or $\begin{array}{r} 682 \\ -371 \\ \hline 311 \end{array}$) I also like doing the same sorts of problems with algebra in them.

a) Explain why you like it and how you created it ?
I like these because ~~often~~ they are fairly easy.

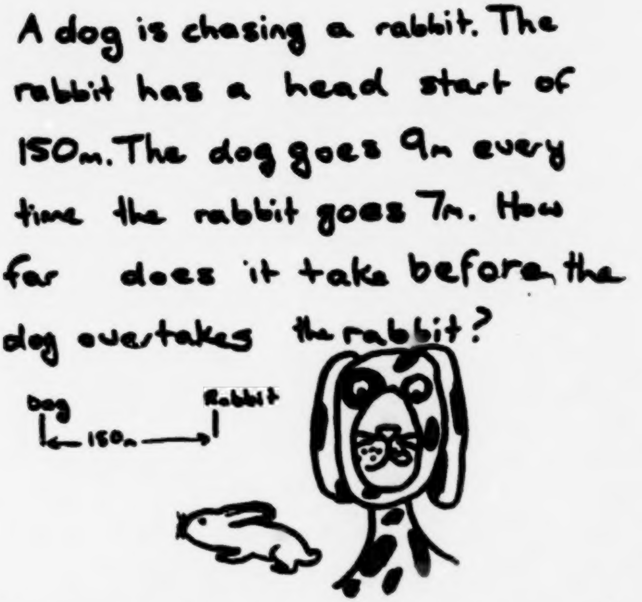
3. Give an example of a problem similar to one you enjoy solving.

Find the radius.

a) Explain why you like it and how you created it ?

I like it because I like geometry and I created it because I learnt about tangents.

Figure 2. Mathematics problems created by two students to illustrate problems they enjoy.



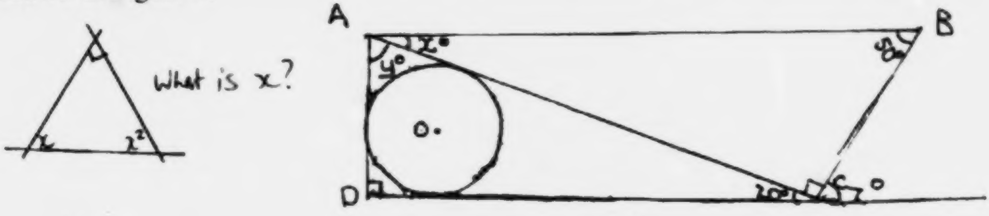
A dog is chasing a rabbit. The rabbit has a head start of 150m. The dog goes 9m every time the rabbit goes 7m. How far does it take before the dog overtakes the rabbit?

Dog
Rabbit
150m

Figure 3. Problem posed by a student when asked to create a question that she would find difficult to solve.

Semi-Structured Problem-Posing Situations

The semi-structured problem-posing situations used in the study ranged from situations incorporating unfinished structures to posing sequences of interconnected problems. We will mention only a few here. For example, students were asked to pose a problem which involved the use of a concept of the right-angled triangle—two responses are shown in Figure 4.



(a) (b)

Figure 4. Two problems posed by different students involving right-angled triangles.

Responses to another form of semi-structured problem posing are shown in Figure 5. These illustrate the posing of a class of problems related to a specific solution method—such as the use of the Pigeon-hole Principle, permutations, combinations, and working backwards.

$a * b = (a+b) \times \frac{ab}{a} - \left(\frac{ab}{1} - b\right)$

What does $3 * 4 =$

In how many ways can 12 boys and 10 girls stay in a line if the girls occupy the middle two places?

(a) (b)

Figure 5. Two problems posed by different students which involve (a) constructing a new operation by working backwards, and (b) permutations.

Semi-structured problem-posing situations can also involve giving the students unfinished problem structures, and asking them to describe what kind of problems could be created on the basis of the information given. The unfinished problem structures can be given either by a picture, equation, calculation or inequality. Figure 6 gives a student's response to the question "Make up as many problems as you can using the following calculation: 3 x 25 + 15 + 5 – 4."

- Around which two digits could you place brackets so that the answer is 80?
- What is the prime factors of the answer to this calculation.

Figure 6. Problem posed in response to an unfinished problem structure.

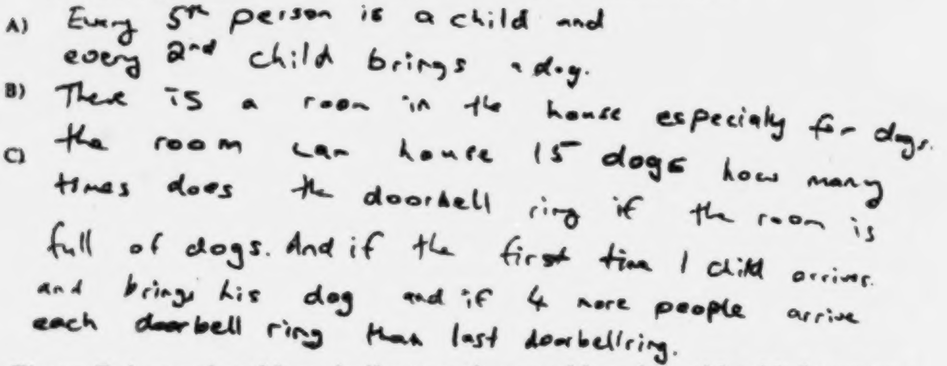
Asking students to restate a problem when its solution is given, or presenting students with a part of the problem statement and a set of possible answers was another form of semi-structured problem-posing situation which was given to the students.

Structured Problem-Posing Situations

Three categories of structured problem-posing situations, based on a specific problem, were trialed. They were aimed at helping students to understand particular problems and solution structures, and to explore possible interrelationships between problem statement and solution ideas.

The section of a problem in which the question was asked was omitted, and students were asked to pose a series of possible questions and to put them in a suitable order. In other cases they were asked to add structure and to pose a question (see, for example, Figure 7) or to find the surplus information and to improve the problem structure. The following situation provided a starting point for the student's response in Figure 7:

> Last night there was a party and the host's doorbell rang 10 times. The first time the doorbell rang only one guest arrived. Each time the doorbell rang after that, three more guests arrived than had arrived on previous ring.
> Ask as many questions as you can. Try to put them in a suitable order.



A) Every 5th person is a child and every 2nd child brings a dog.

B) There is a room in the house especialy for dogs.

C) the room can house 15 dogs how many times does the doorbell ring if the room is full of dogs. And if the first time 1 child arrives and brings his dog and if 4 more people arrive each doorbell ring than last doorbellring.

Figure 7. A posed problem similar to a given problem, but with added structure.

During, before and after solving a specific problem students were asked, on a regular basis, to suggest changes in the problem which might (or which do not) affect the solution method. Students were also asked to suggest a problem which resembled a given problem but might have a different solution method, and to pose a problem which is the inverse of the given problem

Restating a problem on the basis of its solution was another problem-posing situation used in the study. Students were asked to pose problems with different task formats, including "regular" problems, and multiple choice questions. In some cases the problem statement was given by a series of pictures. Improving the characteristics of a written solution by determining the main steps in the given solution and improving the language was another of the problem-posing activities in which the class engaged.

Conclusion

Although problem posing has had greater attention and recognition in recent years, the lack of a research framework which links problem posing, problem solving and school curricula has reduced the credibility of research in this area, and has delayed any systematic implementation of problem-posing situations into mathematics classrooms. This paper, with its three categories of problem-posing situations, and its set of problem-posing examples in each of these categories, is a step towards bridging the gap.

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Identifying Teacher Education Students' Views of Quality Mathematics Teaching

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This report arises from a project which uses an interactive multimedia resource to support learning about teaching mathematics. As one of a range of data collection instruments, pre-service teacher education students completed a card sorting task (using Q sort technique) before and after using the resource. The technique provided data to supplement other more general qualitative data. Results indicate that use of the resource helped focus students' attention on some key characteristics of quality mathematics teaching, and that the instrument was useful as one of a range of sources for qualitative data.

Learning About Teaching From the Study of Particular Examples

Initial education students bring to their courses a wealth of experience and some firm ideas about what good teachers do. It is necessary for teacher educators not only to build on those experiences but also to stimulate critical reflection on them. Discussion of abstract concepts and generalisations or the presentation of theories in the absence of practical contexts may simply result in the reinforcement of pre-conceptions about teaching. The basic proposition which underlies this project, of which this report is a part, is that the study of particular exemplars of quality practice can both stimulate reflection on key components of teaching and allow prior conceptions to be challenged.

In 1995, pre-service students used an interactive multimedia (IMM) resource to observe and analyse teaching exemplars. This was a work requirement of a second year undergraduate course on the study of mathematics teaching. Interaction with the resource was in small groups, for approximately 20 hours. Study and discussion of particular cases were used to prepare the students for, and to supplement, practicum experience. Merseth and Lacey (1993) suggest that the use of teaching exemplars in this way can develop skills of critical analysis and problem solving, can represent the complexity of teaching situations, can foster multiple perspectives and levels of analysis, and can offer students opportunity to engage directly in their own education. The course of study also included more conventional components such as reviews of the literature, lectures, and tutorials on aspects of mathematics teaching.

Prior to the development of the IMM resource, this project identified and developed a framework for describing elements of quality teaching. It used analysis of recent literature and a survey of 200 practitioners, teacher educators and other education professionals from several countries (Sullivan & Mousley, 1994). Full lessons which exemplified the components of quality teaching were partially scripted, then taught and videotaped. These tapes were examined using several techniques, including a qualitative analysis of unstructured reviews of the lessons by over 30 experienced teacher educators (see Mousley, Sullivan, & Gervasoni, 1994), then transferred to CD-ROM disc. An IMM computer environment was authored to provide flexible access to all aspects of the lessons.

Merseth and Lacey (1993) argue that the potential of multimedia includes the possibility of introducing the complexity of teaching to novices, and that the non linear capability distinguishes multimedia from conventional videotape since they allow the use of multiple perspectives and opportunities to review situations. The IMM resource under study includes videotapes of a mathematics lesson, other video records such as pre and post lesson interviews with the teachers, procedural documents and readings associated with the lessons, graphic representations of data, and other appropriate resources. It is an extensive information bank which can be accessed in flexible ways to support detailed classroom observation and analysis. Indexing and the interactive nature of the program allow scenarios to be accessed and linked to other data, enabling users to focus on specific teaching skills, moments of interaction, selected sets of incidents, sequences of events, links between written theory and action, and so on.

The larger project aims to evaluate the effectiveness of this approach in stimulating analysis of aspects of teaching. In 1995, data were sought through structured survey items, the writing of lesson critiques, responses to formulated scenarios, unstructured interview items, observation of students' interactions with each other and with the program as they worked, journals maintained throughout the process, and reflective essays written after the completion of the program.

This paper presents the results for one particular instrument and discusses both the results and the effectiveness of the instrument.

Identifying Teacher Education Students' Views of Quality Mathematics Teaching

The particular research tool reported here—a card sort—was used to provide some fixed data about beliefs of the student teachers prior to their interaction with the IMM resource and after their use of the resource. The procedure used was an adaptation of the Q-sort technique (Sax, 1979; Best & Kahn, 1986) which had been structured for intensive work with few persons. In this process, statements are written on cards and respondents are asked to place each card, in piles, along an 11 point linear scale from a strong negative to strong positive response. A variation on this procedure was used by Pelto (1970), who asked his subjects to arrange factors written on cards in a priority order, to determine the most influential member of a community. He then sought explanations on the reasons for such a choice of rankings. The sorting of factors written on cards has also been used to determine extremes for a semantic differential instrument (Clarke, 1989).

The advantages of using a similar procedure for this investigation were that:

a) the presented factors or components could be determined by teacher education students themselves, to maximise the understandability and communicability of the terminology;

b) it facilitated the arrangement and ranking of factors more readily than could be done with a written list;

c) it allowed the inclusion of various components from the investigation and to test, by non-inclusion, least important components ;

d) in addition to ranking, it allowed the subjects to distinguish between ranks; and

e) it facilitated comparisons of the perceived components prior to interactions with the resource with those after interactions with the resource.

The first step was to conduct interviews with teacher education students who were similar in background to the subjects of the study. They were asked to indicate what factors they saw as the features of quality teaching. Some of these students were interviewed in small groups, others were interviewed individually. The ideas presented by the students were separated into discrete categories and recorded.

Independently of this process, the researchers themselves determined a number of factors and wrote these down in separate categories. These factors included points identified in the original survey of teacher educators and used as a basis for the development of the IMM resource. From these collective ideas a single set of cards was prepared, each of which showed one factor seen as a key feature of quality teaching.

These cards were presented to the students during an interview which was seeking views of components of quality teaching through a range of question types. The purpose of the cards was explained to the students, as was the source of the factors contained on the cards. The students were asked to arrange the cards in order, from most important to least important. They were told it was possible to have equal placings. After they had arranged the cards in order, they were asked to indicate a point below which the cards were of very little or no significance to them. These cards were then separated out. The students were then asked to explain their understanding of the top five cards in their list, and also to indicate what influenced them to choose those five cards in particular.

The descriptors on the cards used were as follows:

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control and management
communicate with students on their own level
discipline
good communication skills
good organisation
getting the children to think for themselves
respect needs of children
inspire children to create and succeed
variety
clear explanations of what to do
saying children's names when asking questions
honesty
open body language
monitoring student understanding
flexibility
meaningful and relevant
telling children on what they are being assessed
creativity
providing clear work examples for the c'n to copy
lesson fun for students
confident, caring, but strict
catering for all abilities
ability to develop rapport with students
approachability
encouragement

Figure 1: *Factors used as the basis of the Q-Sort technique*

The main purpose of collecting data in this way was to provide a foundational base against which other data collected in response to more open-ended items could be established and verified. It also allowed comparison between the subjects and appreciation of the significance of each of the factors whether determined through open-ended type responses or through structured instruments, such as this one. A further use of the instrument was to try to obtain more substantive basic data which could be compared both prior to, and subsequent to, interactions with the resource. The intention of this structured data collection was to both enrich and triangulate data sought on unstructured items. It is certainly not suggested that such instruments be used on their own, but only to the extent that they support other data. One advantage of using the cards in this way is that it provides a meaningful prompt for the subjects and it also allows more substantial links to be made between what the subjects believe to be key factors and the terminology used by the researchers.

In the project overall, ten undergraduate teacher education students used the IMM resource as a trial group, and were the focus of the study. For this instrument data from only eight students are available. One student did not rank the cards ("They are all important") and merely sorted the cards into categories. One other student had one component of the data missing and so their data are not included in the analysis.

In order to convey the type of data available and the results themselves, the results from four of the eight available data sets are presented in Tables 1 and 2. The data presented are from the interview prior to interaction with the resource, and also from the interview after completion of the resource but prior to teaching practice. The highest ten ranked cards are listed in each case. Dot points (•) are used to indicate cards which were given a particular rank. A forward slash (/) is used to indicate cards which are assigned equivalent ranks. Also presented are the explanations given to the interviewer on the meaning of some of these cards and the justification for their selection. The cards which were not ranked in the top ten cards prior to the interactions but were included after the interaction with the resource are shown in italics.

Table 1: Responses of students N and M to the Q-Sort technique

	N	M
Pre interaction factors	• Inspire students to create and succeed / Control and management / Good organization / Catering for all abilities / Getting the children to think for themselves • Meaningful and relevant / Good communication skills / Discipline /Encouragement / Creativity / Respect needs of children	• Clear explanations of what to do • Good organization • Encouragement • Variety / Flexibility • Good communication skills • Providing clear worked examples for the children to copy • Getting the children to think for themselves • Meaningful and relevant • Communicate with students on their own level
Explanation of pre interaction factors	Well the whole point of kids being at school is to 'inspire' them. You've got to be able to 'manage and control' them, 'good organization' and 'cater for their abilities' so they're going to be 'inspired to create and succeed'. So they're … not going to turn around and get to high school and say I hate school …So sort of 'cater for their abilities' but, and 'getting them to think for themselves' because that's the whole point, I mean they're not going to learn if you're just going to get up there and tell them everything.	Clear explanation of what the children are to do' because I feel that if they haven't got that then they get confused and frustrated and, ah, they won't do as well in the lesson. Ah, 'organization' … because you're more confident, you know what to do, and that shows to the children. Encouragement, because once you encourage a child if they answer a question correctly you praise them and then they feel good about themselves and they're more likely to respond again.
Post interaction factors	• Monitoring student understanding / Good communication skills / Encouragement / • Catering for all abilities / Good organization / Flexibility / Control and management / Meaningful and relevant / Creativity / Getting the children to think for themselves / Clear explanations of what to do / Open body language	• Good organization • Clear explanations of what to do / Getting the children to think for themselves • Variety • Catering for all abilities / Monitoring student understanding • Meaningful and relevant • Communicate with students on their own level / Good communication skills • Encouragement / Inspire students to create and succeed
Explanation of post interaction factors	'Monitoring student understanding' . That's evaluating what was taught and also evaluating how you have taught it. 'Getting children to think for themselves', … they are there to think for themselves. … if they think for themselves it shows they have an understanding of what is going on. 'Clear explanation of what to do', .. she or he can understand what they're teaching. 'Flexibility',…being able to change the lesson if it's going on the wrong track.	On my first teaching round I found that the lessons that were organised and that I had spent time on were definitely a lot more successful that the ones I had planned the night before or the recess before. And again, if you're organised you are able to give 'clear explanations to the children of what you expect of them' and that helps them because then they're not confused and all that. 'Variety' so they don't get bored and restless with the lesson.

Table 2: Responses of students A and K to the Q-Sort technique

	A	K
Pre interaction factors	• Catering for all abilities / Encouragement • Good organization / Lesson fun for students / Flexibility • Clear explanations of what to do / Providing clear worked examples for the children to copy / Open body language • Meaningful and relevant / variety / creativity / honesty	• Lesson fun for students / Variety / Good organisation / Creativity • Good communication skills / Confident, caring but strict / Communicate with student on their own level / Catering for all abilities • Honesty / Encouragement / Providing clear worked examples for the children to copy • Meaningful and relevant / Clear explanations of what to do
Explanation of pre interaction factors	Well, 'catering for all abilities' means that you can't have a lesson where some students find it really easy and they can just breeze through it, and others find it really hard. 'Good organization', I think that's just planning everything, having a developed plan, just so you know what you're talking about. 'Lesson fun for students', ah, they're not going to enjoy it if they have to just sit there and take notes, … And with 'flexibility', like, just say you're catering for all abilities. You've got to be able to not just stick to the one thing, you've got to be able to plan for the students who are good and for the students who may not finish as quickly.	Lesson fun for students so that they've got interesting lessons. 'Variety' so that they're not doing the same old boring lesson everyday like for maths, not doing tables or multiplication or that. Ah, 'good organisation', so that I'm organised, the teacher's organised and knows exactly how it's going to shape out so she answers any questions and she's able to work through it. 'Creativity', making the lesson fun and exciting. And, 'catering for all abilities' not just going after the high level ability in your class, catering for everybody, of every level.
Post interaction factors	• Good organization / Good communication skills • Getting the children to think for themselves / Clear explanations of what to do / Providing clear worked examples for the children to copy • Creativity / Variety / Control and Management / Respect needs of children / • Catering for all abilities	• Catering for all abilities / Meaningful and relevant / Getting the children to think for themselves • Variety / Flexibility / Lesson fun for students / Good organisation / Good communication skills / Communicate with students on their own level • Encouragement / Monitoring student understanding / Inspire students to create and succeed
Explanation of post interaction factors	'Good organisation' means that you are well prepared for class. And you know what you are talking about and so that ties in with 'good communication skills' so you can teach the students clearly. 'Getting students to think for themselves' means that you are not up there just to dictate to them, you have got to allow them to basically think for themselves, to work it out, without just giving them the answers all the time. 'Clear explanations of what to do' so the children know what they have to do so you are all sure of that. And 'providing clear worked examples for the children to copy' they can, the teacher can do one or two so the children know how to set it out and how to go about working it out, but then they can do them for themselves.	'Catering for all abilities' so you provide the lesson so that you've got the 'slow' ones and the 'strong' ones. 'Meaningful and relevant', again, so that they think it has purpose. That you are not just saying it and then forget it. 'Getting the children to think for themselves' so that they feel that they are involved in the lesson, so that their lessons they have got to provide the answers to the questions. 'Communication on their own level' so that you are not talking above them, so that they are able to understand. 'Good organisation', you don't, so that when you get in the class, you don't want to have to be running backwards and forwards, and losing all the children.

Space does not allow full presentation of the data, but these are a fair indication of the results overall. The following discussion refers to the information from all eight subjects.

One advantage of collecting data in this way is that it is possible to formulate overall impressions of the beliefs of the subjects. It is clear that considerable care should be exercised before generalising from this group to any other, and indeed that is not the intention of the subsequent analysis. It is also recognised that the numbers of participants is relatively small. However, a summary of the data as a whole gives insights into both the beliefs of these students and impact of the resource.

There was a somewhat unexpected commonality in the responses prior to any interaction with the resource, which also happened to be prior to any teacher education studies. The students had predominantly studied only units in the discipline major and minors up to this phase of their course.

Table 3 presents those cards which were selected by at least half of the students before and after the interactions with the resource.

Table 3: Common factors selected by the students

Factor	Pre	Post
catering for all abilities	5	7
clear explanations of what to do	5	5
control and management	5	4
creativity	7	5
encouragement	7	6
getting children to think for themselves	4	6
good communication skills	4	6
good organisation	8	8
inspire children to create and succeed	5	4
meaningful and relevant	5	4
variety	5	4

One impression gained is that there is stability and indeed good sense in this ranking of factors prior to any teacher education studies. Not only was the list of factors, predominantly suggested by such students themselves, both comprehensive and meaningful, but the ranking of the factors overall is compatible with what we might expect that experienced practitioners might believe. Even though these beginning teacher education students have significant experience with classroom teaching, these experiences may well be generally productive and oriented toward a desirable teaching ethos.

A further impression is that even after significant student-focussed input, there was little change to individual students' impressions as well as students' impressions overall. To the extent that there was a change, this was in a direction compatible with the goals of the resource. The cards which were selected after interaction with the resource, but not beforehand, by more than one student were *Clear explanations of what to do* (by 3 students), *Getting children to think for themselves* (3), *Communicating with students on their own level* (2), *Discipline* (2), *Flexibility* (2), *Good communication skills* (2), and *Open body language* (2). To the extent that there has been a change, it has been in the direction of a focus towards clarity and good communication and also towards getting the children to think for themselves. As it happens in the lesson presented in the resource the teacher explains clearly what the students should be doing, and communicates effectively with the children.

One apparently contradictory point is that a good number of the students selected both *Giving clear explanations* and *Getting the children to think for themselves* among their factors. In the lesson illustrated in the IMM program, the teacher pays particular attention to the clarity of the instructions about the processes which the children should follow while working in groups, deciding how to record their answers and communicating their findings. However, the teacher does not tell the children how to do any of the mathematics. In other words she makes it clear that mathematically they need to think for themselves, but she provide them ample support on procedural matters.

It is interesting to examine which cards were never or only seldom selected. *Telling children on what they are being assessed*, and *Saying children's names when asking questions* were selected by nobody either before or after. Only one student selected *Monitoring student understanding* and *Open body language*. before using the resource. Afterwards only one selected *Ability to develop rapport with students*, *Confident and caring but strict*, and *Honesty*.

Summary

There were two components to the use of this technique in this way. One was to examine the impact of the multimedia resource via the responses of students to this card sort. It appeared that these teacher education students had well formed views of the task of teaching at the start of their course. These views were generally appropriate, but it illustrates that the task of challenging rather than merely reproducing or reinforcing the status quo is complex and developmental. Because the resource engages the students directly in reflection on, and discussion of, specific examples of teaching, it has the potential to foster this longer term consideration of teaching processes by the students themselves.

It seems from these data that student teacher's beliefs about the important components of teaching are stable. To the extent that there was a trend in the changes in beliefs observed, the influence seems to have been in the desired direction of good communication and the transferring of responsibility for learning to the students themselves.

A second purpose of this paper was to examine the use of such an instrument as a data collection tool. It seems that it is an effective tool that could be used to support other qualitative data. It is certainly not argued that such a tool should be used by itself for data collection. However, it does make a meaningful contribution to qualitative research which can facilitate the sharing of research results with readers. In the case here, readers are invited to form their own impressions of the significance of the cards which we used, and readers can gain some insights into the meaning attributed to the cards by the subjects of the study. Given that these aspects of the data are presented and clear, readers can interpret these data in the light of their own background, or in the light of teacher education students with whom they may be familiar. In this case the data provide a substantive base for communication, which can be used as a baseline for discussion of other, less directed qualitative data.

It seems that the cart-sort technique is worth trialing further as a tool for investigating students' beliefs about teaching and learning, as well as how they might change as a result of a particular learning experience.

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Mathematical Misconceptions Commonly Exhibited by Entering Tertiary Mathematics Students

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In terms of subsequent learning of mathematics, the importance of mastering a limited number of basic concepts cannot be over-estimated. This paper examines the nature and frequency of mathematical misconceptions which have been commonly exhibited in tests at two tertiary institutions. The frequency of mathematical misconceptions seen is of great concern. Hopefully information about which misconceptions occur commonly will lead mathematics educators to place greater emphasis on the teaching of these concepts, thereby causing a decrease in the frequency of the related misconceptions.

Introduction

Due to the sequential nature of many branches of mathematics, it is extremely important that individuals attempting to learn mathematics at a particular level first master the fundamental mathematics which is pre-requisite knowledge for success at that higher level. The mastery of this pre-requisite knowledge depends very heavily on individuals having a thorough understanding of a limited number of basic concepts. As the learning and understanding of these concepts as well as the ability to use this knowledge is so crucial to subsequent learning of mathematics, it is of great interest to mathematics educators (teachers, lecturers and tutors) to be able to ascertain information about the types of misconceptions commonly held by students and the frequency with which these misconceptions occur.

This paper will consider these mathematical misconceptions from the perspective of mathematics students making the transition from secondary mathematics to tertiary mathematics. It is argued that the understanding of the basic mathematical concepts which should be acquired in secondary school is crucial to the preparedness of mathematics students to undertake a tertiary course involving some mathematics units. The preparedness of mathematics students is important from both the viewpoints of prospective tertiary mathematics students and that of the Victorian Government which had a stated policy that it was committed to increasing participation in post-secondary mathematics education (Victorian Government 1987, p. 89) and that educational programs from Years P--12 should provide all students with a sound preparation for further schooling (Ministry of Education (Victoria) -- Ministerial Paper Number 6 1984, pp. 6.11).

A National Statement on Mathematics for Australian Schools (Australian Education Council 1990) hereafter referred to as the National Statement, also recognises the importance of this issue. One of the goals for mathematics in Australian schools included in the National Statement was that 'as a result of learning mathematics in school, all students should possess sufficient command of mathematical expressions, representations and technology to continue to learn mathematics independently and collaboratively' (p. 18).

This goal is particularly significant when one considers that 'contrary to a widely held belief, 90% of all HSC students do apply for tertiary entrance. It is reasonable to infer from this that most students see HSC as a preparation for tertiary studies' (Blyth and Calegari 1985, p. 312). Also, 'statistics collected by the MAV (Mathematical Association of Victoria) show that 75% of all tertiary courses require a pass in HSC mathematics' (Blyth and Calegari 1985, p. 312).

A great deal has been written on the subject of mathematical misconceptions. There are many types of mathematical misconceptions, 'and a complete list may not even be practical' (Davis 1984, p. 335). A number of authors have written about how a consideration of pupils' misconceptions in mathematics can be used to develop teaching techniques which are directed towards diagnosing and eliminating these misconceptions (Bell 1982, Farrell 1992, Margulies 1993, Perso 1992).

Papers such as this one have the potential to assist in the direction of the teaching of mathematics in such a way as to help to correct some of these misconceptions, or at least to decrease the frequency of their occurrence. This desirable outcome may come about by making mathematics teachers aware of some of the more common misconceptions and how frequently they occur. By providing mathematics educators with information about which misconceptions are common and how frequently they occur, it is hoped that this may lead to a greater emphasis being placed on the teaching of areas in which misconceptions occur frequently, and therefore provide a better mathematics education for students.

In this paper, the mathematical misconceptions have been drawn from a Diagnostic Test used at the University of Melbourne, a first year examination at the University of Melbourne, and a test given at LaTrobe University (LaTrobe). A list of these misconceptions and their frequencies of occurence is provided, along with a discussion of why they may have occurred.

The 1992 Diagnostic Test

Since 1989 a Diagnostic Test has been used at the University of Melbourne to identify the level of preparedness of students to undertake a tertiary mathematics course and to pinpoint areas of weakness in the backgrounds of the students (Barrington and Carbone 1991). Results from the tests are used to advise each student which first year mathematics unit is most appropriate for that student and also to determine whether a student has a need for extra assistance.

While analysing the 1992 Diagnostic Test (1992 DT), a number of misconceptions were seen to have occurred with a relatively high frequency and a list of these was compiled. This research does not provide a list of all of the mathematical misconceptions which were exhibited, but considers a relatively small number of mathematical misconceptions with high frequency of occurrence (this was taken to be greater than 10% of those who attempted the question, or about 5% where a question already had a misconception with greater than 10% frequency recorded). A tally was kept of the number of students who attempted each question which had a high rate of misconceptions, how many gave the correct answer to these questions, how many exhibited the misconception, and how many gave another incorrect answer. A summary of these statistics is given in Table 1 below. The items referred to in Table 1 are listed and discussed in following paragraphs.

In Table 1, and in the discussion which follows below, the proportion of misconceptions refers to the frequency of that misconception with respect to the total number of students who attempted that question i.e. if 1000 students had sat a test, and, of these, 800 had attempted a particular question with 200 demonstrating a misconception, then that misconception will be said to have occurred in 25% of cases (not 20%). The reason for adopting this procedure is that it is not clear why a student did not attempt a particular question -- there is no evidence which suggests the presence or the absence of the misconception being considered.

Table 1 -- Frequencies/percentages of Misconceptions

Table 1 summarises the frequencies of the answers given by the 394 V.C.E. (Victorian Certificate of Education) mathematics subject 'Change and Approximation 3/4 Extensions' (C & A 3/4 X) students who sat the 1992 Diagnostic Test. '% Att.' refers to the percentage of students who had attempted that question, % 's in each of the other categories refer to the percentage of students in that category of those who had attempted the question. Numbers in brackets after question numbers indicate the first and second misconceptions.

Question	Attempted (% Att.)	Correct (% Correct)	Misconception (% Miscon.)	Other (% Other)
A3	332 (84.3)	148 (44.6)	91 (27.4)	93 (28.0)
A5	341 (86.6)	227 (66.6)	45 (13.2)	69 (20.2)
A7(1)	378 (95.9)	215 (56.9)	78 (20.6)	85 (22.5)
A7(2)			18 (4.8)	
A8(1)	363 (92.1)	215 (59.2)	48 (13.2)	100 (27.6)
A8(2)			20 (5.5)	
A9	283 (71.8)	127 (44.9)	70 (24.7)	86 (30.4)
B3	374 (94.9)	235 (62.8)	67 (17.9)	72 (19.3)
D4 i	229 (58.1)	20 (8.7)	82 (35.8)	127 (55.5)

The misconceptions and their frequencies of occurrence contained in this section have been drawn from the test booklets of the students who had completed C & A 3/4 X in 1991 and then attempted the 1992 DT. All questions on the test were 'short answer' and only the answer was marked.

It should be noted that not all of the so-called mathematical misconceptions shown here are, literally speaking, misconceptions. Some may simply be a result of misreading or misinterpreting the question. These mistakes are nevertheless important, the level of their importance being governed by the nature of the mistake and the frequency with which it occurs. The question numbers shown below are those used in the 1992 DT.

In each case below, the correct solution is shown immediately after the question. It should be noted that the method shown is the one which was anticipated, and other correct solutions may be possible.

A3. Factorise $(2x+y)^2 - x^2$.

$$\begin{aligned}(2x+y)^2 - x^2 &= ((2x+y)-x)((2x+y)+x)\\ &= (x+y)(3x+y).\end{aligned}$$

Of those who answered this question, 27% gave the answer $3x^2+4xy+y^2$. It is unlikely that all of them had the same misconception. Since many students had demonstrated (in other questions) that they knew the meaning of the word factorise, this answer may be due to some not recognising that a difference of squares was involved, some expanding the given expression and subsequently forgetting the purpose of the question, while others may have factorised the difference of squares and then expanded having forgotten the purpose of the question.

A5. Simplify $\log_{10}45 + \log_{10}2 - \log_{10}15$.

$$\begin{aligned}\log_{10}45 + \log_{10}2 - \log_{10}15 &= \log_{10}(45 \times 2 \div 15)\\ &= \log_{10}6.\end{aligned}$$

Some 13% of those who answered the question solved it in the following way :

$$\log_{10}45 + \log_{10}2 - \log_{10}15 = \log_{10}(45+2-15) = \log_{10}32.$$

The common misconception exhibited here is that $\log A + \log B = \log(A + B)$. This idea may be due to an overgeneralisation of the process used to find the sum of real numbers. It may also be that the students who have exhibited this misconception have overgeneralised linearity of operation regardless of the operator. Two other examples of this type of misconception are given below :

(i) $\sqrt{x^2 + y^2} = x + y$ (where $x + y \geq 0$); (ii) $\sin(A + B) = \sin A + \sin B$.

A7. Find $\{x:(x-1)(x^2-3x)=0\}$.

$(x-1)(x^2-3x) = (x-1)x(x-3) = 0$; so $x=0,1$ or 3.

About 21% of those who answered the question missed the value $x=0$ and gave the answer $x=1,3$. Another common mistake (5%) was to miss the values $x=0,3$ and give the answer $x=1$. This appears to be due to the second bracket in the question being ignored or disregarded.

A8. $\{x:2x+4<5x+10\}$.

$-6<3x$; $x>-2$.

Many students (13%) appear to have written down the inequation $-6<3x$ and then incorrectly that $x<-2$. It is a matter of conjecture as to the proportions who were careless or who had misconceptions about the meanings of < and >. It is also possible that some students have tried to get x on the left side of the inequation by first writing $-6<3x$ and then dividing both sides by -3 without reversing the inequality. This error may have been due to a misconception or a lack of care. A further 6% of students gave the answer $x=-2$ to this question. This may have been due to being careless with < , > and = or to a misunderstanding of what these symbols mean or what is expected of them in this type of question. It may also be that this incorrect answer is a result of confusing the method of approach to this type of question. When sketching the graph of an inequality, students are often taught to sketch the equality first, and then consider which region is required. Some students may have adopted this approach to the given inequation, but, having solved the equation, have forgotten to complete the solution.

A9. Solve for x : $\frac{3}{x} - \frac{4}{a} = \frac{5}{b}$ where a and b are positive real numbers.

$$\frac{3}{x} = \frac{5}{b} + \frac{4}{a}$$

$$= \frac{5a+4b}{ab}$$

$$x = \frac{3ab}{5a+4b}$$

Of those who answered this question, 25% appear to have first written down $\frac{x}{3} - \frac{a}{4} = \frac{b}{5}$ (or similar). Apparently these students believed that combinations of fractions could be 'flipped over' without any consideration of a common denominator. This misconception may be a result of believing that you can do anything to an equation as long as you do the same thing to both sides of the equation. Here students have taken the reciprocal of each term rather than each side of the equation.

B3. Write down the derivative of $f(x)$: $f(x) = 2x^3e^x$.

$$f'(x) = \frac{d}{dx}(2x^3) \times e^x + 2x^3 \times \frac{d}{dx}(e^x)$$

$$= 6x^2e^x + 2x^3e^x.$$

Some 18% of those who answered the question wrote down the answer incorrectly as $f'(x) = 6x^2e^x$. This may be a case of many students forgetting the product rule for differentiation. Alternatively, some of the students may have treated e^x as a constant while others might have thought that the derivative of a product was the product of the derivatives i.e. if $f(x) = g(x) \cdot h(x)$, then $f'(x) = g'(x) \cdot h'(x)$.

D4i. Given that $\sin A = \frac{3}{4}$ and that $\frac{\pi}{2} < A < \pi$, evaluate the following : $\cos A$.

$\cos^2 A = 1 - \sin^2 A = 1 - \frac{9}{16} = \frac{7}{16}$; $\cos A = \pm\frac{\sqrt{7}}{4}$ but we are given that A is in the second quadrant, so $\cos A = -\frac{\sqrt{7}}{4}$.

The answer $\cos A = \frac{\sqrt{7}}{4}$ was given by 36% of those who attempted this question. This appears to contain the misconception that $x^2 = C^2$ has only one root : $x = C$; the negative root is missed. It is also likely that some students either have not read the question carefully enough or have disregarded the fact that A is in the second quadrant.

A Cautionary Note

When considering the frequency of mathematical misconceptions exhibited by students attempting the 1992 DT, it is important to note that they are some of the better students in the state of Victoria (according to their Tertiary Entrance Scores), who had succeeded in what was considered to be the most demanding Victorian Certificate of Education (V.C.E.) mathematics course, that being C & A 3/4 X. The Tertiary Entrance Score or Anderson score as it was commonly known, was calculated for each student by adding together the student's marks for their four best subjects and adding to this ten per cent of the marks for the other subjects attempted for which the student had scored at least 40%. The students being considered here had an average Anderson score of about 328 which corresponds to the thirteenth percentile of all students who completed the V.C.E. in 1991, and they had an average C & A 3/4 X mark of about 76%. It is worth considering that if the misconceptions described in this paper occur with such frequency amongst the better students in the state, with what frequency could we expect them to occur amongst students of lesser standing?

It should however be acknowledged that as the 1992 DT was held at the end of February, some of the misconceptions exhibited may be due to the length of the break from studying mathematics.

Another Important Misconception

The misconception discussed in this section was exhibited in the first semester 1992 Mathematics 100 Examination at the University of Melbourne. The question asked on the examination required students to find the area between two cubic curves. To find the points of intersection of these curves, students equated them to give the equation $x^3 - 3x + 2 = 0$. Students were given that one of the solutions was $x = 1$.

$$(x-1)(x^2+x-2) = 0$$

(the step above could be done by inspection or by long division)

$$(x-1)(x-1)(x-2) = 0$$

$$(x-1)^2(x-2) = 0$$

so $x = -2$ or 1.

The markers, which included the author, discussed the performance of students subsequent to the completion of marking. One misconception was highlighted by each of the markers. This had to do with the incorrect use of what is sometimes referred to as the null factor law -- if the product of two (or more) numbers is zero, then at least one of those numbers must be equal to zero, i.e. if $x(x-3)=0$, then $x=0$ or $x-3=0$; and so $x=0$ or 3. This law appears to have been modified to yield the following : if the product of two (or more) numbers is equal to a number, then at least one of those numbers must be equal to that number, i.e. if $x(x-3)=1$, then $x=1$ or $x-3=1$; and so $x=1$ or 4. The equation was solved incorrectly by many of the students in the following way :

$$x^3-3x+2=0$$

$$x^3-3x=-2$$

$$x(x^2-3)=-2$$

so $x=-2$ or $x^2-3=-2$

$$x^2=1$$

$x=1$. Some of the students did get the two solutions $x=\pm 1$ but many demonstrated the same misconception as that shown previously in D4i.

The markers discussed the frequency with which this misconception occurred and each stated that the frequency was somewhere between 20 and 25% of students. The nature of this misconception shocked the markers, but not nearly as much as the frequency of its occurrence. No exact count was taken of the frequency of occurrence of this misconception, but all markers reported a similar estimate of the frequency quite independently of one another. It is nevertheless quite likely that there could be a considerable error in these estimates. Even if this is so, the fact that the misconception occurred with such frequency so as to be noticed by the markers as a common problem amongst a large number of the students, and to cause such a widespread level of concern amongst the markers, means that it is important. No other misconceptions caused widespread concern or discussion.

It was surprising that this misconception had persisted to this stage especially with this group of 'well above average' students (as seen by their tertiary entrance scores). The null factor law is not taught in Mathematics 100 as students are expected to have this knowledge prior to entry in this unit.

It is ironic to note that students who exhibited both of the misconceptions shown above, had the correct two solutions. In other words, those students who overgeneralised the null factor law when the right hand side of the equation was equal to -2, and then put 1 instead of ± 1 when solving $x^2=1$, obtained the two solutions $x=-2$ or 1, and these were the only two correct solutions.

Some Other Misconceptions

At a similar time to the 1992 DT, students at LaTrobe University were required to sit a mathematics test. As these tests were administered at approximately the same time, it is of interest to consider the misconceptions exhibited on the LaTrobe test. The misconceptions below are drawn from an analysis of a test given to students studying mathematics at LaTrobe. No information is provided which indicates which V.C.E. mathematics courses had been attempted in the previous year, or what levels had been achieved. However, the misconceptions exhibited can be considered to be important as the test was comprised of 'very simple questions -- no question was above early year 11 standard' (Worley 1993, p. 1). In this analysis it is stated that 'the lack of preparation of our students in pre-calculus skills has become a problem of major concern for us at LaTrobe' (Worley 1993, p. 1). It is interesting to note that many of the misconceptions shown below appear on the list of misconceptions provided in 'Algebraic Atrocities' (Margulies 1993, p. 41). Examples from the analysis of the test are shown below :

Question : Given that $-\cos\frac{\pi}{6}=-\frac{7\sqrt{3}}{2}\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$, evaluate $\cos\frac{7\pi}{6}$. Students were expected to know that $\frac{7\pi}{6}$ is in the third quadrant, and the correct answer is $\cos\frac{7\pi}{6}=-\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2}$. The most common response was $\cos\frac{7\pi}{6}=\frac{7\sqrt{3}}{2}$. This misconception is similar to that in Question A5 of the 1992 DT -- students have overgeneralised linearity of operation regardless of the operator.

Question : Simplify $\frac{x^2-5x-6}{x^2-2x-3}$.

$$\frac{x^2-5x-6}{x^2-2x-3}=\frac{(x-6)(x+1)}{(x-3)(x+1)}=\frac{(x-6)}{(x-3)}\ ,\ x\neq-1\ .$$

There were a wide variety of answers given, two of which are shown below :

$$\frac{x^2-5x-6}{x^2-2x-3}=\frac{-5x-2}{-2x-1}\ ;$$

$$\frac{x^2-5x-6}{x^2-2x-3}=-3x-3\ .$$

In both cases, 'cancelling factors is clearly not understood by the majority of students. Clearly most have developed a general rule that any term on the numerator can be cancelled with any term on the denominator' (Worley 1993, p. 2). In the second case, it appears that the rule about subtracting indices when dividing numbers has been overgeneralised.

Question : Simplify $2^{-1}+3^{-1}$.

$$2^{-1}+3^{-1}=\frac{1}{2}+\frac{1}{3}=\frac{2+3}{6}=\frac{5}{6}\ .$$

About 24% of students put either 5^{-1} or 5^{-2}, to this question on indices showing 'a total lack of understanding of the rules' (Worley 1993, p. 2).

Question : Simplify 2^x+2^x . Either 2.2^x or 2^{x+1} was accepted as a correct answer to this question. However, '$2^x+2^x=4^x$' or 2^{2x} were overwhelmingly the most common errors, though 2^{x^2} also featured' (Worley 1993, p. 2). The same lack of understanding of index laws is evident here.

An approach to the problem

The vast majority of the misconceptions reported in this paper appear to be due to what Dina Tirosh (1990) refers to as external inconsistencies which she says 'occurs when a student's existing mathematical concept is incompatible with newly presented information' (p. 122). Insufficient understanding is likely to cause an inappropriate transfer of theorems which apply to one mathematical subculture into another in which they are invalid. As a result, students have over-generalised incorrect or inadequate rules. This categorisation is likely to be the major cause for most of the misconceptions exhibited in A5, A9 and B3 on the 92DT, the question taken from the Mathematics 100 exam, and in each of the questions from the LaTrobe test.

An approach which has been successful in helping many students overcome misconceptions is the 'conflict teaching approach' based on Piaget's notion of cognitive conflict. In this approach, teachers discuss with the learners the inconsistencies in the thinking of the learners in order to have the learners realise that their conceptions were inadequate and in need of modification (Tirosh 1990). Shlomo Vinner (1990) supports this approach and states that 'there is no doubt that if inconsistencies in the students' thinking are drawn to their attention, it will help some of them to resolve some inconsistencies in a desirable way' (p. 97).

Summary

A number of mathematical misconceptions have been exhibited (sometimes on very simple material) in tests at two different of tertiary institutions. The frequencies of these misconceptions are a matter of great concern for tertiary mathematics educators (lecturers and tutors).

Most of the basic concepts referred to in this paper are not taught in most of the tertiary mathematics units considered herein. Students are expected to have learned these concepts prior to entry to the respective unit. As a result, students who have not mastered those basic concepts can be severely disadvantaged in those tertiary mathematics units.

Many of the concepts shown previously *should* have been mastered long before a student gets to university. The fact that that is not the case with a large number of students (and a large number of the better students in the State) provides a very strong argument that the school system is not satisfying the needs of these students. These needs ought to be able to be satisfied and these concepts taught more effectively. The problem yet to be solved, is *how*?

It is the belief of the author that an effective method of minimising the frequency of mathematical misconceptions is to teach with emphasis placed on the understanding of a 'rule' rather than just remembering that rule. If misconceptions do occur (as they will) despite the best efforts of the teacher, the approach outlined above is a good starting point from which to begin to remedy the problem.

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Critical Moments in a Teacher's Attempts to Introduce Discussion-Based Teaching into a Secondary Mathematics Programme

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The purpose of the project is to document the progress, over one school year, of a mathematics teacher in a secondary school in Hong Kong, as he attempts to enhance mathematics learning through the introduction of a discussion-based approach. The results of the study will give insights into the kinds of difficulties experienced by a teacher trying to implement a non-traditional teaching approach, as well as giving some understanding of the role of discussion in a Hong Kong mathematics classroom. This paper presents a progress report of the teacher's feelings about the innovation after the first five months of its implementation, and the factors affecting these feelings.

Introduction

With recent reforms in mathematics education (NCTM, 1989; Australian Education Council, 1991; Curriculum Development Council, 1994), mathematics teachers in many countries are faced with the dilemma of whether to, and how to, change their teaching approaches. Despite sound research evidence supporting these recent innovations, "it is sobering to note that the history of reform in mathematics education is largely one of failure" (Gregg, 1995, p.444). A pattern of teachers reverting to traditional practices has persisted even if they are initially enthusiastic about implementing a new approach (Gregg, 1995). Consequently, it is important to gain insight into the factors which can influence teachers' decisions to adopt, or not to adopt, changes to their teaching. Simon (1993) suggested that an understanding of factors which can contribute to, or inhibit, successful change can best be achieved by "careful study of teachers as they transform their practice and of their developing conceptions" (p.107).

The dilemma faced by a young teacher in Hong Kong is perhaps even greater than in most western countries. On one hand, curriculum documents are recommending that teachers "ask more open-ended questions... and in general play the role of facilitator, so that students must do the thinking and justify their conclusions (Curriculum Development Council, 1993a, pp.26-27), and "involve students actively in their learning process, to train them to organize ideas and information, and to develop the skill of discrimination and judgement....interact with each other [and]... exchange views and form tentative conclusions" (Curriculum Development Council, 1993b, pp.48). On the other hand, the standard of examination performance is exceptionally high in Hong Kong and teachers are faced with pressures from senior colleagues, parents, and students to cover the syllabus and prepare the students to succeed in knowledge-based examinations. Many secondary mathematics teachers believe there is little place to implement reforms, because there is so little time available to spend on any one topic. Furthermore, there is a very strong cultural belief in the role of the teacher as conveyor of knowledge and the pupils as recipients, which is very much conducive to traditional approaches to teaching (Leung, 1995).

To enable them to resolve this dilemma, it is important for teachers to explore ways of finding a balance between the traditional approach which has achieved good results for generations of Hong Kong students, and newer strategies which curriculum reformers are suggesting can enhance these results, and make mathematics more accessible to a wider range of people (NCTM, 1989). Hence, this project focuses on one teacher as he attempts to address these conflicting issues and explore how and when new approaches can enhance the quality of the mathematics learning in his classroom.

The issues which can influence the successful implementation of change are complex, and raise questions about the strategies the teacher needs to use to encourage maximum student participation, the respective roles of the teacher and the pupils, and the extent and nature of changes in these over time, the contribution of the reform to the students' understanding of mathematics topics and their ability to transfer knowledge to unfamiliar contexts, and the teacher's and the students' beliefs about the value of the reform. The purpose of this paper is to address one aspect of the project, that is the changes in the teacher's feelings about the innovation, and to try to identify the critical moments and major factors which can influence these feelings.

The particular mathematics reform which has been chosen for examination in this project is that of co-operative discussion. The rationale for this is based on theories which advocate the use of discussion in helping learners to build up conceptual understanding of mathematics and become mathematically literate (for example, Schoenfeld, 1985, 1987; NCTM, 1989; Von Glaserfield, 1991). It is linked to the social constructivist notion that people learn most effectively by constructing knowledge in social settings (Vygotsky, 1962). The use of discussion is fundamental to this notion (National Research Council, 1989), and there is research evidence (for example, NCTM, 1989; Webb, 1991; Cobb, Wood and Yackel, 1993; Hart, 1993) which suggests that the appropriate use of discussion improves students' mathematical achievement. Yackel, Cobb and Wood (1993), for example, believe that "collaborative discourse can help children clarify their own understandings by talking....and by reconceptualising their own cognitive structures as they attempt to make sense of their partner's explanations" (p.35). Not only can it enhance learning, discourse can also be useful in enabling the teacher to gain insights into whether the students have inaccurate or incorrect understanding (Wood, Cobb and Yackel, 1993). Models for classroom interactions have been proposed which include whole-class discussion, co-operative grouping, and pair problem solving (Cobb, Wood and Yackel, 1993). The teacher's role is critical as catalyst and coach, in designing tasks and asking appropriate questions to guide the students' understanding (NCTM, 1989).

Methodology

A case study methodology is being used for this project for the purpose of portraying "an educational problem in all its personal and social complexity (Stake, 1988, p.254). Accounts are being collected from the points of view of three stakeholders: the teacher, the student groups, and a group of observers who include the teacher's colleagues and the researcher.

The teacher in the case study is a full-time third-year teacher who is enrolled in a part-time Postgraduate Certificate of Education course. He has a background in philosophy and psychology, and spent a year tutoring in a university philosophy department before commencing his career as a secondary mathematics teacher. This experience had shaped his philosophy about how he thought mathematics should be taught and, following exposure in the course to issues concerning social interaction and discussion in the mathematics classroom, he expressed a belief in the value of this kind of approach and an interest in incorporating it into his teaching.

The classes which the teacher is using the discussion approach are two Form 4 classes in a Band 5 school. Form 4 is the fourth year of secondary schooling, and Band 5 is the lowest level in the selective secondary system. One class is in the Science strand and the other the Arts strand, but both groups have below-average mathematics skills, poor motivation, and fluctuating behavioural problems.

Data are being collected over the period of the 1995/96 school year, to explore the nature and extent of change as the teacher learns more about the use of discussion, and the students become more accustomed to the approach. Data sources include the teacher's lesson plans and evaluations, written and tape-recorded samples of students' work, written and tape-recorded reflections from the students, observations by the researcher, and monthly interviews with the teacher.

Results

Background and Description of the Activities

During the first two months of the teaching experiment, there were two main topics taught: graph plotting and quadratic equations. Typically, between twenty and thirty minutes of each forty minute lesson were spent in discussion, which involved students working in pairs or groups of four to solve tasks posed by the teacher, then presenting their solutions for class discussion. During the third and fourth months, the teacher found it necessary to reduce the amount of time spent in discussion and return to a more teacher-delivery oriented approach, due to the need to rush through two chapters to complete work in time for examination revision. The topics covered during this time were polynomials, remainder theorem, factorisation theorem, indices, logarithms and sets. While the main mode of delivery was traditional, the teacher conducted three major discussion-based lessons and several shorter discussions of about ten minutes with each of the two classes. As well as setting some specific questions for discussion, the teacher would often ask the students to work in groups to find solutions to the text-book examples set as class exercises.

The fifth month was spent in revision of examination topics, and in the introduction of one new chapter, on ratio, proportion and variation. During the revision time, the teacher introduced the idea of giving the students solutions which he had prepared to old exam questions, and asking them to evaluate these and locate errors. This approach was used at least twice a week. The students would work in small groups, then participate in a teacher-led discussion at the end of the lesson. During the introduction of the new chapter he continued with the approach of presenting a question and conducting whole class, large group and small group discussions of the solutions. During the fifth month, he also commenced an extra curricula discussion/revision group which met twice a week and was attended by an average of six volunteer students.

Changes in the Teacher's Feelings

By the end of the second month, the teacher felt that some small progress had been made, and that it was worthwhile to persist with the discussion-oriented teaching approach. By the end of the fourth month, however, he had mixed feelings about whether or not to continue with the experiment. While he still had a fundamental belief in the value of discussion to enhance learning, he was beginning to feel that the experiment was being made increasingly difficult by the combination of a number of factors. This suggests that he may have reached a critical stage, where it would be very easy for him to abandon the reform. One month later, he felt that a corner had been turned, and that the experiment was at last beginning to succeed. He estimated then that 70 per cent of the Science class and 40 per cent of the Arts class were participating actively in discussions.

Because the teacher came so close to abandoning the reform after four months, it is important to examine the factors which were predominant at this time, and those which influenced his change of feelings a month later. These factors will be examined from the points of view of the different stakeholders.

Contributing Factors

Students' Attitudes and Beliefs: One group of factors exerting considerable influence on the teacher's feelings, were the students' attitudes and beliefs about the discussion approach. Simon (1993) has suggested that students who have long-established traditions of being told "how to do it" resist an approach which encourages them to take responsibililty for their own mathematical thinking and decision-making, and this was certainly the case in this study.

During the first two months, both the teacher and an observer noted that most students in the class lacked confidence, willingness, and ability to discuss or to share ideas, and most of the students were not doing anything at all to contribute to the group's solution. The observer's comment was that "only a few students were involved with learning". Some students were prepared to give answers, but had no self-confidence and were neither prepared, nor able, to defend their points of view. In interview, the students indicated that when the teacher asked for clarification with a question like, "How do you know that?", they felt that he was criticising them.

Most students just watched or copied what their more active peers were doing, and made no attempt to understand. The teacher also believed that they were sometimes just lazy, knowing how to do it, but finding it easier to let someone else do it for them.

After two months, the teacher conducted an interview with ten students, and their comments confirmed many of his suspicions. The students in the Science strand did not react favourably to the expectation that they participate in discussion. They did not either understand or accept the approach he was using, and lacked confidence in his new approach. They firmly believed that the teachers should convey knowledge, and that getting them to talk was not worthwhile. The majority preferred to learn passively, and thought that it was a waste of time to include discussion. Furthermore, they wanted the lesson to be oriented towards their examination, and the teacher had not yet introduced any examination questions. They were not interested in the teacher's explanation, that he needed to develop their foundation knowledge before they could move towards the examination content. During the third and fourth months, the teacher still expressed concern about the general lack of willingness to participate.

After four months, sixteen students completed a questionnaire and open-ended written responses regarding their attitudes and beliefs about the innovation, which suggest a slightly more favourable attitude than two months earlier. Those who responded to the questionnaire were evenly distributed in their acceptance of the discussion approach. They were divided as to whether or not they would like to use the approach again in the new semester. They seemed to think that the use of discussion helped their understanding at least to some extent, although their preference depended on other factors, such as the topic, and their store of background knowledge to draw on to answer the questions. The students felt that the new approach had helped them with at least some of their examination preparation. While they were evenly spread in their opinions as to whether the use of discussion aroused their interest in mathematics, they did indicate that it made them more attentive, and also that it helped them to remember the concepts better. Some students said that they found great difficulty with presenting points or developing discussion, with the consequence that there was either no discussion at all, or that it ended quickly and students daydreamed or did other things. The three brightest students in the Science class were particularly enthusiastic about the value in the innovative approach, and formed their own study group to continue their discussions after class.

By the end of the fifth month, the teacher had observed substantial changes in the students' attitudes towards participating in discussion, particularly in the Science class. He attributed this, at least partly, to the fact that, after a discussion with the class about the importance of their feedback, the students had responded well to the opportunity to be partners in the decision-making. At their request, he made the questions more exam-oriented and they responded well to this. Even some of the badly behaved students wanted to share discussion with their classmates. Students who were previously unwilling to ask questions in their small groups were prepared to ask their group mates for explanations when they did not understand. Whereas, previously, they had felt threatened by the question, "How do you know that?", they were now willing to defend their thinking.

Students' Knowledge: After the first two months of using discussion, the teacher believed that some of the responses of about one quarter to one-third of the class indicated that the discussion activities had contributed to the students' knowledge and understanding of the topic. Nevertheless, one of the major problems the teacher observed was that the majority of students lacked the conceptual background which would enable them to participate in discussions successfully. An observer noted, in the second month, that only one group had the necessary knowledge to be able to complete the discussion task successfully. This problem had become particularly evident with the Arts group by the end of the fourth month. Questions which required proof statements and use of the strategy of finding contradictions were not answered well. The students were interested at the start, but lost interest when they found that they did not have the knowledge base to be able to answer the question. The teacher suspected, and the students confirmed in interview, that they often found it difficult to construct their own knowledge easily from the class activities, because they could see little linkage. Despite the fact that the teacher tried to make the activities exam-oriented, the students still thought that there were gaps between these activities and their need to get good exam results. There seemed to be a link between the students' knowledge and the extent to which they were prepared to participate. It was recorded by the teacher, and by the students, that if they felt confident that the activity was within their capabilities, about one half of the class would work actively on the task.

A further problem which the teacher identified was that the students did not have very good explanation skills when presenting their groups' findings to the rest of the class. Often, they would just write the main points on the blackboard, with no elaboration of what they were doing.

After five months, the teacher was beginning to feel that the tasks he was setting were more consistent with the students' knowledge and experience. He thought that the strategy of giving them pre-prepared solutions to evaluate had an unexpectedly good effect because it gave them some background support instead of expecting them to start from their own knowledge. The students reported that they also felt more comfortable with this approach, because they felt previously that he had been scolding them for their lack of knowledge. The teacher found that, as the students became more confident, they also became better at constructing their own mathematical thinking, and the brighter students actually gave some insightful responses which could then be used as a basis for further discussion.

Management: After two months, and again after four, the teacher described several management issues which he had been addressing systematically, but did not feel that he had fully resolved yet. One was that teacher supervision could not run smoothly, because there were too many groups for the teacher to take care of. The students and an observer also indicated their concern that the teacher could only take good care of one group at a time. Because the teacher could not be present at all groups, some students thought that they were being neglected. The observer commented specifically on the students' lack of self-discipline.

Another of the management issues which the teacher faced was that there was not a sufficiently large distribution of high ability students, capable of taking leadership roles in the discussion, to allow for every group to have an effective leader. Furthermore, he observed that the students did not have a very high level of self-control to be able to engage in group discussions, and that it was difficult for him to maintain control of all of the groups. The teacher was observed during January by the vice-principal of the school. His particular concerns were that the class order was sometimes not good and that the noise level was sometimes unacceptably high.

After five months, the teacher reported that he had negotiated some classroom management issues with the students, with favourable outcomes. At the students' request, he had taken disciplinary measures with some who were not behaving in class activities, with the result that the previous non-participants had begun to take his expectations more seriously. He also experimented, successfully, with having a smaller number of groups for some sessions, which made the students feel that he was sharing his attention more evenly.

School Policy: During the January interview, the teacher expressed his concern that he did not want to conflict with school policy in his attempts to change his teaching approach. He had received some advice, particularly from the vice-principal, that what he was doing was inconsistent with the school policy of a silent classroom, with just one voice from the teacher. The vice-principal also suggested that it was contradicting school policy to include too many class activities, which might have prevented him from completing the required work in time. This mis-match between the school philosophy and his own was a critical factor which could easily have influenced him to abandon the project.

Support from Colleagues: After two months, the teacher was sufficiently enthusiastic and confident about the project to work independently. However, two months later, he was feeling the need to engage in professional dialogue with his colleagues about what he was doing, and was discouraged by their lack of willingness to respond to his need. While the researcher and his classmates were available to give support, it was the collegial support within his own department which was most important to him and which was not forthcoming. This need for emotional support is one of the necessary components described by Weisglass (1994).

At the end of the fifth month, the teacher reported that he had been sharing some of the students' insights with his colleagues, with the result that they were beginning to show an interest in the students' thinking. He was beginning to feel that this was opening discussion channels, and that he was less isolated from his colleagues than he had felt himself to be earlier.

Syllabus Constraints: Gregg (1995) mentioned the impact of syllabus constraints on mathematics teaching: "the text will continue to be viewed as a mathematical authority, and the teacher's role will continue to be that of covering the formalized mathematics presented in the textbook" (p.464). The fact that the teacher had to modify his approach to cover the syllabus content in time for the examinations indicates that this was a major factor influencing his feelings about the project. Furthermore, students' comments suggested that one of their objections to using the new approach was that they were afraid the syllabus would not be completed in time, and the vice-principal expressed the same concern. At the end of five months, the same syllabus and time constraints applied as previously. However, both the teacher and the students indicated that they were happier with the discussion topics being more directly related to exam questions.

Discussion

It must he emphasised that this paper presents a half-way report of research in progress. Nevertheless, some interesting, if predictable, changes have occurred in the teacher's feelings about the innovation. He has gone through four stages in the first five months of implementing the new approach. The first stage was one of *enthusiasm and confidence*. He believed in the value of the approach, had a sound theoretical knowledge of the principles, and was confident that the teaching experiment would succeed. The next phase was still one of *hopefulness*, that the obstacles would be overcome. Four months into the experiment, he was expressing *mixed feelings* about whether to continue. This, then, was a critical time, when it would have been very easy for him to abandon the project. Given the interaction of the many factors discussed above, it is not surprising that his feelings were mixed. A slight shift in the students' attitudes and beliefs, from hostile to ambivalent, was not enough to over-ride the other factors. The most critical factors in influencing the teacher's negative feelings at the time when he was most ready to give up were difficulties in: gaining the students confidence in the value of what he was trying to do; identifying appropriate tasks which would cater for the students' level of knowledge and experience; developing classroom management strategies appropriate for the new teaching approach; and having access to collegial support and professional discussion. The teacher believed that the factor which contributed most significantly to the fourth stage, *renewed enthusiasm*, after five months was that he had finally achieved the co-operation of the students in negotiating the activities with him. This, combined with the fact that he had found suitable discussion tasks to account for their restricted background knowledge, seemed also to lead to positive changes in the other factors which had contributed to his earlier negative feelings.

In the first five months of the project, there has been one "critical moment" where the teacher has come close to abandoning the new approach, followed by renewed enthusiasm. It is anticipated that, in the remaining five months of the project, there will be more highs and lows. Understanding the factors which contributed to this critical moment has given some useful insights which may help the teacher to cope with future fluctuations in his feelings and which may also be useful to share with other mathematics teachers going through the process of reform.

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'Oh, the Usual': The Usefulness of Curriculum Genres to Explain Students' Interpretations of Mathematics Lessons[1]

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Although learning must arise out of students' own interpretations of their classroom experiences, their voices are not often the focus of educational research. This paper locates students' experiences of mathematics lessons within the framework of 'curriculum genres', the term Christie (1993) uses to define the distinctive ways teaching-learning activities are staged and socially structured through language. The usefulness of this language education construct for analysing mathematics learning is discussed.

During recent years there has been a major shift in educational research emphasis towards the learning which occurs within a normal school environment (Doyle, 1990). As a result the interactions of teachers and learners as they go about their usual business of teaching and learning have become of more central concern to mathematics research (Kanes & Atweh, 1993; Pegg, 1989). This interest in the learning environment as a whole has meant that the more fundamental social processes of classroom culture are now seen to be directly relevant to mathematics education (Bauersfeld, 1992; Weissglass, 1992).

Mathematics researchers have begun to look to other disciplines for different ways of conceptualising learning. Investigations into the role of language in learning mathematics has provided some useful new insights (Ellerton & Clarkson, 1992). For example, Atweh (1993) used Halliday's Functional Theory of Language to analyse language interactions in two mathematics classrooms and reported distinctive differences in the way language structured the social context of the learning, even though the same curriculum content was dealt with.

While such research reports clearly recognise the importance of students as the objects of these teaching-learning processes, there appear to be few studies which centrally focus upon the students' own experiences or interpretations of the classroom interactions (Erickson & Shultz, 1992). And yet, it is surely the students' interpretations of the teaching-learning activities which determine entirely the learning that has occurred. While it is clear that teaching is directly concerned with communicating the contents of a curriculum to students, to focus so much attention on what teachers *intend* to convey to students misses the crucial aspect of the communication process.

> If a person wants to communicate, and expresses a message, if there is no-one to interpret, or if the interpretation is impossible for some reason, no communication occurs. Similarly, the very nature and content of the message depends, not upon the expression, but on the interpretation. Whatever the interpretation is, that is what the message becomes. (Garner, 1994, p127)

In this paper, students' reported experiences of mathematics lessons will be analysed using insights from Christie's (1993) work with 'curriculum genres', the term she uses to define the distinctive ways teaching-learning activities are socially constructed through language. The concept arises out of systemic linguistic theories and makes use of functional grammar analysis. Although Christie's work has been largely associated with language education, the principles involved may well be just as relevant to mathematics learning and provide a useful way of contextualising students' experiences and interpretations.

What is a Curriculum Genre?

Anyone who has had even a passing involvement in the language learning areas of the curriculum in the last few years would have encountered the term 'genres' in reference to particular forms of written text, such as a 'narrative genre' or a 'report genre'. The term can also apply to oral forms of language.

Christie (1985) suggested that there are two kinds of general purposes which feature in the language used for most situations. One purpose is 'to initiate, maintain and foster relationships' and the other is 'for the negotiation of *content* of some kind' (p7). Genres are the various, yet distinctive, ways that those two kinds of purposes come together to form socially constructed meanings from the particular patterns of language interaction in which people engage. The key features of a genre are that it is socially constructed through language, has a goal, and develops through stages to reach that goal. Different genres have their own distinctive language features which help to communicate the particular meanings intended.

Christie (1993) points out that the classroom is a place in which language plays a central role in structuring the teaching-learning activities in which students are engaged, and that 'the social processes in which they [teachers] participate with their students in the pursuit of various goals are structured in distinctive ways, making them instances of particular genres' (p155). She applies the term 'curriculum genres' to distinguish the different types of lessons which she describes as the 'staged, goal driven, purposive activities in which students are initiated into ways of working, thinking and dealing with experience valued in an English-speaking culture' (p155).

Of importance for genre identification and analysis is the 'schematic structure' of a particular instance of curriculum genre. This includes reference to 'obligatory stages', which are so named because they are essential to any instance of a particular genre, and 'optional stages', which refers to those stages which may or may not be evident (Christie, 1993, p166). A schematic structure provides a framework for constructing the distinctive features which make a particular curriculum genre recognisable as such.

Teachers, consciously or unconsciously, communicate the conventions, modes and social behaviours associated with a particular curriculum genre via two language registers. Christie claims these are built in different ways through the language choices made by the teacher while guiding students into new ways of knowing and working. These registers appear to broadly equate to the two general language purposes previously mentioned.

First there is the *pedagogical register*, so called because it is concerned with structuring the pedagogical goals of the teaching-learning activity, such as those associated with the students' behaviour, the way in which tasks are to be engaged with, the movement from one stage of the lesson to the next, and the general management of the classroom environment. Secondly there is the *content register* which is concerned with the lesson 'content'; the knowledge and ways of working associated with the fields of experience encompassed by a particular curriculum genre (Christie, 1993, p156).

While any instance of a curriculum genre must begin by bringing to the fore the pedagogical register, at some point the content register must be foregrounded, if relevant learning is to occur. Christie (1993) claims that the two registers must 'converge for a sustained period of time'(p156) if the curriculum genre is to be successful in supporting this learning. In other words, the parts of the lesson which are concerned with organising students and explaining how to go about the tasks (the pedagogical goals), should support and must not detract from the actual understandings that the lesson is intended to develop (the content goals).

Of particular interest for mathematics education is Christie's (1993) description of the processes which structure the content registers operating in curriculum genres. Christie describes how the content register for any particular curriculum genre is drawn from the relevant field of experience outside the school but, in order to make it accessible to novices, it changes in character when it is relocated in the social contexts of schooling. So the 'insights, skills and ways of going about dealing with problems' (p156) which characterise the way mathematicians (and other users of mathematics) operate in the social contexts of the wider community have been transformed in ways which were intended to be appropriate for the initiation of learners into this field of experience, forming a mathematics content register.

The Students

The data analysed for this study were taken from those obtained in an earlier investigation of students' understanding and attitudes towards mathematics. Twelve students were interviewed on two occasions, once towards the end of year seven primary school and again half way through year eight. Several questions in each of the interviews asked students to talk about the characteristics of their mathematics classrooms and the mathematics they were learning. They were also asked if they had any advice for their teachers.

Each of the twelve students was from a different year seven classroom in nine metropolitan schools. In year eight they attended two secondary schools and were placed in eight different class groups. Five of the class groups were taught two separate maths units by different teachers, although in several cases, two class groups were taught by the same teacher. Overall it has been calculated that the twelve students were reporting their direct experiences of a total of twenty two different teachers of mathematics in eleven schools.

Data Analysis

Initially the notion of the existence of a distinct type of curriculum genre which could be classified as a typical 'mathematics curriculum genre' was examined. If such a genre was identifiable, one would expect that students would respond easily to questions that asked them to describe their mathematics lessons.

One would also expect there to be some recognisable schematic structure in the students' descriptions of their lessons, through which one might identify some obligatory stages and perhaps some optional stages as well.

Clearly this study cannot attempt to analyse the actual language features through which a mathematical curriculum genre might be constructed, given that the data consists of students' descriptions of what happens in mathematics lessons and not lesson transcripts. However, it may be possible that some evidence of the distinctions between the pedagogical register and the content register may be revealed in the language used by the students in their descriptions.

Mathematics Lessons as Curriculum Genres

The consistency with which students described what they would expect to happen in a mathematics lesson was striking. In no case did a student seem confused by the question or be unable to respond. Furthermore, it was clear from the responses of some students that they considered this to be common knowledge. When asked what was expected in their mathematics lessons, one student responded, 'Oh, the usual!' clearly assuming that the interviewer would understand. Several others responded with variations of 'the normal sorts of things'. One student stated, 'Mental!', clearly expecting the one word to fully convey information about that activity and how it was organised.

It was very clear from these responses that the students themselves believed there was a standard format for mathematics lessons and that the interviewer already knew what that was. This in itself suggests that the students had experienced instances of a particular kind of mathematics curriculum genre.

All students went on to describe a set of consistent features which could clearly define the schematic structure of a curriculum genre. Most responses revealed two obligatory stages. The first would be what Christie (1993) refers to generally as the 'task orientation stage' (p166) where students find out about the way they are to be engaged with the learning activity, and the second involves engagement with the actual task. The following examples are typical of the descriptions given by the students (the task orientation stage is shown in italics):

> *Well our teacher just tells us to get out our pads and pencils and stuff like that. Then turn to page whatever,* and then we have to do the questions from the book.
>
> *She just reminds us how to do it,* then she gives us some on the board
>
> *Oh, we have a book and he tells us the exercise and how to do it* and then you copy it and do an answer statement. *The teacher shows us* and then we do it.
>
> *She gives out the book and she tells us what page and all that and then she explains how to do it* and then we have to do the exercises.

There were some cases in which no explicit task orientation stage was mentioned, but it seems clear that the teacher had previously structured this aspect. Students themselves had no doubt as to what was to happen, for example:

> Well, we just automatically take out our pads and we rule up in three columns and then he puts up some maths on the board and then we write down the answers and that.

Only one student went on to describe a closure stage; 'They call out the answers and we mark it'. All the rest finished their description at the task engagement, or 'doing' stage. A number of students also mentioned what could be called an optional stage in the structure. This described what students were to do if they could not proceed with the task for some reason:

> We put our hand up and she comes around and helps us.
>
> Well, you go up to the teacher and then she explains them.

There were a number of students who went on to mention tasks other than the more usual written exercises, though in every case these were predicated by the word 'sometimes'. For example: '*sometimes* we measure, we go out and have to measure things, perimeters and areas and everything'; '*sometimes* we have competitions on the board'; '*sometimes* we just play maths games'.

There seems to be little doubt that the kinds of lesson descriptions given by the students are consistent with the characteristics Christie (1993) associates with instances of a specific curriculum genre. What is unavoidably disturbing is the experiences of mathematics that these students appear to be accumulating. It seems that a very restrictive but consistent type of mathematics curriculum genre has dominated in these classrooms.

The above analyses have been concerned primarily with those aspects of the genre which would have been constructed through the pedagogical register, and it is of interest that little if any content has been mentioned in that context. Most mentions of content in the data were in response to an initial question, 'What sorts of things do you do in maths?' which was originally intended to find out about the range of learning activities engaged in, not the content as such. In almost all mentions of content, the verb 'to do' was employed - students were 'doing' or had 'done' the mathematical processes and concepts mentioned:

> We're just *doing* revision now, but division of decimals, fractions, time, we've *done* a little bit of litres and millilitres.
>
> We are *doing* time on twenty four hour clocks and in the other one we are *doing* statistics.
>
> We *do* percentages and fractions and decimals

The language used in these references to the content register probably suggests something of the way in which this is constructed in the lessons. It appears that no matter what the mathematical content, it is always 'done' in lessons. No student spoke of 'using' fractions, decimals or percentages for any purpose, or of being engaged in constructing, investigating, planning or even calculating.

One student even referred to the teacher's pedagogical activity as 'doing'; clearly, in the content register of this particular curriculum genre, teachers, as well as students, 'do' mathematics:

> Some teachers *do* easy maths and they *do* all the same, and some teachers *do* all different sorts, like Mr Brown, he *does* addition, multiplication, division and all that we *do*, like in one subject.

Some students had very strong feelings about the ways they thought teachers could improve their mathematics learning for them. These seemed to fall into two categories. There were those who thought teachers could make mathematics more interesting through games and competitions:

> Make it fun and not boring. Teach it with some enthusiasm, not like you are dead on two legs, 'cause Mr Smith's a bit like that. He's a bit boring to listen to. He could do like competitions on the board like we used to have in year seven and that was a lot of fun.

And others whose concern was that teachers' explanations were insufficient:

> He doesn't really help us. He just says, 'work out of the book', and he just goes and does his own stuff. If we ask him for help, he just like, just does it for us, but doesn't explain how to do it, so we don't really know how to.
>
> Our teacher, she says we ask too much when she gives us an exercise to do and like some people don't understand it and they ask me because I'm pretty good at maths, and then we get in trouble. She should say it more, she should explain it more, like finding the area of a rectangle, she just says 'width times height' and she just does one and then we've got to do about seventeen out of the book.

Within both these categories though, it was the pedagogical register rather than the content register that students were asking teachers to adjust, suggesting perhaps that the students had a particular way of understanding the relationship between the two structures; namely that teachers could only adjust the pedagogical register because the content register was immutable.

Implications for future research

At the time these interviews were carried out, the concepts associated with Christie's (1993) notion of 'curriculum genres' had not yet been encountered. However applying some of those principles of analysis to the students' reported experiences revealed curriculum genre features which were easily identified from the data and suggests that the concept is a useful one for mathematics education research. Furthermore, the analysis has been helpful in contextualising the students' interpretations of their mathematics lessons and implies some mechanisms by which their experiences have been structured. The results of this study suggest a number of ways this form of analysis might usefully be developed in future research.

It is difficult not to be concerned about the contrast between the way teachers describe their pedagogical intentions in the research literature (Lubinski et al, 1994) and the way these students express their experiences. Considering these children's experiences directly reflected over twenty different classrooms, can we conclude that the restrictive curriculum genre reported by the students arose from a group of untypical classrooms, in which the teachers made no attempt to develop the mathematics learning required of the syllabus? Might it not be more realistic to suggest that the students' reported experiences were restrictive because their interpretations of the 'message' were very different than the 'message' their teachers intended to convey. In any case, we are still left with Garner's (1994) point that 'whatever the interpretation is, that is the message'(p127). Christie (1993) says that the ways teachers structure experience often operate at a subconscious level and are not visible for the 'close scrutiny or control' needed to enable their teaching to be a 'deliberate act'(p155). It is possible that insights gained by the use of curriculum genre principles to analyse actual lesson interactions, as well as students' reported experiences of the same lessons, may better illuminate the way teaching practices and student learning is structured in language, so helping teachers to better construct their mathematics curriculum genres.

This study has also exposed some important implications for the process of change from the students' viewpoint. There is some evidence to suggest that students are not at all passive recipients of changes in their classrooms and schools and they do not always enthusiastically endorse and accept changes which their teachers and research suggests will benefit them and make their schooling more relevant (McQuillan & Muncey, 1994, p273). The concepts associated with curriculum genres may help clarify the sources of students' resistance to change.

It was evident in this study that students' advice to teachers was most appropriate for the type of curriculum genre typified in the students descriptions; 'Explain what I have to do in more detail, and make the exercises more fun' summarises their suggestions. However if, for example, a teacher wished to change to a genuine problem solving approach, any degree of real success would be highly unlikely unless both registers of the curriculum genre in which these students had been operating were completely restructured. Research into those processes from a student's point of view could help teachers gain insight into the way language can purposefully be used to structure a new curriculum genre for the mathematics they wished to teach.

Another way in which an awareness of curriculum genres could assist research is in providing a framework for understanding the processes by which the content register has been historically built for a particular purpose out of the wider mathematical field of knowledge. Analysis of the range of mathematics curriculum genres in operation may expose the ways in which they need to change, and provide mechanisms for effecting such changes.

NOTES: 1. This paper was presented at MERGA 18, 1995 in Darwin. However doe to an oversight it was not published in that conference's proceedings.

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Computers in the Mathematics Classroom: A Survey

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In this paper a quantitative analysis of the results of a nationwide survey into the use of computers in New Zealand primary and secondary classrooms is described. The hardware and software distribution and pattern of classroom use in mathematics is outlined. The reasons teachers perceive for not being able to make even greater use of both calculators and computers in their mathematics teaching raises some important issues which have strong implications in terms of school funding and teacher training.

Introduction

The computer's potential for improving students' understanding of mathematics has long been discussed. In spite of this potential, a review of recent research in mathematics education in the 5-16 year old age range, (Askew and Wiliam, 1995, p.34) said of the UK position "Although computers have been in use in mathematics education in this country [UK] for well over twenty-five years, the pattern of usage is still very varied and very sparse."

A recent United Kingdom department of education report (DFE, 1995) reported the same low level of usage of computers in mathematics, with an average of 15.6 minutes of lesson time per week spent using the computer. In the United States the position on computer use, described by Ely (1993) is very similar. This study aimed to address the question of why the computer has not been more widely influential in the teaching of mathematics by discovering the pattern of use, and the factors influencing it, in New Zealand schools.

Method

A questionnaire about the use of, and attitudes to, calculators and computers in mathematics teaching was sent out in January 1995 to every school in New Zealand. Although having much in common, the primary and secondary school questionnaires were tailored to the different circumstances in each type of school. The questionnaires were in two parts; one for information on the school and the other for the perspectives of the teachers. Part one was filled in by the person responsible for mathematics in the school and they were encouraged to ask all the other mathematics teachers to fill in part two. The quantitative data from the questionnaire's closed response questions was coded and analysed using the spreadsheet Excel. The qualitative data gathered from the open response questions has been entered into the NUD.IST software package and is currently being analysed there.

The Samples

Replies were received from 480 of the 2471 schools (19.4%) in the primary/intermediate sector and 90 of the 336 schools (26.8%) in the secondary sector. For the 480 schools represented in the first questionnaire responses, 45% were from primary schools, 44.6% from full primary, 6.3% intermediate. The mean number of students was 179, with 24.7% of schools having fewer than 50 students and 20.7 % having more than 300. Of the 90 secondary sector schools, 13% were all boys schools, 25% all girls and 59% co-educational. Their mean size was 622 students, with 17.8% of schools having fewer than 300 students and 28.9% more than 900 students.

This discussion is based on 1500 replies from teachers in 480 primary/intermediate sector schools and 339 teachers in 90 secondary sector schools. Of the primary school teachers 25.2% were male and 74.8% female, with a mean age of 39.9 years, whilst the secondary teachers were 51.5% male and 48.5% female, with a mean age of 41.5 years. This sample is large enough to be representative of what is happening in schools, and it was pleasing from this standpoint to see that those not using computers in their teaching were well represented.

Results

When the teachers were asked if they use computers at all in their teaching, 38.7% of primary and 32.8% of secondary teachers said that they did not, however this primary/secondary difference is not statistically significant. When those who did use the computer were then asked how often they used it in their teaching the results were as shown in table 1.

Table 1. Primary and secondary teachers using computers in their mathematics teaching

Amount of Use	% of primary or intermediate teachers (n=1500)	% of secondary teachers (n=339)
At least once a week	31.9	5.9
At least once a month	13.3	19.2
At least once a term	11.5	26.5
At least once a year	3.2	16.2

This summary gives a rather less encouraging picture of computer use than the broad question of whether or not one uses a computer. Only 45.2% of primary and 25.1% of secondary teachers use the computer at least once month. We note that significantly more of the primary teachers than secondary use the computer at least once a week (χ^2, $p<0.0001$). One reason for this higher use in primary schools is likely to be the different accessibility of the computers. In primary schools, 93.3% use computers in their own classroom, whereas 89.1% of secondary teachers usually use a computer room. An obvious conclusion is that if the computer is available in the mathematics room it is more likely to be used in teaching. The pattern of computer use among the teachers, the curriculum areas are they using the computers for and the types of software are they using are shown in tables 2 and 3.

The emphasis on number work in the primary school is to be expected, though it is pleasing to see a sizeable proportion using computers in each curriculum content area in these schools. In secondary schools the emphasis is on graphical work and statistics, and the decline in use for measurement is not surprising.

Table 2. Curriculum areas where primary and secondary teachers are using computers

Area of Use	% of primary or intermediate teachers (n=915)		% of secondary teachers (n=229)	
	Some Use	Most Often Used	Some Use	Most Often Used
Number	82.7	59.2	41.0	12.2
Geometry	50.9	10.1	34.1	4.8
Statistics	44.9	11.1	75.1	38.0
Graphical work	44.6	13.3	74.2	35.4
Measurement	29.7	2.4	16.6	1.3
Algebra	24.8	2.0	32.3	4.8
Trigonometry	–	–	22.7	3.1
Calculus	–	–	24.0	3.9

Table 3. Types of software primary and secondary teachers are using with computers

Area of Use	% of primary or intermediate teachers (n=915)		% of secondary teachers (n=229)	
	Some Use	Most Often Used	Some Use	Most Often Used
Mathematical Programs	77.9	55.8	61.1	25.8
Word Processing	42.6	18.6	23.1	3.5
Graph Drawing Package	27.4	7.0	61.1	22.3
Database	22.5	4.5	14.0	0.9
Spreadsheet	22.2	5.9	67.2	31.9
Statistics Package	11.6	1.3	44.1	11.8

In addition 7.4% of secondary teachers said that they used desk top publishing software.

Many of the aspects of this pattern of software use are not surprising. Specific mathematical programs are widely used at primary level and continue to enjoy widespread use in secondary schools. The significant (χ^2, $p<0.0001$) increase in the use of the spreadsheet in the secondary classroom is a move from content-oriented software toward a generic type. What is not clear and requires investigation is the types of use to which spreadsheets are being put. A previous year long study (Barton *et al*, 1994) suggested strongly that teachers may have little idea of how to extract the mathematical value of spreadsheets. The results also show a significant increase in the use of graph drawing packages (χ^2, $p<0.0001$) and statistics programs (χ^2, $p<0.0001$) in the secondary years. It is interesting too to note that 44.9% of the primary teachers use the computer for statistics but only 11.6% use a statistics package. Similarly in secondary schools, the corresponding figures are 75.1% and 44.1%. Thus quite a proportion of the statistics is taking place on generic or other software rather than specifically written statistics software.

How do teachers organise their lessons around the computer use? We can get some idea of what is happening in the classroom by looking at table 4, which describes the methods of using the computer. One might expect teachers to use the computer either to attempt to improve students' skills or to allow them to investigate and solve problems, but not to do both. However, it would appear that a substantial proportion of teachers in both primary and secondary sectors use both methods and do not see themselves as on one side of a dichotomous ideological fence. The continuing decline in the time spent on programming in the secondary school is evident from the low percentage of teachers (8.7%) promoting this. Although this is understandable there is evidence that programming may encourage the formation of mathematical thinking (e.g. Pea & Kurland, 1984).

Table 4. Methods which primary and secondary teachers are using computers in their teaching

Method	% of primary or intermediate teachers (n=915)		% of secondary teachers (n=229)	
	Some Use	Most Often Used	Some Use	Most Often Used
Skill Development	68.3	36.3	67.7	37.6
Free Use	63.0	19.2	34.9	3.1
Investigations/PS	62.6	28.7	68.6	38.4
Demonstrations	19.3	2.7	40.6	10.9
Programming (inc. Logo)	23.1	7.8	8.7	1.3

The significant drop (χ^2, $p<0.0001$) in the amount of free use allowed by secondary teachers and a corresponding significant increase (χ^2, $p<0.0001$) in the use of demonstrations is interesting. The reasons for this need further investigation but may involve both classroom management and computer access issues.

Reasons For Not Using the Computer

It was very encouraging to see that 92.6% of the primary teachers and 93.5% of the secondary teachers responded that they would like to use computers more in their mathematics teaching. Hence a large proportion of the teachers at both levels who are not currently using computers would like to. What factors do they perceive as preventing them from doing so? They were asked several questions designed to discover what they saw as being at the root of the problem. Firstly they were asked about obstacles to their use of the computer. Their replies are summarised in table 5.

Table 5. Obstacles which primary and secondary teachers perceive as preventing them personally from using computers in their teaching

Obstacle	% of primary or intermediate teachers (n=1500)		% of secondary teachers (n=339)	
	First Mentioned (%)	Mentioned (%)	First Mentioned (%)	Mentioned (%)
Available Software	28.3	64.9	17.4	52.5
Available Computers	33.4	56.9	43.7	67.8
Lack of Training	22.2	55.5	17.4	45.4
Lack of Confidence	12.6	41.0	12.7	34.8
Government Policy	2.6	12.0	4.1	12.4
School Policy	0.3	8.7	0.6	8.0

In addition 13% of secondary and of 8.3% of primary teachers mentioned some other obstacle.

The pattern here is dramatically clear, with a very uniform response. As far as teachers are concerned there are two areas where they would like to see improvement in order to reach their goal of using computers more. They are the provision of resources, in terms of available hardware and software and the increasing of their confidence through satisfactory training. However, there is a slightly different emphasis at each level. In primary schools lack of software is significantly (χ^2, $p<0.001$) more of a problem than in secondary schools, where computer availability is significantly worse (χ^2, $p<0.001$). Also lack of training and confidence is considered significantly more of a problem (χ^2, $p<0.001$) for primary teachers. These figures indicate that primary school teachers feel in even greater need of assistance and training than secondary teachers. The provision of suitable inservice training for primary teachers is thus a particular area which needs to be addressed. Later in the questionnaire the teachers were also asked rate on a Likert scale their level of agreement with what might prevent mathematics teachers in general from using computers. Some of the responses here are summarised in table 6.

Table 6. Obstacles which primary and secondary teachers perceive as preventing teachers in general from using computers in their teaching

Obstacle	% of primary or intermediate teachers (n=1500)		% of secondary teachers (n=339)	
	Strongly Agree (%)	Agree (%)	Strongly Agree (%)	Agree (%)
Lack of good ideas	4.7	31.2	7.1	33.9
Classroom management	7.3	30.3	7.1	39.8
Lack of software	16.1	44.2	13.9	36.9
Keeping discipline	1.4	5.4	1.5	5.6
Teacher confidence	17.1	55.1	15.0	56.0
Assessment not compulsory	–	–	2.7	16.8
Computers of little benefit	0.5	0.9	0.3	4.4

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Lack of Confidence and Training

A comparison of tables 5 and 6 reveals that whilst 41% of primary teachers think that lack of confidence is an obstacle to their use of the computer, 72.1% agree that it is a problem for teachers in general. Similarly 34.8% of secondary teachers admit to this as a problem for themselves but 71% agree that it is a problem for teachers in general. These figures could well show a reluctance to admit to personal feelings of lack of confidence, but they certainly highlight the way that teachers in both sectors rate this as a serious obstacle to computer use in the teaching profession. The type of training which might help to alleviate this feeling can also be discerned in the above table. Teachers do not see keeping discipline in the classroom as a problem, but 37.6% of primary and 46.9% of secondary teachers do think that the management and organisation of their classroom around the use of the computer is an area they would like assistance with. Similarly 35.9% of primary and 41.0% of secondary teachers would welcome some training which would give them good ideas to use in their classroom, and 60.3% and 50.8% respectively want to know about good software they can make use of. These figures agree well with findings of Thomas *et al* (1995), who, in a year long study of teachers using computers in their classroom found that the upheaval resulting from the computer presence had several components, including finding a place (in more than one sense) in the classroom for the machine, considering the need to extend one's classroom management techniques and recognising some of the implications of electronic technology for school mathematics.

This survey did uncover some gender differences in what are perceived to be obstacles to computer use, as table 7 shows. We see from this summary that, in primary schools, while there is no significant difference between genders on the need for more software and more training, significantly more female teachers than males see their lack of confidence as a problem (χ^2, $p<0.01$), while significantly more male teachers than female view a shortage of computers as an obstacle (χ^2, $p<0.01$). In secondary schools significantly more female teachers than males cite each of the obstacles: software (χ^2, $p<0.05$); computers (χ^2, $p<0.01$); training (χ^2, $p<0.001$); and confidence (χ^2, $p<0.001$). It is difficult to attach any reasons to these differences without further investigation and this will be something which the interviews with the teachers will address.

Table 7. Gender differences in perceived obstacles to computer use

Obstacle	% of primary or intermediate teachers (n=1487)		% of secondary teachers (n=299)	
	Female (%)	Male (%)	Female (%)	Male (%)
Available Software	64.5	68.3	65.5	53.9
Available Computers	54.9	65.1	84.8	69.4
Lack of Training	56.9	53.3	66.2	37.7
Lack of Confidence	43.5	34.9	51.7	27.9

Hardware Availability

56.9% of primary and 67.8% of secondary teachers cite a lack of available hardware as an obstacle to its use. In primary schools, the mean number of computers per school reported was 7.0, giving 1 computer for every 25.6 students, and while 2.9% of the schools claimed to own no computers at all, 22 schools (4.6%) had 20 or more and two schools, of more than 500 students, had 40 and 41 computers respectively. Although 40 of the schools have a computer room, of the 61.3% of teachers who used computers in their teaching, 93.3% use computers in their own classroom, with 82.4% of these usually having only one computer available in their room. The teachers generally see this as insufficient and are asking for more computers in their classroom.

In secondary schools the distribution of computers is rather different. The mean number of computers per school reported was 39.9, giving 1 computer for every 15.6

students, and this compares well with the 1994 New Zealand department of education figures of 1 for every 17 students. However, of these 40 computers in the school, only 1.7 are in the mathematics department, the rest are in the 1.8 computer rooms which the schools average. Thus 89.1% of secondary teachers using computers usually do so in a computer room. Clearly access to these rooms is a problem for many though. 44.5% said that the room was difficult to book and so they seldom used it. This issue of access to the computer is very important. It is apparently unhelpful to fill a school with computer suites full of computers if the mathematics staff find that competition to use the room is so strong that they are often unable to take their classes in.

When heads of departments and those responsible for mathematics were asked, in an open response question, what they saw as obstacles to the implementation of computers in their schools they gave the answers recorded in table 8.

Table 8. Obstacles which primary and secondary teachers responsible for mathematics perceive as preventing them from implementing computers in teaching

Obstacle	% of primary or intermediate teachers (n=480)	% of secondary teachers (n=90)
	Mentioned by (%)	Mentioned by (%)
Access to Computer Room	0.0	51.1
Lack of Training/Confidence	49.6	45.6
Lack of Time	30.0	32.2
Lack of Money	34.4	25.6
Lack of Software	16.7	18.9
Lack of Computers	20.4	16.7

Apart from confirming the pressing need to increase teachers' confidence by proper training, these replies demonstrate another important reason why school teachers are saying in such large numbers that a shortage of hardware and software is an obstacle for them. Many say that the funds are not available to purchase these items. Whilst this is not a new problem for schools it is an issue which clearly needs to be addressed by those who allocate funds. Allied to this deficit in funding is the pressure of time which these teachers feel under. Nearly a third of them feel that they do not have time to implement computer use to the extent which they would like. This problem too appears to be one which is primarily due to funding constraints. If wider and better computer use is to be achieved then adequate funding needs to be made available to schools to allow teachers to be released for both training and implementation. Clearly teachers view this as desirable, since so many would like to use computers more and, as we see from table 6 above, only 1.4% of primary teachers and 4.7% of secondary teachers see computers as being of little benefit in their teaching. Research has suggested that if the resources they are asking for are not provided then it leads to frustration and a loss of enthusiasm on the part of teachers (Dunn & Ridgway, 1994), so that the matter is urgent.

This data analysis raises many other questions which are being explored. For example, are there any significant differences in the pattern of computer use between different types of schools, schools with different types of computers in use, teachers of different ages, teachers who have been involved in professional development in computer use, and other variables, etc.?

Discussion

What then are the implications of this study? Outstandingly it seems that more funding for computers and for training seems necessary. However, it has been observed that computers alone will not bring change in the classroom. Rather, the beliefs and attitudes of the teachers themselves seem to be major elements, since teachers have a crucial role to play (Lai, 1993; Collis, 1993). Teacher factors appear to outweigh school factors (Veen, 1993), and, as Maddux (1994) describes, information technology will remain relatively unimportant until all teachers incorporate it successfully into their teaching. Carey and Sale (1993, p.191) agree with the need for both training and support for teachers as a primary concern:

> . . .the study. . .does give educators reason to doubt that the computer is a sufficient catalyst for changing teachers into coaches and facilitators without deliberate, well-planned efforts to provide training and support for such teaching modes and for moving towards teacher education that addresses the role of the teacher as well as the role of the computer.

Veen (1993) too makes the observation that change which requires teachers to alter their beliefs and practices will only come very slowly. Thomas *et al* (1995) showed that it took a year for teachers to change their mindset, even when they were using the computer in their classroom.

What type of training is needed? I have outlined above some things that the teachers said that they thought they needed from training:

- improved confidence in using computers
- better classroom management with computers
- clearer understanding of good software available
- good ideas for the classroom use of computers

Dunn and Ridgway (1991) have also identified a lack of teacher confidence as a major reason for teachers not using computers. They found that when student teachers actually used the computer in their teaching practice 59% of them gained in confidence and they conclude that "the college course should seek to build up personal confidence to the level where students are happy to use computers during their teaching practice, and that confidence is likely to be increased by classroom uses of IT." (ibid, pp. 13,14). Monaghan (1993) concluded that teacher training should encourage students to avoid viewing computer use as an optional extra. Clearly teacher training must address the vital issue that it is primarily the classroom use of computers which can improve teacher confidence. According to Mellar and Jackson (1992, p.242), the type of training teachers need agrees with that described in this study, namely ". . . courses aimed at the educational issues - the ways computers can be used in teaching, managing the use of the computer in the classroom, and the role of the computer in aiding learning."

Dunn and Ridgway (1994, p. 48) also describe how teachers need training in the educational use of software: "More remains to be done, however, in improving students' understanding of the educational purposes of the software they are using, and to broaden the range of programs with which the majority. . .feel confident." They suggest that increasing the confidence and expertise of the tutors in the training colleges is an important element of this process. Maddux (1994, p.132) goes so far as to say that, in some countries, this is the primary goal:

> The only solution involves convincing the entire teacher education staff and administration that computers can and should play an important role in all teacher education courses. This goal should be the primary goal of those of us in teacher education who believe that technology has the potential of revolutionizing teaching and learning. . . It seems obvious that until all or most teacher educators in teacher education institutions take technology seriously, our student teachers are unlikely to do so.

It is worth noting that of the 36 newer secondary teachers in this survey, between 21 and 30 years of age, only one would not like to use the computer more in their teaching but 33.3% cited lack of confidence and 44.4% lack of training as obstacles to such use.

Although change may come slowly, with evidence that it takes teachers up to a year to change their view of computers, the direction of change can be influenced if all involved act decisively (Fullan, 1991; Thomas *et al* , 1995). Teachers and teacher educators need to be aware of the changes that are needed and implement them together, so that the full benefits of computers will accrue in our mathematics classrooms.

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Problem Solving in Algebra: The Role of Arithmetic Thinking and Affective Factors

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In this paper we discuss the roles of, and the relationship between, arithmetic and algebraic thinking in algebraic problem solving. We report the results of a study where we recorded students' mode of thinking, problem solving ability, mathematical anxiety and self-concept. We found a positive correlation for girls between anxiety and problem solving performance, but for boys the positive correlation was with self-concept. We also found a reluctance on the part of students to use algebraic thinking.

Background

Many students have had difficulty with algebraic problem solving. There have been a number of studies conducted by mathematics education researchers to attempt to understand the nature of these difficulties. Kieran (1992) sees generating equations from words as the major area of difficulty for high school algebra students. Much research (e.g. Lochhead, 1980; Clement, Narode and Rosnick, 1981; Kaput & Sims Knight, 1983; MacGregor & Stacey, 1993; Thomas, 1994) has analysed the difficulties faced by students while solving algebra word problems and have offered suggestions to overcome these difficulties. Brown *et al* (1988) expressed the opinion that students appear to cover their inability to translate the relationship between the words into symbolic form or to solve resulting equations resorting to memorising rules and procedures that they eventually come to believe represent the essence of algebra.

Others have tried to compare arithmetic and algebraic modes of thinking in the context of problem solving. Filloy (1985) stated that algebra is the continuation of arithmetic. Kieran (1992) delineated arithmetic and algebraic thinking by referring to arithmetic operations carried out on numbers to yield numbers as procedural and a set of operations carried out on algebraic expressions as structural. While differentiating two types of operations, Kinch and Greeno (1985) see the difference as mainly due to the complexity of algebraic schemata compared with arithmetic set schemata. In the same context Herscovics and Linchevski (1994) stated, from a study where students solved fifty algebraic equations of different types, that a cognitive gap exists between arithmetic and algebra which is characterised by the students inability to operate spontaneously on the unknown. There is evidence emerging (Thomas, 1994; Crowley, Thomas & Tall, 1994) that many students, described as *process-oriented*, are locked into an arithmetic, procedural way of thinking even when doing algebra.

Affective Factors

Problem solving can be seen with a different perspective, as an interaction between cognitive and affective variables. McLeod (1989) claims that the cognitive processes involved in problem solving are particularly susceptible to the affective domain. From a Piagetian perspective this is not surprising since the process of acquisition of knowledge involves the constant interaction between the learning subject and his or her environment, and affective variables result from this same interaction. Boekaeris *et al* (1995) stated that mathematical problem solving can be seen as a process in which cognitive and affective variables interact and Lester (1994) has described this interaction as involving several closely related independent categories of factors such as knowledge acquisition and utilisation, beliefs, affects and socio-cultural contexts. Hart (1989), too, summarised the affective domain as comprising a wide variety of beliefs, attitudes and emotions including anxiety which typically contain a strong emotional feeling. Self-concept, a reflection of how one sees oneself, is also part of the affective domain. Skaalvik and Rankin (1995) define self-concept colloquially as a composite view of one self, the hierarchical structure of which Shavelson, Hubner and Stanton (1976) and Shavelson and Bolus (1982) have modelled emphasising its multidimensional nature. Self-concept and anxiety have been taken in some mathematics studies as more specific components of attitudes and shown to have direct, consistent positive and negative correlations with mathematics achievement which are also gender specific. Reyes (1984), for example finds consistent positive correctional association between self-concept and academic achievement and Marsh (1993) demonstrated the gender differences in self-concept. Similarly the effect of mathematics anxiety on mathematics achievement has interested researchers, with Richardson and Woolfolk (1980) mentioning that certain features, such as an emphasis on problem solving, logic, etc. particularly provoke anxiety in students.

However the relationship between self-concept and anxiety in the affective domain and student word problem solving ability in algebra, an area of particular difficulty for students, has not been a specific area of study. This paper documents the start which we have made in looking at the importance of affective factors such as these on students' mathematical problem solving ability.

Method

Subjects

The data for the present study was collected from 116 form 3 and 4 students (age 13 and 14 years) comprising 48 boys and 68 girls. The students belong to average ability groups from one boys, one girls and one co-educational school in the Auckland region of New Zealand, and all were reported by their teachers to be following the New Zealand mathematics national curriculum.

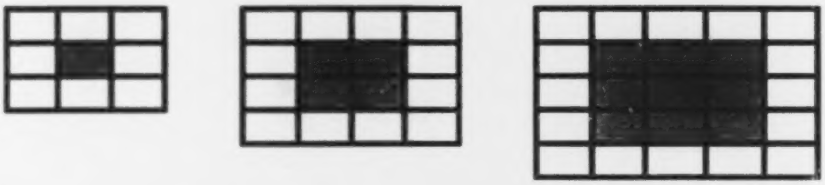
Instruments

The mathematics anxiety scale of Fennema & Sherman (1976) served as the students' mathematics anxiety measure. Fennema & Sherman scales have been among the most popularly used scales for about two decades and recently Melancon *et al* (1994) reported favourably regarding the validity of the scores obtained using the scales. The mathematical self-concept scale developed by Gourgey (1982) and having 27 items was used to measure mathematical self-concept. The reason for using this scale was that the definition given by Gourgey for self-concept constitutes a general view of the affective domain and the individual subjects report regarding their perceived capability in performing mathematics. Both scales are in a 5-point Likert format.

In addition to the attitude scales, three parallel algebraic tests of 5 problems each were administered to measure the students' problem solving ability. Examples of the problems from the three tests used are:

1. Julia had \$560. After buying a gold chain she had \$151 left. How much did she pay for the gold chain ?
2. The number of girls in the class is 12 less than the number of boys. The total number of students in the class is 34. How many girls are there in the class ?
3. To hire the pizza house for a birthday party costs a basic rate of \$70 plus \$3.50 per person. If the total bill is \$252.00, how many people attended the party ?
4. If the total amount of money collected by selling 230 tickets to the school play was \$764 and the cost of the ticket was \$4 per adult and \$3 per child, how many adult tickets and how many child tickets were sold?
5. A square garden is surrounded by square paving stones. Where N = the number of paving stones and L = the length of the side of the garden.

Complete the table to find the number of paving stones for a garden 4 long and 100 long. Write an algebraic equation to find the number of paving stones for a garden L long.



Length of the side of the garden	L	1	2	3	4 . . . 100 . . . L
Number of paving stones	N	8	12	16	

These problems were selected to correspond to examples in the published New Zealand mathematics curriculum for each of the forms 1 to 5. It was arranged that they have the same achievement objectives and difficulty level as described in the mathematics curriculum followed in the schools. Problems were taken from levels both above and below that of the students so that there would be questions that were genuine problems for the students, being outside their experience, and we would be able to monitor the methods they used to tackle these novel problems.

The instruments were administered under the supervision of the mathematics teacher concerned within a 1-week period during the middle of the 1995 academic year. To ensure the uniformity of administration the teachers were provided with clear instructions and directions.

Results

Affective Factors

The results of the 116 students, 48 boys and 68 girls, who completed all three algebra tests and both affective factor scales were analysed for the influence of these factors. The individual students' scores on each of the three algebra tests we used, as described in table 1, show a fairly high level of correlation between the tests. Since they were designed to be of a similar content and standard this is what we would expect. We conclude that there is a fairly stable level of attainment for each student over the tests, and thus together they constitute a fairly reliable measure of ability on these types of algebra word problem. The results of the ANOVA show that there is, however, a significant difference in the overall results of the three tests (df =2, F=31.65, $p<0.0001$). Analysis of this difference showed that overall the results on test 3 were significantly lower than those on the other two tests. Also, on two of the tests the girls performed significantly better than the boys (Test 1: $t = 0.943$, n.s.; Test 2: $t = 1.892$, $p<0.05$; Test 3: $t = 2.177$, $p<0.05$).

Table 1: Correlations for individual male and female student scores on the three tests

	Overall	Female	Male
Test 1 v Test 2	0.603	0.677	0.488
Test 2 v Test 3	0.482	0.567	0.364
Test 1 v Test 3	0.696	0.756	0.658

An analysis of the correlations between the students' individual algebra test scores and their affective factor scores on the anxiety and self-concept scales used revealed a significant positive correlation between self-concept and the results of all the tests (Test 1: $r = 0.422$, $p<0.01$; Test 2: $r = 0.306$, $p<0.01$; Test 3: $r = 0.394$, $p<0.01$; df = 114). This seems to indicate that those who have a higher level of self-concept generally perform at a higher level on the algebra tests, and vice-versa, confirming the finding of Reyes (1984) study, but in the domain of algebra problem solving.

Gender Considerations: When we divide the students' results into male and female we see some very interesting results emerging which tend to be masked when the results are taken as a whole. As table 1 shows, the correlations between the algebra tests based on individual scores, on a gender basis, are higher in each case for the female students than for the male students. Thus the females perform more consistently than the males across the algebra tests. When we look at the correlations between the algebra scores and the anxiety and self-concept measures on a gender basis we obtain the figures in table 2.

Table 2: Correlations between algebra, anxiety and self-concept scores for individual male and female students

	Female (N=68)	Male (N=48)
Test 1 v Anxiety	0.453	0.034
Test 2 v Anxiety	0.393	0.258
Test 3 v Anxiety	0.340	0.140
Test 1 v Self-Concept	0.415	0.472
Test 2 v Self-Concept	0.366	0.291
Test 3 v Self-Concept	0.327	0.530

For the girls every one of the correlations is significant ($p < 0.01$, df = 66), but for the boys, none of the anxiety results are significant, but all the self-concept ones are (Test 1: $p<0.01$; Test 2: $p<0.05$; Test 3: $p<0.01$; df = 46). We also see clearly that there is a higher positive correlation each time between algebra performance and anxiety for the female students, and on two occasions, a higher correlation between algebra performance and self-concept for the males. When we looked at the overall differences between the scores for anxiety and self-concept between the genders we found no significant difference in scores (anxiety, df = 99, $t = 0.653$, $p< 0.515$ and self-concept: df = 94, $t = 1.152$, $p< 0.252$). Thus, while there is no significant difference for these two factors between the genders, it appears that girls who are anxious about mathematics are more likely to perform better in the algebra problem solving whereas this is not true for boys. Likewise both girls and boys who have a higher self-concept are more likely to perform better. The results on anxiety are, at first, surprising, since they are different from the negative correlations published in the research literature (see e.g. Aiken, 1976; Betz, 1978; Adams & Holcomb, 1986; Gliner, 1987; each of whom has established consistent, negative relationships between mathematics anxiety and performance) and one might think that those who are anxious about mathematics would not perform well (and for some boys this may be the case). However, the results show that the opposite is true for the girls, with their anxiety apparently motivating them to perform better. It appears that there is also an age factor involved since when we looked at the girls and boys at single sex schools in our sample, we obtained the correlations given in table 3:

Table 3: Correlations between algebra, anxiety and self-concept scores for individual male and female students aged 13 and 14

	Girls Age 13 (N=25)	Girls Age 14 (N=28)	Boys Age 13 (N=26)	Boys Age 14 (N=19)
Test 1 v Anxiety	0.873	0.481	---	---
Test 2 v Anxiety	0.675	0.243	---	---
Test 3 v Anxiety	0.403	0.013	---	---
Test 1 v Self-Concept	0.702	0.129	0.369	0.759
Test 2 v Self-Concept	0.674	-0.173	0.310	0.447
Test 3 v Self-Concept	0.574	0.070	0.327	0.455

Thus for the younger girls both anxiety and self-concept correlate significantly with algebra problem solving performance in every test, but within just one year this effect has diminished to the point where it has disappeared. Conversely, as the boys age a year from 13 to 14, their self-concept, which at 13 is not significantly correlated with performance, becomes significantly correlated a year later.

Arithmetic and Algebraic Thinking

It can be seen from tables 4 and 5 that, for the first three problems, where although algebraic methods may be useful they are not essential for their solution, an arithmetic approach is easily the preferred option for students of both ages.

Table 4: Percentages of male and female form 3 (age 13 years) students using arithmetic, trial and error and algebraic methods.

	Girls (N=43)			Boys (N=40)		
Problem	Arithmetic	Trial & Error	Algebraic	Arithmetic	Trial & Error	Algebraic
1	61.2	0.8	0	48.3	2.5	1.7
2	32.6	12.4	10.9	25.0	10.8	5.0
3	43.4	4.7	2.3	36.7	4.2	5.0
4	3.1	41.9	5.4	5.8	20.0	7.5
5	9.3	1.1	26.7	0	0	26.2

When arithmetic thinking is not successful then the students first look to trial and error, as may be seen from questions 2, and particularly, question 4, which cannot easily be solved arithmetically and in which we deliberately avoiding giving any direction towards an algebraic method of solution. In question 5, which also cannot easily be solved arithmetically, there was a slight hint of algebra given, with the introduction of a letter for a variable. This question also had a diagram.

Table 5: Percentages of male and female form 4 (age 14 years) students using arithmetic, trial and error and algebraic methods.

	Girls (N=41)			Boys (N=32)		
Problem	Arithmetic	Trial & Error	Algebraic	Arithmetic	Trial & Error	Algebraic
1	71.5	0	2.4	61.5	0	3.1
2	35.0	2.4	14.6	31.3	1.0	3.1
3	53.7	1.6	4.9	53.1	0	2.1
4	5.7	24.4	5.7	5.2	20.8	1.0
5	4.9	1.2	54.9	0	0	14.1

Interestingly, by the age of 14 significantly more girls ($\chi^2=24.0$, $p<0.001$) than boys are using algebraic methods to solve question 5. This may indicate that the girls are progressing from arithmetic to algebraic thinking earlier than the boys. The two groups comprise different students though, so this result needs further investigation, tracking the 13 year old group through to form 4, and this we will look at when they are a year older.

Conclusions

What do these results show? Summarising the results on affective factors we may say that anxiety and self-concept have a positive effect on the problem solving performance of girls, especially younger ones, whereas for boys the positive effect on their problem solving ability is self-concept, and then only for the older students. Further research is needed to see if these effects are only present in single sex schools, as described here, or whether they exist in co-educational schools as well. When doing algebra, we see that even after two or three years, if the relation between the variables in a problem is given in the verbal form, the students show a tendency to use an arithmetic mode of thinking. A failure to solve the given problem using these arithmetic methods leads them to follow intuitive methods such as guess and check or trial and error rather than algebraic ones even though these methods are known to them and they were able to use algebra on the final question. A greater number of the students could express the relationship between the unknowns in an algebraic form if the problem was given with a geometrical figure, even when the problem was totally unfamiliar to them. Clearly the transition from arithmetic to algebraic thinking takes time, and just as there is a gap between acquisition of knowledge, understanding and application it can be seen that there is a gap between learning algebraic methods and applying them to solve problems. This transition may take less time for girls than for boys. There appears to be a relationship between the changing effects of anxiety and self-concept and the changing preference for algebraic methods over arithmetic ones. For the girls, as the correlation between these affective factors and performance disappears their use of algebraic methods increased. However, for the boys, as the correlation between self-concept and performance increased, their use of arithmetic methods increased. The underlying reasons for what appears to be a relationship are not clear and will be the subject of continued investigation.

It may be that the reluctance to use algebra can be lessened by using geometric figures to express the relationship between unknowns simultaneously with verbal forms. That mathematical thinking may be encouraged by the use of visualisation encouraged by multiple representations is well known. Thomas (1995) has suggested a theory of *cognitive integration* which may begin to explain the mechanisms by which this may take place.

Further, our results may be consistent with an information processing model of cognition, such as that explained by Silver (1987), in which there are two kinds of information processing, automatic and controlled. Automatic processing involves activation of cognitive nodes, but under the control of a particular input for working memory (internal or sensory) rather than the subject's control. In controlled information processing a sequence of nodes is activated by a controlling subject. In mathematics, by the time student reaches secondary stage, solving problems using arithmetic procedures has become automatic, though it started as a controlled process at primary school level. For this reason children might have the tendency to use arithmetic procedures for some time, even after they formally learn algebraic methods. There is a possible role played by both affective factors, anxiety for girls and self-concept for boys in this transition from an arithmetic mode of thinking to an algebraic one, or from automatic to controlled processing. Further empirical studies are needed to clarify whether the apparent time lag between the learning of algebraic methods and the application of them to solve problems does fit such a theoretical model, and, if it does, to assist in analysing possible reasons (either cognitive or affective) for this time lag.

It remains to find out if the algebraic methods ever become the preferred method of solution for students and if so at what age and for which students. Further, do affective factors such as anxiety, self-concept and other social factors influence the transition in their thinking and if so, how and why?

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The Riemann Integral in Calculus: Students' Processes and Concepts

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Students' ability to understand calculus has been of concern for some time. Students are often locked into a process-oriented style of thinking which is an obstacle to their understanding of important concepts. In this paper we describe the results of an integral calculus questionnaire given to senior secondary school students and designed to measure their understanding of concepts associated with the Riemann integral. We describe some misconceptions in their understanding and the instrumental, process-oriented thinking underlying them.

Introduction

Calculus reform programmes are now well underway in some parts of the world in an attempt to address the difficulties which many students have had with understanding calculus. Much of the research into these difficulties has uncovered students' conceptual difficulties with limits, differentiation and integration (see e.g. Tall, 1985, 1986a, b; Steen, 1988; Barnes, 1988; Li & Tall, 1993; Thompson, 1994) and considerable effort has also been put into finding ways to improve such understanding, often using computer software such as symbolic manipulators (e.g. Small & Horsack, 1986; Palmiter, 1991; Barnes, 1994; Hubbard, 1995). Heid (1988), for example, suggested giving meaning to basic concepts, by using graphical software and symbolic manipulators to build conceptual insights based on mathematical and cognitive principles. She found that, using these methods, the learning of fundamental concepts was greatly improved in her experimental class. These students made use of a large variety of concept representations and learned how to reason with them. They learned to think for themselves and to construct their own ways of handling the concepts. It became apparent that they had integrated the ideas into their own knowledge structures.

One recurring theme from the research is that students have a tendency to be process-oriented rather than concept-oriented in their approach to calculus. They are much happier following an algorithm or manipulating symbols than they are with dealing with concepts such as limit or rate of change. This is not too surprising when one considers that, until recently, calculus teaching in school and university has often concentrated on process skills rather than conceptual understanding, or, in the words of Skemp (1976), it has been instrumental not relational. A key distinction between relational understanding and instrumental understanding is that the former can be described as learning 'why to', while in contrast, the latter is learning 'how to' and involves learning by rote, memorising facts and rules.

Background

In the research on the calculus there has been a tendency to concentrate on differentiation and its concepts rather than integration. Although this situation is now changing, the purpose of our research is to investigate student thinking and misconceptions when dealing with the Riemann integral. Here, a definition of definite integral can considered whereby a set of n rectangles, which can be freely constructed, is used to approximate the area. This requires that, for a given set of well-defined rectangles, the tops of the rectangles intersect with the graph of the function, the widths of the rectangles partition the horizontal distance involved, and as n‡∞, the width of the rectangles approaches zero. But a definition of the definite integral based on these concepts is often quickly forgotten once the fundamental theorem of the calculus linking areas and antiderivatives is introduced. This is not unnatural, because the Riemann sums used in the definition are tedious to calculate by hand and require the taking, and understanding, of a limit, whereas the antiderivatives involve clear algorithmic processes.

Processes and Concepts

Much has been written in recent years about the relationship between processes and concepts in mathematics. Piaget (1985, p.49) described how "actions or operations become thematised objects of thought or assimilation" and this idea has been further clarified by researchers such as Dubinsky & Lewin (1986) who describe the *encapsulation* of a process as an object. Sfard (1991) too has emphasised that there is a conceptual change when this occurs, involving the conversion of a dynamic process into a static object. Encapsulating both the differentiation and integration processes seems to be an essential prerequisite for understanding the fundamental theorem of calculus. Gray & Tall (1993) have introduced symbolism and define the notion of *procept* as an amalgam of three things - process, symbol and concept. Thus a symbol, such as:

$$\int_1^x \frac{\sin x}{1+\log t}\,dt$$

evokes both the process of integration and the concept of integral, with the cognitive combination of all three being a procept. Procepts are first met through a process, then a symbolism is introduced for the product of that process, and this symbolism eventually takes on the dual meaning of the process and the object created by the processes. The limit is an important example of a procept in the calculus. For example, both of the symbols: $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{n\to\infty} a_n$ may represent either the process of getting close to specific value, or the value of the limit itself. It seems that much of the symbolism used in mathematics carries for the mathematician the dual role of process and concept. This distinction between the usage of symbolisation to stand for a process or a concept or conceptual structure depending on one's point of focus is clearly an important one mathematically. In addition, Gray & Tall (1993) define *proceptual thinking* as the ability to be able to switch one's focus between these dual roles of the symbols as necessary. Someone who has the ability to think in this way may be described as a *versatile* mathematician (Tall & Thomas, 1991). One major difference between the *versatile* mathematician and the learner is that the mathematician already has a global picture of the concept, so that when she/he breaks it down into a number of stages, she/he can see each stage as part of the whole concept. The learner, on the other hand sees only the part in the context of limited understanding.

Method

In the light of the above discussion, our aim is to use computer-based work to improve students' relational understanding of, and proceptual thinking about, Riemann integration. In this aim we agree with Tall (1993), who suggests that the computer relieves the learner of the tyranny of having to encapsulate the process before obtaining a sense of the properties of the object. By using software which carries out the process internally, it may become possible for the learner to explore the properties of the object produced by the process before, at the same time, or after studying the process itself. This new flexibility in curriculum development he has called the *principle of selective construction*.

The first stage of the research, documented here, was to discover what type of thinking and understanding students are developing with current styles of teaching. To this end we developed a questionnaire containing fifteen questions, each of which sought to probe understanding of one aspect of integration, but doing so in a way which often would not enable students to carry out processes or algorithms in order to obtain answers.

This questionnaire on the concepts of the Riemann integral in calculus was given to 47 Form 7 (age 18 years) high school students early in 1996. These students had been studying calculus for about one year and were from 3 schools, one for boys only, one for girls only and one co-educational. Due to advanced promotion through school, the actual ages of the students ranged from 16 to 19 years and there were 16 male and 31 female students.

Questionnaire and Results

The questions we gave to the students produced answers which highlight qualitative differences in thinking. We shall now describe these questions and the responses to them.

Conservation of Integral

The question here was:

Given that $\int_9^{16}\sqrt{x}dx=\frac{74}{3}$, what is $\int_{16}^{9}\sqrt{t}dt$?

This was intended to determine whether students understood the concept of conservation of integral under change of variable and could apply this concept in a situation where the limits were reversed. 34.0% of the students were able to correctly answer this question. We gave the students an actual function, $\sqrt{x}$, to see whether some process-oriented students would calculate the integral(s) in order to answer the question, but in the event, only two students attempted to do this. However, 17.2% of the students invented an algorithm to deal with their lack of understanding of the reversed limits. Since the values of the limits appeared inverted they inverted the answer too and obtained $\frac{3}{74}$. This provides evidence of the way in which a substantial number of students, who are instrumental in their thinking and understanding rather than versatile, conceptual thinkers, attempt to deal with problems which have concepts they do not understand.

The Riemann Integral

A question, dealing with Riemann sums, which also revealed conceptual gaps in understanding was:

If f(x) is an integrable, strictly decreasing function on [1, 5], what is the value of $\lim\limits_{\Delta x\to 0}\sum\limits_{i=1}^{n} f(x_{i+1})\Delta x-\lim\limits_{\Delta x\to 0}\sum\limits_{i=1}^{n} f(x_i)\Delta x$, where x_i defines the start of the i-th subinterval, and Δx is its width?

(a) very small and positive
(b) 0
(c) very small and negative
(d) not possible to say
(e) very large and positive
(f) very large and negative

This was designed to test understanding of the concept of the equivalence of the limit of the upper and lower sums. Since no function is explicitly stated it was not possible for a process-oriented student to calculate the sums or limits in order to answer the question. A summary of the responses to this question is given in table 1.

Table 1: Summary of question responses for equivalence of the limit of upper and lower sums (N=47)

	No Response	a	b	c	d	e	f
Number of students	19	14	5	5	3	1	0

We see that 40.0% of the students were unable to answer the question (which is admittedly difficult) and only 10.6% answered correctly. The 40.4% of the students who gave a) or c) as the answer do not have the conceptual understanding that the limits of the rightsum and leftsum will be the same, if f(x) is an integrable function. A second question on Riemann sums was:

If f(x) is a strictly decreasing function, an approximation to $\int_1^4 f(x)dx$ using a Riemann sum (i.e. of rectangles under the curve) with left endpoints and number of strips n=10 works out to be 7.615. Will an approximation with left endpoints and number of strips n=50 be:

(a) more than 7.615	(b) less than 7.615
(c) equal to 7.615	(d) less than or equal to 7.615
(e) not possible to say	(f) more than or equal to 7.615?

Explain your answer.

Unfortunately only 20 of the students were given this question which looked at the process involved in the area approaching the Riemann limit. Of these 10% gave the correct answer whilst 50% gave no response. Of the others, 30% opted for a) or f) and 10% for d). In spite of being asked to explain, only two students attempted any form of reason for their answer, stating "it was because the curve is lower and wider" and "with left endpoints some might not get included with more strips. So >10 this would have the strips taking up more of the graph." There seems to be a distinct lack of understanding of these types of sums among these students with many students not having encapsulated the process of taking the limit of the sums of the rectangles as an object, namely the definite integral of the function.

Integration and Transformations

We wanted to see if the students had grasped conceptually the relationship between the area under the graph of a function and transformations of the graph. Once again we thought it important to try and move students away from the processes which we thought they would want to use, and so we gave them general integrals in terms of a function f(x) rather than an explicit function which they could integrate directly. The following two questions were given in this area:

If $\int_1^3 f(t)dt = 8.6$, then write down the value of $\int_2^4 f(t-1)dt$ and

If $\int_1^5 f(x)dx = 10$, then write down the value of $\int_1^5 (f(x)+2)dx$.

Students were asked to show their working. Responses to these two questions are summarised in table 2.

Table 2: Summary of responses for transformation questions (N=47)

	Females (N=31)			Males (N=16)		
Transformation Direction	No Response	Correct	Wrong Answer	No Response	Correct	Wrong Answer
Parallel to the t-axis	21	3	7	7	5	4
Parallel to the y-axis	13	2	16	6	4	6

Whilst 17.0% of the students were able to answer the first of these questions correctly, 59.6% were unable to make any response. This could be because they did not recognise the f(t-1) as representing a transformation and there was evidence that this concept was lacking. For example, one student gave 7.6 as the answer, thinking that f(t - 1) implied the value of the integral minus 1, and 5 students (10.6%) attempted to integrate the t - 1 and obtained $t^2 - t$ or $\frac{(t-1)^2}{2}$ leading to a wrong numeric answer, such as 4. Another 2 students (4.3%) who, like those above, were locked into a process-oriented view of the integrals rather than a conceptual one, found another way of introducing a function they could work with. Their idea went as follows:

Let f(t) = ax + b (interestingly, but not surprisingly, a function in x not t)

then $\int_1^3 f(t)dt = 3a + b - a - b = 8.6$ so $2a = 8.6$ and $a = 4.3$.

Thus $\int_2^4 f(t-1)dt = 4a + b - 1 - 2a - b - 1 = 2a - 2 = 8.6 - 2 = 6.6$

The errors in this last line, including miscalculating f(t-1), apart, this demonstrates the lengths students will go to to introduce a known algorithm into a question where they lack the conceptual understanding necessary. In the second of these questions, although 12.8% answered correctly, their preferred method was still to perform an integration, rather than visualise the transformation as creating an extra rectangle of area 4x2 = 8. They usually wrote a version of:

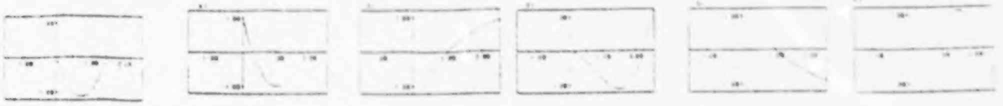
$$\int_1^5 (f(x)+2)dx = \int_1^5 f(x)dx + \int_1^5 2dx = 10 + [2x]_1^5 = 10 + 10 - 2 = 18.$$

There were no significant differences on the basis of gender in the results.

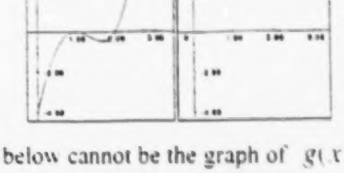
Sketching the Integral Function

We gave the students several questions which tested their understanding of concepts associated with drawing graphs of integral functions from a consideration of area. These were:

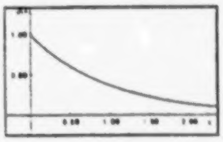
The graph of a function f is shown. Which graph can represent an antiderivative of f? Give your reasons.



The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that F(0) = 0.



Explain why the graph below cannot be the graph of $g(x) = \int_0^x f(t)dt$ for any $f(t)$.



For the first of the questions, 23.4% were successful with several giving data which showed that they had considered a process approach rather than looking at the sign and magnitude of the area under the curve. The results are summarised in table 3. One of the successful students wrote down the function as $x(x - 1.5)$ in order, no doubt, to integrate it. Typical of the thinking used were other successful students who wrote "it must be a positive cubic"; "the antiderivative of f must be a cubic"; and "the antiderivative of f is a ~ shape", again working from a quadratic function with a positive x^2 term and integrating.

Table 3: Summary of question responses for the graph of the antiderivative (N=47)

	No Response	a	b	c	d	e
Number of student responses	10	2	12	11	5	5

In addition 2 students gave other answers

The second question proved rather more testing and, as table 4 shows, only 2 students (4.2%) answered correctly.

Table 4: Summary of responses for the antiderivative sketch question (N=47)

Females (N=31)			Males (N=16)		
No Response	Correct	Wrong Sketch	No Response	Correct	Wrong Sketch
18	2	11	4	0	12

It was again noticeable that a number of the students felt that the only way to answer this question was to find a function which matched the graph, integrate it with respect to x and then sketch that. The concepts of the calculus associating area with integration appear to be little known or applied. Three students, all unsuccessful, wrote down the function $(x - 1)^2(x - 2)$ (one differentiating it) and another wrote $(y + 4)(x - 1)(x - 2)$, confirming that a sizeable number of students rely on the process of integration rather than any conceptual understanding they have built up.

In the third question, 8.5% of the students arrived at the correct solution, but 74.5% were unable to make any attempt at an answer. It may be that they are not used to seeing the concept of the area function written in the form given. We recognise that a written solution gives insufficient insight for this type of investigation and will be interviewing students in the future to ascertain exactly what their difficulties with these types of questions are.

The process-oriented response here was "the function should be with respect to x . . . the graph should be parabola" compared with one student who wrote, very concisely: "$\int_0^0 f(t)\,dt = 0$ not 1", thus demonstrating the power of conceptual thinking.

Conclusions

The evidence we have presented shows that students often lack certain conceptual understanding from current mathematics learning. The high numbers of students who were unable to make any attempt at solving these problems where there is no obvious process to carry out shows that their experiences to date have left important gaps in their conceptual understanding. They have a tendency to see integral calculus as a series of processes with associated algorithms and do not develop the grasp of concepts which would give them the necessary versatility of thought. Thus, instead of having a proceptual view of the symbols in integration they have only a process-oriented view. However, it is not surprising that many students find concepts such as limits and area functions difficult when they are unable to experience these processes directly in many classrooms. It should be possible to design curriculum materials which give an improved cognitive base for a flexible proceptual understanding of limit and other concepts. Thus it may be possible for the student to develop a more balanced view of, say, limits and areas, dually as process and concept by using a computer software. Although the fundamental theorem of calculus doesn't require taking a limit, symbolic manipulation, for example, can allow students to see what is done at each stage, giving them understanding of the concepts as well as the ability to calculate integrals. We hope that the appropriate use of such computer software will improve the students' conceptual understanding of calculus.

One advantage of this method should be that it will engage the student in appropriate activities which require them to think through the process involved. With respect to the graphical aspects of calculus, spreadsheets and symbolic manipulators can be used to develop an understanding of the fundamental concept of calculus, avoiding a passive classroom environment. One of the principal aims of the mathematics educator should be to provide a range of experiences that develop the mathematical ideas in a cognitive manner so that the learner both knows and understands. The computer may focus students on the concepts and ideas of calculus rather than routine computations in a way which may help them to gain understanding which our survey has shown to be lacking.

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Teaching Children to Draw Diagrams in Solving Word Problems: An Exploratory Study.

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This paper discusses the results of diagram drawing instruction on pupils' ability to understand and solve word problems. The results show that after instruction, diagrams drawn were more relational and resulted in better performance. There was a conscious attempt on the part of the pupils to illustrate the relationship among variables in their diagrams. The pupils also demonstrated positive attitude towards this approach of solving problems.

Introduction

In recent years, there has been a shift in Brunei Darussalam towards a greater emphasis on mathematical applications and problem solving in school mathematics. The new lower secondary mathematics syllabus issued by the Curriculum Development Department (CDD,1995) in Brunei clearly states that *"teachers should help students develop problem-solving skills besides striking a balance between concept development and mastery of basic skills."* (p.2). Translating this into practice involves the use of word problems as a means of engaging students in practical and realistic application of mathematical concepts and skills learned in class.

Analysis of solution strategies used by lower secondary school children in national mathematics examination in Brunei Darussalam(Veloo, 1995) reveals that, pupils solve word problems by first attempting to formulate algebraic expressions using information contained in the word problem. This is the most difficult step and a large proportion of pupils had little success at this stage. In another study on primary school children, it was noted that less than 1% of the pupils drew any kind of diagram in solving word problems (Lopez-Real, Veloo & Maawiah, 1992).

Many researchers have reported that an important facilitator in the process of solving a word problem is the translation of information and conditions given in the problem statement into a visual diagrammatic form. According to Larson (1985) a first step in understanding a problem is to draw and label a figure, diagram or a graph. Children have been found to perform better in mathematical problem solving situations when diagrams are used by teachers to elicit appropriate mental images (Yancey,V. et al,1989). Norman Webb (1979) reported that pictorial representation was one of the component scores that accounted for most variance in problem-solving scores.

Although diagrams have been recognised as useful in problem solving, not much has been reported on how to teach children to generate their own diagrams. This study is a follow up of two earlier studies (Lopez & Veloo, 1993; Veloo & Lopez,1994) conducted in Brunei Darussalam. In these earlier studies, it was found that (a) primary school pupils performed better on word problems after being asked to draw a diagram before solving the given problems (b) primary school pupils tended to draw diagrams that were concrete representations of the problem, whereas secondary school children were more inclined to draw diagrams that were abstract, (c) pupils generally have not been taught how to draw and use diagrams in solving word problems and (d) the diagram drawing strategy once understood was readily accepted by pupils. In these earlier studies no instruction on how to draw diagram was given.

This exploratory study examines the effectiveness of diagram drawing instruction on pupils' understanding and successful solution of mathematical word problems.

Methodology

Sample

The research was conducted in a secondary school in Brunei Darussalam. Fifty students from two secondary 4 classes, comprising 28 girls and 22 boys(average age of about 15 years) were involved in the study. The study consisted of three phases, a pre-test, an instructional phase and a post-test.

Instrumentation

A pre-test and a post-test consisting of word problems and open ended questionnaire were prepared for administration in this study. The word problems used in the pretest and posttest were obtained from a survey of school textbooks, past year examination papers and mathematics education literature. The open ended questionnaire consisted of 5 questions on diagram drawing and solving word problems was prepared and administered as part of the posttest.

The pre-test

Ten problems that were deemed suitable were selected for the study were selected for the pre-test. The test booklets were divided into two sets, Test A and Test B. The questions in both Test A and B were identical but differed only in the instruction given to the candidates. Pupils attempting Test A were asked to read the question carefully and then solve the problem showing all working in the space provided. Test B candidates were asked to read the question carefully, draw a diagram that illustrates the information given in the problem and then solve the problem showing all working. Pupils in both classes were randomly assigned either Test A or B. All students were given as much time as they require to complete test. All question booklets were collected after the test.

The post-test

The post test consisted of 5 questions, selected from the pre-test, whose facility levels were below 45%. The test was given to all pupils to determine the effect of diagram drawing instruction on pupils ability to understand and use diagrams to solve the word problems. The open ended questionnaire was administered as part of the posttest.

Procedure

A week following the pre-test, a two-week instruction on visualization and techniques of illustrating word problems pictorially was given to all pupils involved in the study by the researcher. Topics covered in the instruction included, *Fractions, Percentage, Ratio, Profit and Loss* and *Simple Algebra.* These topics were selected because problems involved can be easily represented by diagrams. Students were initially introduced to the *part-whole model,* the *comparison model* (Kho T. H.,1987) and *ratio model* of representing word problems pictorially. These models were further refined as word problems were discussed during instruction and problems solving exercises.

Results

Table 1 shows pupils' performance on the pre and post-tests on five questions whose facility indices fell below 45%. One mark was awarded for correct response and zero mark was awarded for incorrect response.

Table 1: Pupils' performance on pre and post-test Comparison of pupils' performance and use of diagrams in the pre and post tests (n=41)

Question	**Q3**	**Q4**	**Q5**	**Q6**	**Q10**	**mean**	**SD**	**t**
Pre-test Facility(%)	12.2	28.5	42.8	12.2	22.4	1.27	1.43	1.325
Diagram(%)	50	4.1	8.2	6.1	8.2			
Post-test Facility(%)	21.1	65.8	86.8	13.2	36.9	2.35	1.29	
Diagram(%)	92.1	78.9	92.1	84.2	73.6			

Table 2: Comparison of pupils' performance on 3 questions in the pre and post tests. (n=41)

Question	**Q4**	**Q5**	**Q10**	**mean**	**SD**	**t**
Pre-test Facility(%)	28.5	42.8	22.4	1.05	1.06	2.125*
Post-test Facility(%)	65.8	86.8	36.9	1.90	0.87	

(* significant at $p < 0.05$)

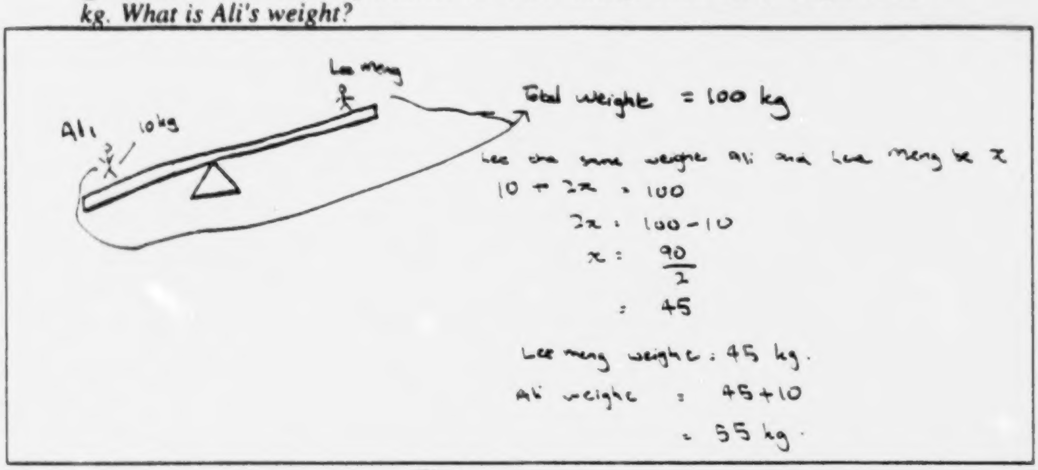
On all 5 questions that were re-tested in the post-test, pupils' performance was better. On three questions (4, 5 and 10) their performance was statistically significant ($p<0.05$), see table 2. A marked improvement in the quality of diagrams was observed. Diagrams drawn were more *relational* that is pupils tried to illustrate the relationship among variables in their diagrams.

Further analyses revealed that most of the diagrams drawn by pupils in the pre-test which the writer considered helpful were observed for questions 3, 8 and 9.

> ***Question 3:*** *A motorist traveled from town A to B at an average speed of 90 km/h. After traveling one-third of the journey in 45 minutes, he continued to travel another 120 km to reach town B. Find his average speed for the second part of the journey.*
>
> ***Question 8:*** *A 48 metre wire is cut into three pieces. The second piece is three times as long as the first. The third piece is four times as long as the second. How long is each piece?*
>
> ***Question 9:*** *A long pole is planted through some water into the mud at the bottom of a lake. Half of the pole is in the mud, one third is in the water and the part above water is 11\2 metres long. How long is the pole?*

A possible reason why more diagrams were observed for these three questions and not the others in the pre-test is that, these questions could be illustrated pictorially without much difficulty. For the other questions in the test, pupils either did not draw diagrams or attempted to draw *literal* representations of information given in the problem. An example of a *literal* diagrams is shown below.

***Question 1:* Ali is 10 kg heavier than Lee Meng. Their total weight is 100 kg. What is Ali's weight?**



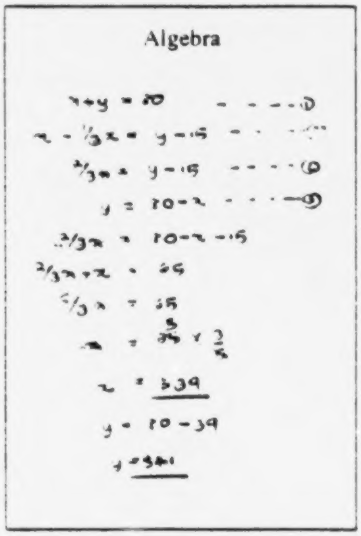
Let the same weight Ali and Lee Meng be x
10 + 2x = 100
2x = 100 − 10
x = 90/2
= 45
Lee meng weight = 45 kg.
Ali weight = 45 + 10
= 55 kg.

Fig 1

The following are examples of diagrams observed for questions 4, 5, 6 and 10 in the posttest.

> ***Question 4:*** *Two brothers A and B together had a total of $80. If A spends one-third of his money and B spends $15, they would have the same amount of money left. How much did each have at the start?*

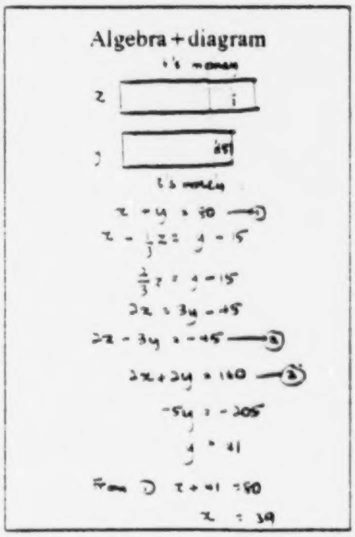
Algebra



x+y = 80 ----(1)
x − 1/3x = y − 15 ----
2/3x = y − 15 ----(2)
y = 80 − x ----(3)
2/3x = 80 − x − 15
2/3x + x = 65
5/3x = 65
x = 65 × 3/5
x = $39
y = 80 − 39
y = $41

Fig 2(a)

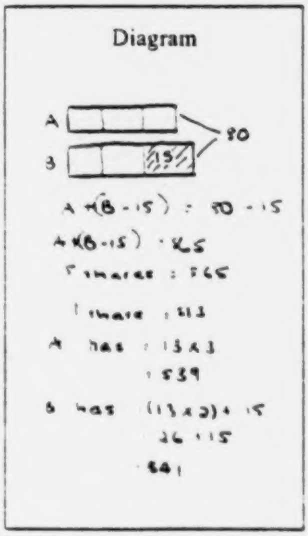
Algebra + diagram



x + y = 80 → (1)
x − 1/3 x = y − 15
2/3 x = y − 15
2x = 3y − 45
2x − 3y = −45 → (2)
2x + 2y = 160 → (3)
−5y = −205
y = 41
From (1) x + 41 = 80
x = 39

Fig 2(b)

Diagram



A + (B − 15) = 80 − 15
A + (B − 15) = 65
5 shares = $65
1 share = $13
A has = 13 × 3
= $39
B has (13 × 2) + 15
26 + 15
= $41

Fig 2(c)

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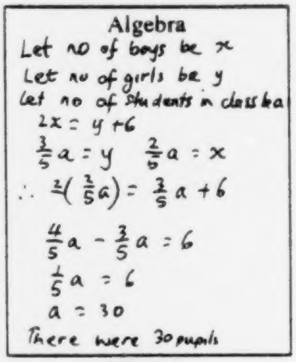
Question 5: In a class three-fifth of the pupils were girls and the rest were boys. If the number of boys were doubled, and 6 more girls joined the class, there will be equal number of boys and girls. How many pupils were in the class at the start?



Algebra

Let no of boys be x

Let no of girls be y

Let no of students in class be a

$2x = y + 6$

$\frac{3}{5}a = y \quad \frac{2}{5}a = x$

$\therefore 2\left(\frac{2}{5}a\right) = \frac{3}{5}a + 6$

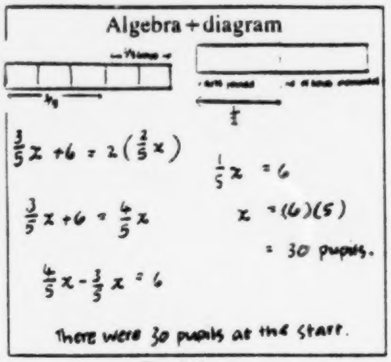
$\frac{4}{5}a - \frac{3}{5}a = 6$

$\frac{1}{5}a = 6$

$a = 30$

There were 30 pupils

Fig 3(a)



Algebra + diagram

$\frac{3}{5}x + 6 = 2\left(\frac{2}{5}x\right)$

$\frac{3}{5}x + 6 = \frac{4}{5}x$

$\frac{4}{5}x - \frac{3}{5}x = 6$

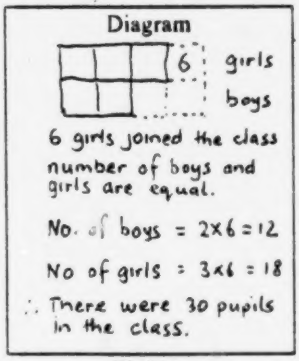
$\frac{1}{5}x = 6$

$x = (6)(5)$

$= 30$ pupils.

There were 30 pupils at the start.

Fig 3(b)



Diagram

6 girls joined the class

number of boys and girls are equal.

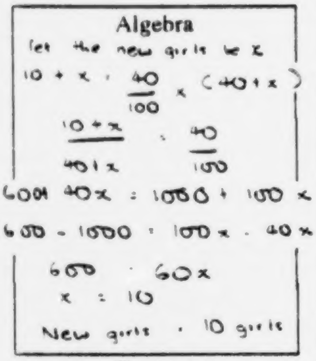
No. of boys = 2×6 = 12

No of girls = 3×6 = 18

∴ There were 30 pupils in the class.

Fig3(c)

Question 6: In a class of 40 pupils, 25% were girls. When some new girls joined the class, the percentage of girls increased to 40%. How many new girls joined the class?



Algebra

let the new girls be x

$10 + x = \frac{40}{100} \times (40 + x)$

$\frac{10 + x}{40 + x} = \frac{40}{100}$

$1000 + 100x = 1600 + 40x$

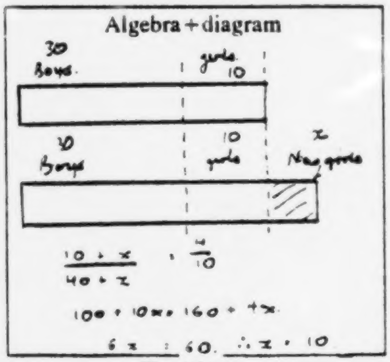
$1600 - 1000 = 100x - 40x$

$600 = 60x$

$x = 10$

New girls = 10 girls

Fig 4(a)



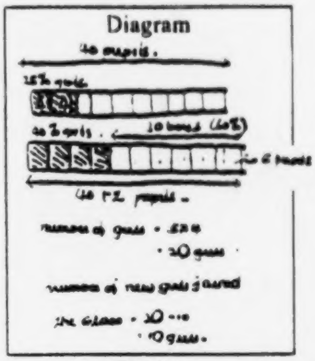
Algebra + diagram

$\frac{10 + x}{40 + x} = \frac{4}{10}$

$100 + 10x = 160 + 4x$

$6x = 60. \therefore x = 10$

Fig 4(b)



Diagram

number of girls = [illegible]

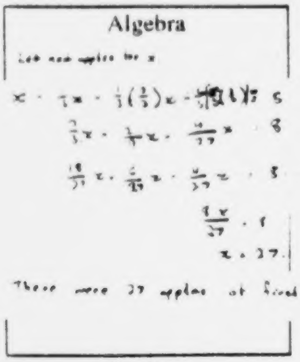
= 20 girls

number of new girls joined the class = 20 − 10

= 10 girls.

Fig4(c)

Question 10: Three tired and hungry men went to sleep with a bag of apples. One man awoke, ate one-third of the apples, and then went back to sleep. Later a second man awoke, and ate one-third of the remaining apples and he went back to sleep. Finally the third man awoke and ate one-third of the remaining apples. When he finished there were 8 apples left in the bag. How many apples were in the bag at the start?



Algebra

Let no. apples be x

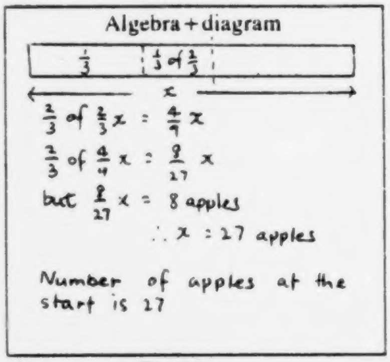
[illegible]

$\frac{8x}{27} = 8$

$x = 27.$

There were 27 apples at first

Fig 5(a)



Algebra + diagram

$\frac{2}{3}$ of $\frac{2}{3}x = \frac{4}{9}x$

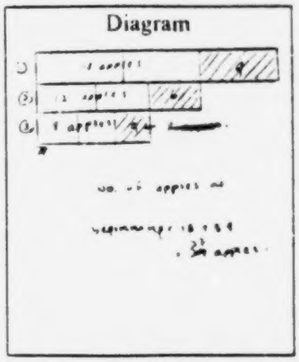
$\frac{2}{3}$ of $\frac{4}{9}x = \frac{8}{27}x$

but $\frac{8}{27}x = 8$ apples

$\therefore x = 27$ apples

Number of apples at the start is 27

Fig 5(b)



Diagram

no. of apples at beginning is [illegible]

= 27 apples.

Fig5(c)

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Discussion

Effect of instruction

The post-test result shows that pupils' performance on all questions retested improved and in three questions, Q4, Q5, Q10, their performance showed significant improvement. For these three problems, solutions could be obtained from the diagram itself once an accurate relational diagram is drawn. This is further discussed under the section on solution strategies.

Pretest results and discussion with pupils immediately after the pretest, revealed that pupils prefer and feel more comfortable with the algebraic approach to solving word problems. There was resistance on the part of the pupils in using diagrams as an initial step in solving problems. This was particularly so among the more able pupils who said in response to a question in the questionnaire, that drawing diagrams tend to *slow them down*. They would *draw diagrams only for the more difficult questions*. However, during and after the instructional phase, most pupils showed a general willingness to apply this strategy in solving word problems. Pupils learned the technique readily and some drew relational diagrams from which the solution to the problem could be easily obtained. Examples of such diagrams are shown under the sub-heading *diagram strategy* for Q4, Q5, Q6 and Q10. There is also evidence from the study to show that an application of diagram drawing approach results in better performance particularly among pupils whose conceptual understanding in weak.

Pupils' general attitude towards this approach can be described as positive. Students initial reluctance to drawing diagrams was mainly due to the fact that they were unfamiliar with this technique. The following is a summary of pupils' responses to a question posed in the questionnaire.

Q. Does drawing diagrams help you in solving mathematical word problems?(Yes/ No) Give reasons for your response.
"Diagrams helps me to understand the question better."
"The question turns out to be easier and I can check the answer by looking at the diagram."
"The diagram makes the problem clearer and I can understand the question better."

Solution Strategies

A variety of strategies was used by pupils in solving the word problems. These can be classified broadly into three categories. The first is the *Algebra strategy*. The second is the *Diagram + Algebra strategy*, where pupils drew diagram as well as presented an algebraic solution to the problem. The third cagegory is the *Diagram strategy*, in which pupils presented a solution to the problem from an analysis of the diagram drawn.

Algebra strategy

Despite the two week instruction on diagram drawing why do some pupils still prefer the *Algebra* approach? It is the method of solution taught in schools and emphasised by the teachers. It is, according to some teachers, the method preferred by the examination board. Pretest results and discussion with pupils immediately after the pretest, revealed that pupils preferred and felt more comfortable with the algebraic approach. The following pupils' responses provides further evidence of what takes place in the mathematics classroom.

Q. Do you normally draw diagrams when solving mathematical word problems? (Yes/No) Give reasons for your response.

A. *"No, because we do not normally use this method in the class. We are more familiar with, writing equations to solve problems."*
"I usually go ahead and calculate without drawing, It is much faster."
"I never felt the need to draw diagrams as it wasn't the practice."

Another reason why the algebra strategy is popular is that it involves the formulation of suitable algebraic expression using information contained in the word problem. There are no fixed rules that a pupil can apply to translate a word problem into algebraic expressions. Analysis of pupils' answer scripts in public examinations showed that this was the stage, where most students fail in the problem solving process(Veloo, 1995). Similar findings were reported by Newman in her study of errors made by sixth grade pupils on written mathematical tasks.(Newman,1977). After this translation stage, not much thinking is involved in the next stage of the problem solving process which involves the application of routine procedures to solve for the unknown. The apparent simplicity of the algebra strategy may be the reason why it is popular and it provides success only to those who are skilled in the techniques of algebra.

Diagram + Algebra strategy

This was the most popular strategy. Diagrams appear to serve as an intermediate step between students concrete understanding of a problem and abstract algebraic model of the problem. The diagrams drawn by pupils in the post-test for questions, 4, 5, 6 and 10 clearly indicates increased understanding of the relationship among variables in the problem. A substantial number of pupils used this strategy to solve the word problems. Once a successful *relational* diagram is drawn, it appears that the next step of translating the problem into a suitable algebraic model becomes easier.

Diagram strategy

This approach was used by some pupils. Problems were solved entirely from the diagram. To successfully use this strategy the pupil must first understand the problem well. Second a relational diagram containing the important information in the problem must be drawn. The diagram helps in clarifying relationships between variables. When students use a diagram to illustrate the information given in a problem, they use a representational system that they are familiar with. The very act of drawing a diagram causes students to focus on relationships among variables in a problem. This stragety was most commonly used by students to answer Questions 4, 5 and 10 although not so common for Q6. While this method may have limited application it is one which requires thinking and analysis of the information contained in the word problem. It challenges one to think and consider the problem as a whole rather than in parts. It can be argued that if pupils can solve word problems using this strategy, then they can easily be helped to see the connection between this form of representation and algebraic representation.

Unsuccessful attempts

Despite drawing diagrams many pupils were still unsuccessful particularly in questions 3 and 6. Analysis of diagrams together with written solution offered by these students clearly showed the flaw in their reasoning. For question 3 pupils have not understood the concept of average velocity. For question 6 the most common incorrect response was 6 was obtained by calculating the total number of girls in the class *after* the increase to be 40% of 40 which is 16. Hence the number of new girls was 6. This incorrect interpretation of the question is clearly shown in the diagrams drawn by these pupils. This information will be of great benefit to the mathematics teacher, who will be in a better position to help these pupils correct their mistakes.

Conclusion

Drawing diagrams can be an important strategy in problem solving but one that is not fully exploited in the classroom. The desire to push children too quickly into abstract manipulation and algorithms may be one reason for this. It is also unfortunate that not many teachers draw diagram as part of the process of solving word problems in the mathematics classroom. Teachers must make a conscious attempt to learn this technique and must challenge themselves and their pupils to solve problems using diagrams. Once pupils develop competence in using diagrams to solve problems, they need to experience the connection between this form of representation and algebraic representation. Burns(1979) describes this as an often overlooked intermediate step. This study has shown that diagram drawing techniques can be taught to children in school. Further work needs to be done to assess long term effect of diagram drawing instruction on pupils ability to solve word problems particulary those involving algebra.

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Fairness and Fractions in Early Childhood

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Responses of 24 children from pre-Grade 1 to Grade 4 were used to illustrate the range of understanding of the partition of continuous fractions in the early years of schooling. The SOLO developmental model with multimodal functioning was used to classify responses and it is hypothesised in this paper that ikonic and early concrete symbolic functioning are exhibited in these grades as fairness becomes a mathematical criterion for work with fractional parts. Implications are drawn for further research and for the classroom.

While there has been a move to diminish the importance of operations with fractions in the mathematics curriculum (Australian Education Council [AEC], 1994), there is still an acknowledged need to build an understanding of what the common fractions are and how they can act as operators on sets or wholes (e.g., AEC, 1994, pp. 40, 56). Often associated with the division of a whole into parts is the idea of fairness. Adults readily accept the nature of a partition being fair if the parts are of an equal size. Discussions of fairness in relation to fractions with young children, however, may not be based on the same shared understanding of the term fair.

While research related to young children's understanding of fractions continues to be detailed (e.g., Saenz-Ludlow, 1994, 1995) and the links are made for teachers with literature at this level (e.g., Conaway & Midkiff, 1994), very little explicit acknowledgment is made of the contribution that the intuitive idea of fairness makes to initial efforts to construct partition of wholes. When mentioned (e.g., Peck & Connell, 1991), it is often as an afterthought since the assumption that children understand what teachers mean seems nearly universal. A notable exception is the work of Streefland (1991) who acknowledges the existence of alternative frameworks by telling a story of a mother who "divided one or two apples among eight children and only the grown-ups had the right to a whole apple" (p. 63). He then develops in his class of fraction researchers a climate where they construct a common shared understanding of fairness in order to solve fraction problems.

The data reported in the present study were collected as part of a larger study of fraction and decimal understanding of students from pre-Grade 1 to Grade 10 (Watson, Collis & Campbell, 1995). The problem chosen for analysis here was used in interview settings with concrete materials. The objective was to document student understanding of the sharing process involving three equal parts in a context which involved continuous partitioning of wholes.

Theoretical model

The model used to analyse the data was the SOLO model with multimodal functioning (Biggs & Collis, 1982; 1989; Collis & Biggs, 1991). This model has been successfully applied to the analysis of student responses to tasks associated with volume measurement (Campbell, Watson & Collis, 1992) early multiplication word problems (Watson & Mulligan, 1990), and the search for associations in a data set (Watson, Collis, Callingham & Moritz, in press), as well as fractions and decimals (Watson, et al., 1995). The model, which grew out of the Piagetian tradition, postulates five modes of functioning which originate in a fixed order but which continue to develop alongside each other throughout life: sensorimotor (from birth), ikonic (from early childhood), concrete symbolic (from the years of schooling), formal (for those capable of higher education), and post formal (associated with research). Of particular interest in this study are the ikonic and concrete symbolic modes, these being associated respectively with intuitive functioning and with the symbolic learning which takes place in school based on concrete materials, and their influence on thinking about fractions. As well, within each mode, there are developmental sequences which are hierarchical and are observed in cycles of response. Each cycle contains:

(i) Unistructural responses (U) where individual skills or constructs can be used separately only;
(ii) Multistructural responses (M) in which several such skills or constructs can be used, usually in sequence; and
(iii) Relational responses (R) in which there is a coordinated mastery of the set of component skills or constructs.

The research of Watson, et al. (1995) observed two UMR cycles in the concrete symbolic mode in relation to fractions, the first building the basic concept and the second applying it in problem solving situations. The work of Watson and Mulligan (1990) observed a UMR cycle in the ikonic mode when young children were first introduced to a basic word problem involving the concept of multiplication, leading to one in concrete symbolic mode for those who were taking on the concrete symbolic idea of repeated addition. A further study by Watson, Campbell and Collis (1993) found evidence for functioning in both ikonic and concrete symbolic modes, in this case with multimodal functioning occurring with "bottom up" support from the ikonic mode to assist problem solving in the concrete symbolic mode. The current study adds to the picture in the development of understanding of fractions by hypothesising an ikonic UMR cycle as well as ikonic support for functioning in the early part of the concrete symbolic mode.

Method

Three children from each of pre-Grade 1 and Grade 1 and six children from each of Grades 2 to Grade 4 were interviewed for approximately 45 minutes each. Concrete materials were provided for the children to use in solving the problem and were chosen to access the children's out-of school experiences. For the Pancake Problem described here, children were given a pancake and asked to cut it so that it could be shared fairly among three dolls. Actual dolls and pancakes were used rather than school-based materials such as blocks and diagrams. At each pause the child was asked if the sharing were fair. The interviews were tape recorded and notes taken by a trained research assistant. Responses were then summarised and similar responses grouped together. The three authors then independently classified the responses in the light of theoretical SOLO model with multimodal functioning. The order of rankings were very similar and discussion in relation to the model resolved the minor discrepancies. The names of all students were changed for this report.

Results

The individual responses will be discussed separately and then an overall model postulated for the development of fractional understanding by young children. In the Pancake Problem there are two issues associated with fair fractional sharing. One is the number of pieces given to each doll and the other is the size of the pieces. The approaches to solving these dilemmas present the bench marks of development of understanding. The figures are stylised computer simulations of the students' actions.

The Ikonic Mode

To be considered as responses in the ikonic mode responses had to acknowledge an appreciation of the splitting task associated with sharing. The conservation of number (typical of the concrete symbolic mode) necessary to distribute parts to the dolls as required by the task, however, was not present.

At the unistructural level of the ikonic mode (IK-U) the response of Josh indicated a single splitting action, cutting the pancake in two parts. These were given to two of the three dolls. The concept present is that of sharing but it cannot be carried out to satisfy the constraints of the problem.

At the multistructural level (IK-M) Jan's response indicated a more complex attempt. She said "three" but in making three cuts of the pancake produced four pieces , did not know what to do with them and hesitated in confusion. In this case a more complex splitting was attempted but the dilemma of sharing could not be resolved.

At the relational level (IK-R) Yvonne also cut the pancake into four roughly equal pieces. She resolved the sharing by giving two pieces to one doll and one to each of the others. At this level, the necessity to split the pancake and share it among all the dolls was understood to the extent that all dolls did receive some pancake. The conservation of number of pieces or quantity required for fair sharing, however was not present.

All of these students were subsequently asked if what they had done was fair. All thought for a while, replied "no" and made another attempt described below. The prompting to consider fairness and the presence of an older "teacher" model, may illustrate the relevance of Vygotsky's suggestion of a zone of proximal development and Lamborn and Fisher's (1988) research supporting the benefit of prompting to achieve optimal rather than functional responses.

The Concrete Symbolic Mode

To be classified in the concrete symbolic mode responses had to exhibit a sharing process which at least demonstrated an appreciation of the need for conservation in terms of the number of pieces given to each doll. As the Pancake Problem was a continuous rather than discrete sharing problem, number of pieces alone was not a sufficient criterion for fair sharing without a consideration of the size of the pieces. All subsequent responses except one were considered to be in the first cycle of the concrete mode indicating the building of the fraction concept (cf. Watson, et al., 1995).

Unistructural level: Six students produced responses considered at the unistructural level (CS-U_1). These exhibited a sharing based on conservation of number, not size, and if there were leftover bits from the distribution this did not create a conflict for the student. Alan's distribution shown in Figure 1 began with "quarters", of which he distributed one to each doll. He did not know what to do with the leftovers but having carried out a single sharing operation, was happy he had fairly distributed the pancake.

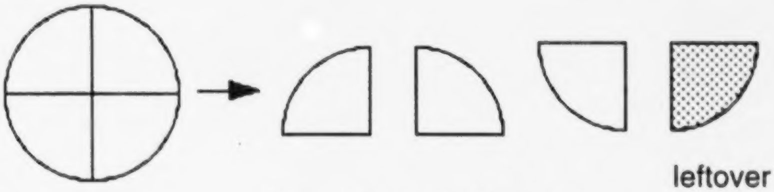

Figure 1. Alan's sharing.

Barb, Karen, Jan (second attempt) and Yvonne (second attempt) each cut the pancake differently but in such a way that each doll received two pieces of unequal size. Karen and Jan each had leftover bits as well, with Jan hiding her spare bit under the table. Jan's action was likely ikonic in nature so that the dolls could not see the part they did not receive. There was no indication that the piece was being hidden from the researcher or that Jan realised a conflict in relation to the distribution of the whole. Each claimed that the sharing was fair because the dolls had two pieces of pancake each. Barb's sharing is shown in Figure 2.

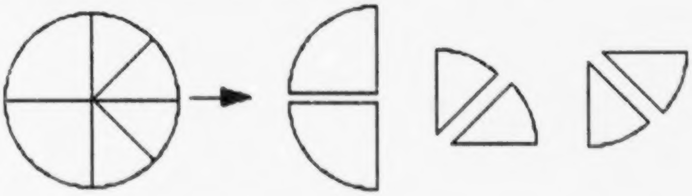

Figure 2. Barb's sharing.

Carla cut three similar pieces from the pancake, then cut the remainder into three very unequal sizes. In distributing the pieces she gave two to each with the largest piece to the doll in the middle, saying "she can decide if she wants to share it." This indicates ikonic support, in including reference to the dolls in the response. Although it may have been the beginning of an acknowledgment of the need for equal sized portions, it was not resolved in the concrete symbolic mode but by reverting to an intuitive ikonic solution.

Multistructural level: The benchmark of responses at the multistructural level (CS-M_1) was the realisation that having the same number of pieces is not enough to constitute fair sharing and that left overs are not possible when wholes are shared fairly. At this level, however, ad hoc methods were used, usually in sequence, to determine fair shares. Kara for example cut the pancake roughly in half and then cut the larger half again. When these pieces were not to her satisfaction she continued making adjustments roughly as shown in Figure 3. Josh's second attempt following his cut into halves was similar. In these cases dolls did not always have the same number of pieces.

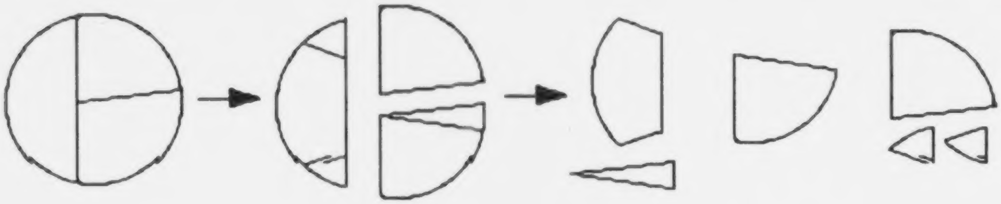

Figure 3. Kara's sharing.

Ned began with cuts similar to "eighths", giving two to each doll. When he realised he had two pieces leftover he retrieved some and adjusted size and sharing until he was satisfied each had the same amount. Eight students began with two roughly equally spaced parallel cuts across the pancake. When questioned on the fairness of the three shares, seven said yes, they were fair because they were the same width. Dan, however, whose cutting was not quite as accurate as the others, cut bits of the corners off his outside pieces to make the shares more fair. His method is shown in Figure 4. At this level the non-integrated nature of adjustments is observed.

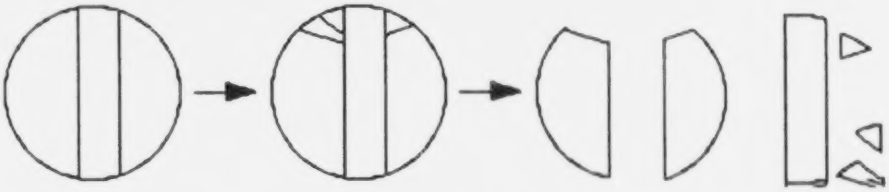

Figure 4. Dan's sharing.

Relational level: At the relational level (CS-R_1), responses displayed an integrated strategy, showing an overview of the problem and an insight into the measurement principles involved and hence beginning to incorporate geometric principles in the consideration of what a fair share should be. Some of these were relatively complex because the task was related to thirds. All solutions however, resolved the necessity to provide both an equal number of pieces and pieces of the same size, totalling the same amount to each person. Carol, Ron and Mark first divided the pancake into quarters (half twice) and then divided one quarter into equal thirds. Each doll was then given one large and one small piece, as shown in Figure 5.

George started in the same manner as the previous three, but then cut the final quarter into six equal pieces, resulting in each doll receiving one large and two small pieces. Saul again began with quarters, splitting three of these into halves and the fourth into thirds. This produced nine pieces to be distributed, with two relatively larger and one smaller, to each doll. These solutions indicate the repetitive use of geometric division into parts (although the terminology was not used) to solve the problem, typical of those who can relate the sharing and the need for equal parts using a relatively complex method. Tara proceeded in two stages but avoided quarters by using thirds at the second stage. Her solution is shown in Figure 6.

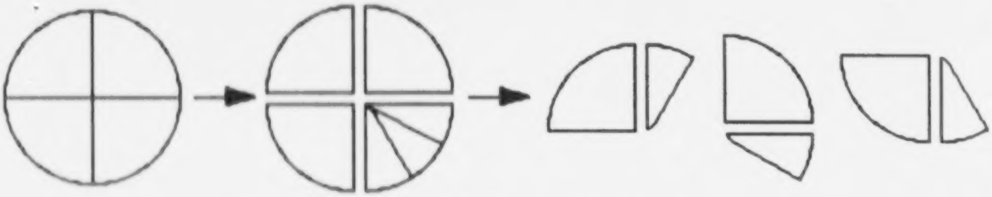

Figure 5. Carol's, Ron's and Mark's sharings.

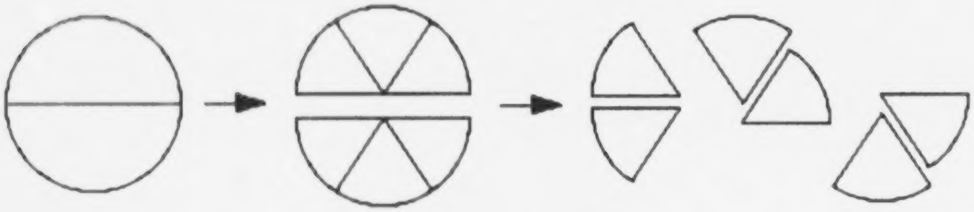

Figure 6. Tara's sharing.

Finally at the relational level, Kay and May (second attempt) realised that parallel cuts of equal width would not produce equal portions and discussed the necessity to make the middle section thinner to compensate. Their solutions are shown in Figure 7. Although it was not possible for these students to measure accurately, the responses reflected a higher level of understanding and relating of geometric principles to the problem than did the other responses involving parallel cuts.

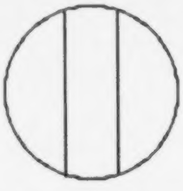

Figure 7. Kay's and May's sharings to compensate for circular shape.

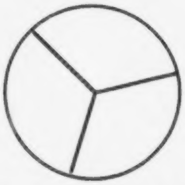

Figure 8. Josh's final sharing.

Unistructural level, second cycle: There was one response which was judged to have reached the unistructural level of the second cycle of the concrete symbolic mode (CS-U_2). Although only allowed one pancake to solve this problem, Josh was subsequently given another pancake for a different problem. This he immediately cut into thirds as shown in Figure 8 to show what he wanted to do now for the earlier sharing problem. This solution is typical of all students above Grade 4 who were interviewed as part of the larger study and indicates again the possibility for optimal performance and learning during a one-to-one interview. It appears that the more complex relationships needed to solve the problem in the first cycle of the concrete symbolic mode are replaced by a single, new, more sophisticated concept.

Summary of results

Table 1 summarises the responses of the 24 children in this study by SOLO level and by grade level. The increased sophistication by grade level is evident, but it is also evident there is potential for a great variety in responses, particularly in Grades 1 and 2.

Table 1.
Summary of responses by SOLO level and by grade level

	Pre-Grade 1	Grade 1	Grade 2	Grade 3	Grade 4
IK-U single idea - sharing			Josh - half → 2		
IK-M divides into parts inappropriately	Jan - 4 pieces, 3 dolls confusion				
IK-R sharing to all but not conservation		Yvonne - 4 bits, shares 2, 1, 1			
CS-U$_1$ sharing based on conservation of number not size - left over bits not a conflict	Barb - 2 each, unequal Jan - after repeated cuts, 2 each hide 1 unequal [+IK] Alan - 1 each 1 leftover	Yvonne - [after 'fair' prompt] 2 each, unequal	Carla - 3+3 pieces, 2 each, bigger for middle doll [+IK] Karen - 2 unequal pieces template for 3 but leftovers		
CS-M$_1$ ad hoc adjustment for equal size -uses appearance		Ned - near $\frac{1}{8}$ths then adjusts	Josh - half of half, smaller and smaller Cam - judges equal width (parallel) Dan - judges parallel then compensates	Kara - half, half then adjust; Alf, Marg - judge equal width (parallel)	Carl, Judy, Owen, May - judge equal width (parallel)
CS-R$_1$ geometric 'principles' to adjust for equal size		Carol - 2 each; 1 *l*, 1*s*	Ron - 2 each; 1 *l*, 1*s*	George - 3 each; 1 *l*, 2 *s* Tara - 2 steps, 2 equal each Saul - 3 each; 2 *l*, 1 *s*	Mark - 2 each, 1 *l*, 1 *s*; Kay, May - parallel cuts; middle thinner
CS-U$_2$ consolidation			Josh - $\frac{1}{3}$s "as a pie"		

The salient features which appear to distinguish the development of fair sharing in relation to continuous fractions in early childhood support the following SOLO development in the ikonic and early concrete symbolic modes.

IK-U	There is a single idea of sharing or partitioning into two parts only.
IK-M	The idea of sharing can cater for more parts but resolution is difficult in cases such as four shares for three dolls. Conceiving the task but not being able to resolve such conflict successfully is typical of multistructural responses.
IK-R	Sharing is understood in more complex environments (such as three dolls) but without conservation the distribution which takes place is inconsistent with the mathematical condition set in the problem. Hence at this level while a resolution is reached using an ikonic (and mathematically

idiosyncratic) notion of fair sharing the lack of conservation precludes the response from the concrete symbolic mode.

CS-U_1 The criterion for fair sharing at this level is based on conservation of number and requires that a fair distribution is based on the same number of pieces (for each doll). Any conflict relating to the size of shares is not acknowledged and leftover parts of the whole are likely to be ignored.

CS-M_1 Sharing at this level recognises the need not only for an equal number of parts but also for an equivalent amount (of pancake) in each share. The method of achieving this outcome is ad hoc, often requiring a sequence of attempts and estimation. Appearance may be an important factor.

CS-R_1 The realisation that sharing requires equal portions is also the significant feature at this level but the method of achieving it utilises other mathematical principles (e.g., geometry and measurement) which are integrated and applied in a coherent fashion. Elementary geometric divisions of the circle starting with halves and/or compensating for the width of parallel cuts are typical examples.

CS-U_2 The consolidation of the idea of third in relation to a circle and equal sharing affords the opportunity to cut the circle directly from the centre in "pie-shaped" pieces. The need for a more complex relationship, typical of the previous level, is replaced by a single simplified concept.

Discussion

The importance of these findings relates both to our understanding of how mathematical ideas develop in young children and to suggestions for classroom teachers. These will be considered in turn.

The response of students at the CS-M_1, level support results obtained by Watson, et al. (1993) in relation to the Mars Bar Problem. In that problem students were asked if a Mars bar cut in half lengthways or widthways would produce equal shares. There the multistructural level was reached when students could visually or physically reorganise one shape of half a bar to fit another shape of half a bar and accept the fairness of the shares. The context of that problem meant that students who did not consider that the shape of a half could vary were classified as CS-U_1 . This is consistent with the lack of consideration of size in the context of this study. The "bottom up" ikonic support observed in the previous study also occurred here.

The progression of attempts to share in the ikonic mode which fall short of mathematical fairness are similar to those of Watson and Mulligan (1990) in relation to an early multiplication problem. Although there is some understanding of the task and the ability to manipulate concrete materials in the general context, lack of conservation means that the task cannot be completed in the way required by the mathematics curriculum.

The relationship of fairness to other topics in the mathematics curriculum, such as chance, would indicate that early concepts should be compared in different settings. The work of Lidster, Pereira-Mendoza, Watson and Collis (1995) suggests an ikonic cycle of fairness in relationship to dice which grows from being ego-centric to other-oriented to world-view based. This is followed by a concrete symbolic construction which grows out of considering physical features of dice. Combining the analyses from these studies and further research may help us develop a better understanding of the ideas on fairness which children bring to school and how they may be fostered to become a useful mathematical construct.

It is clear that when teachers discuss fairness in classroom settings they need to be aware of the multiple understandings which may be present. Detailed questioning of responses which accept a fractional sharing as fair may help find the base lines for tasks to aid conceptual development. At least two important aspects arise from this research. One relates to leftovers which may remain from a whole when fractional parts are divided. While in an out-of-school setting it may be entirely reasonable to have leftover pieces of cake, in later mathematics the division of a whole into equal parts will implicitly assume that there are no leftovers. This distinction will need to be made for some students as they begin work with fractions.

The other aspect relates to the discrete and continuous nature of partition possible of a whole. For discrete sets of equal-sized objects, number of pieces is adequate for determining fair shares. For continuous wholes, however, the amount or portion of the whole is the salient feature to be considered. Fair sharing may not depend on the number of pieces in the share, but the amount of the whole must be the same. For the students in this study, a combination of approaches was the usual response. Ideas such as those of Streefland (1991) which build a consideration of fairness into initial work on fractions may be very helpful in confronting difficulties and developing consistent definitions in the classroom.

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A Study of Children's Probability Judgements

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This paper reports some preliminary results of an ongoing study of children's intuitive understanding of basic probability concepts. The conceptual framework of the study is outlined. Task-based interviews, conducted with fifty K-6 children in N.S.W., have revealed some developmental trends as well as some interesting insight into the types of strategies applied by children when attempting to quantify their perceptions of random devices and to compare probabilities.

In the words of Kapadia and Borovcnik (1991, p2). "Probability can be thought of as the mathematical approach to the quantification of chance, just as rulers measure distance." The concept of chance is an awkward one; it does not sit well with the causal, logical and deterministic thinking commonly applied in the world of mathematics. Nevertheless, probabilistic thinking provides a significant means of applying mathematical ideas in realistic situations (Kapadia and Borovcnik, 1991). Just as with learning to measure distance, children first need to understand the essential concepts of probability and develop the accompanying skills before they are able to make meaningful quantification.

Through life experiences and the use of language children develop intuitions about such things as chance, luck, fairness, faith, certainty, possibility and impossibility. However most social and cultural situations are well beyond the average person's ability to quantify or explain mathematically. Instruction in probability usually begins with more tangible situations, through the use of random generators, like dice, spinners, raffles and coin flipping. This type of activity allows the crucial concept of sample space to be modelled in a concrete way. Random generators not only include the sample space but also the random action that, by chance, produces an outcome. Consider, for example, a box containing three blue blocks and two red blocks (the sample space), and the random action of drawing a block without looking. Knowledge of the structure of the sample space allows the observation that it is more likely that a blue block will be drawn than a red block, but an understanding of chance adds the realisation that a red block could be drawn. It is this type of thinking that forms the foundation of probabilistic reasoning.

The quantification of chance, the actual probability of an event, requires comparison of a different kind and usually the construction of a fraction. The sample space must be considered in terms of the subset of the items representing the desired outcome (say a blue block) in comparison to the total set of possible outcomes in the sample space, which results in placing 3 over 5.

This creation of a numerator and denominator by examining the structure of the sample space becomes crucial when two different sample spaces are being compared with the purpose of determining which sample space has the greater potential for producing a specified outcome. This task is not difficult when the denominators are the same, but becomes quite complex when the denominators are different (ie. the sample spaces contain a different total number of elements). It becomes necessary to construct either fractions or ratios and apply proportional thinking to compare the two relations.

One additional factor that should be considered is that each successive use of the random generator (eg. drawing of a block) produces an outcome. Repeated draws (with replacement) create a set of frequency data which, due to the element of chance, may or may not be representative of the structure of the sample space (eg. 10 draws might result in all blue blocks, which is not representative of the 3 blue, 2 red structure of the sample space). This can be a powerful distracter in the development of mathematically appropriate thinking (Shaunessy, 1992). The understanding that each random action leading to an outcome (eg. drawing of a block) is an independent event is therefore another factor in probabilistic reasoning.

A Conceptual Framework

When designing interview tasks with the purpose of investigating children's probabilistic thinking it can be helpful to consider some factors of the situation as being external to the child (interviewee) and some cognitive and affective factors being internal (Goldin, 1993). The researcher's goal is to access some of these internal factors. Random generators, and the sample spaces imbedded in their structure, presented in the context of a particular task are physical in nature and therefore external factors in probabilistic reasoning. The way in which an individual perceives the physical structure is an internal factor determined by a range of influences. These influences can include belief in the fairness of the situation, an understanding of chance and randomness, previous experiences associated with the situation, and social/cultural background including religion. Age (or cognitive development) is also an influence in terms of the development of thought processes that allow consideration of multiple outcomes and all possibilities, a functional understanding of proportions and ratios, and the consistent application of logic (Piaget & Inhelder, 1951). All of these influences can determine what mathematics, if any, can be extrapolated from the situation by the individual. Thus, when an individual is called on to make inferences from a random generator's structure (eg. predict most likely outcome) or to make comparisons between sample spaces, both external and internal factors are involved.

Language must also be considered because it is usually the vehicle for communicating ideas between the people involved. The use of the game context assists in aligning the researcher's goals with the interviewee's goals and so reduces the dependence on words to convey meaning.

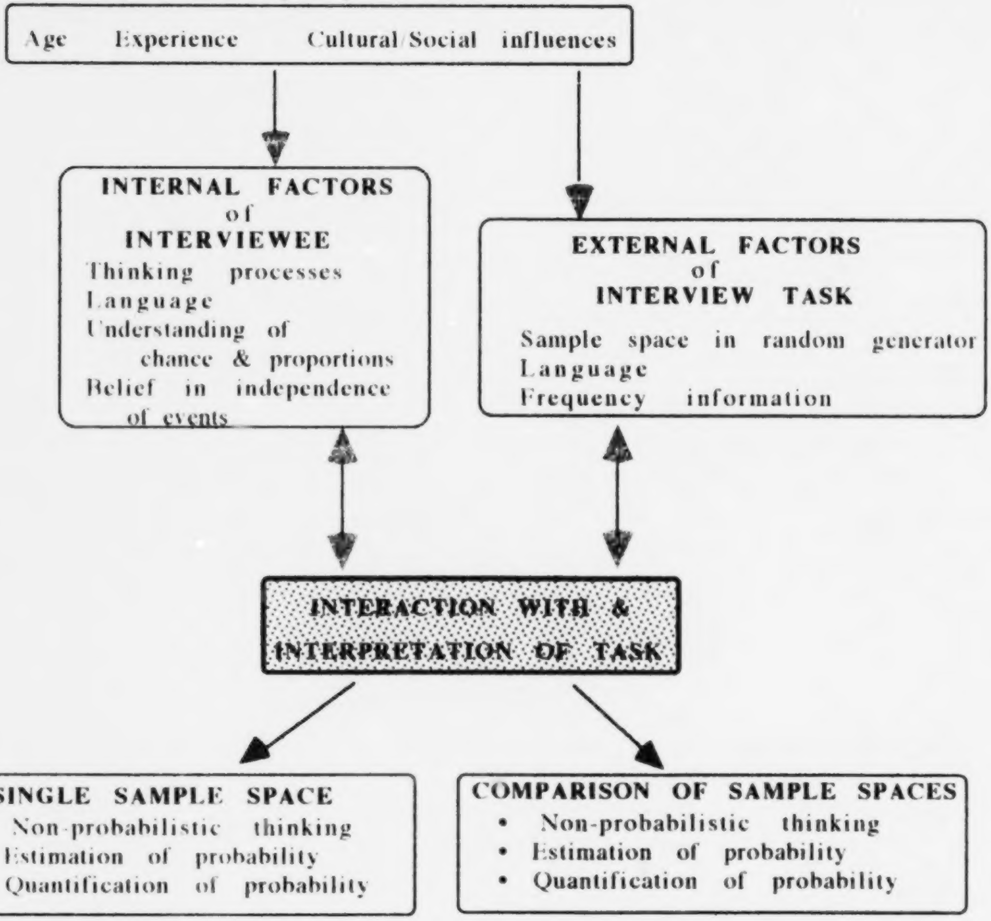
When the interviewee indicates a choice of outcome or a choice of sample space the researcher is witnessing an external action, but can only guess at the internal thought processes that have influenced this choice. Asking the interviewee to give reasons for the choice permits the researcher to gain some insight into the otherwise hidden internal influences. Of course, there is no guarantee that the interviewee will truthfully explain their thinking, or will have the appropriate language available to accurately communicate their thinking, nor that the researcher will correctly interpret what the subject has said. However, cautious interpretation of the information is definitely more useful than no information at all.

There is as yet no one comprehensive theory on the cognitive and psychological components of the acquisition and application of probabilistic reasoning. Therefore the theoretical parameters of this study are based on several established theories that may, in combination, provide a framework for understanding this field. Firstly, there is an underlying constructivist perspective which permeates the whole research design. Secondly, there is an assumption that there will be a change in performance with age, which is consistent with Piaget's developmental theory. This does not exclude the interpretation of result via the more recently developed model of the SOLO Taxonomy (Biggs & Collis, 1982). Thirdly, Fischbein's (1975) ideas about an individual's progress from primary intuitions to secondary intuitions as a result of experience and instruction might provide explanation of some results where Piaget's theories cannot. Finally, the interpretation of the results of some tasks, particularly with the youngest subjects, may be assisted by considering information processing theories such as Brainerd's (1981) working memory model.

As explained in the introduction to this paper, comparing sample spaces can require higher level thinking than considering just one sample space, and therefore these two have been separated in the conceptual diagram. Within these two categories there can be different levels of responses; where probability is not actually considered, where a non-numerical estimate is made, and where quantification actually occurs.

Two major points of contention arise from the research literature. One centres on the age at which a basic understanding of probability arises (Carlson 1970; Falk 1980, Fischbein 1975, Green 1983; Perner 1979; Pumphrey 1968). The other involves a debate

Figure 1. A model of the conceptual framework


Age Experience Cultural/Social influences
INTERNAL FACTORS of INTERVIEWEE
Thinking processes
Language
Understanding of chance & proportions
Belief in independence of events
EXTERNAL FACTORS of INTERVIEW TASK
Sample space in random generator
Language
Frequency information
INTERACTION WITH & INTERPRETATION OF TASK
SINGLE SAMPLE SPACE
Non-probabilistic thinking
Estimation of probability
Quantification of probability
COMPARISON OF SAMPLE SPACES
Non-probabilistic thinking
Estimation of probability
Quantification of probability

about the validity of certain research tasks in determining the presence or level of probabilistic understanding (Acredolo et. al.(1989), Hoemann & Ross 1982, Falk et al, 1980; Perner 1979). The two issues are linked because the latter has been used to explain discrepant results related to age.

Method

The research project being reported here aims to take advantage of the situation that probability is not yet part of the NSW K-6 Mathematics Syllabus and hence has not been formally taught in the schools. Therefore it can be assumed that the understandings that the children hold will be intuitive in nature. The study is exploratory and seeks information about the basic notions of probability discussed above. Of particular interest is the children's perception of the relationship between sample space and the probability (likelihood) of certain events; and any differences in the children's performance in one-sample-space tasks and two-sample-space tasks.

Interview Protocol

A structured interview protocol was developed consisting of four tasks with accompanying questions presented within game contexts. A fifth task was added for half the sample. The tasks utilised two different random generators and required children to consider the structure of sample spaces, consider probabilities of events (without numerical probabilities) and to compare sample spaces (with and without numerical thinking encouraged). The tasks were always presented in a single session (about half an hour) and in the same order The questions were asked in the same sequence each time. Some variation arose due to the random nature of the outcomes in the games, and also due to extra probe questions being used when necessary. All interviews were conducted by the same researcher and were tape recorded. The recorded interviews were transcribed directly into a data-base matrix, where information could be retrieved for individual children, particular tasks, specific questions or for various groupings (such as age groups).

Task 1 'Bears in a Box': Each child selected two colours of plastic bears which were placed into a cardboard box in the ratio of 3:1 (eg. 3 blue, 1 red). The researcher asked the child to state the most likely result of a random draw and explain the reason for their choice. A draw was then made by the child and the result recorded by placing a matching bear on the table. Six draws were made, with the bear replaced each time and the box shaken before each draw. The 'prediction' question was asked before every second draw. After looking at the results of the six draws, the children were asked to say whether another six draws would result in the same, or a different sequence. Another six draws were then made. Finally the children were asked why the box was shaken each time, and whether the game reminded them of anything they had done before.

Task 2 'Non-replacement': Using the same four bears as the previous game a 'prediction' of the colour bear most likely to be drawn was sort, together with the reason for the choice, but after each draw the bear was left out. If the single-colour bear was drawn on the first or second go, the game was repeated to allow the child the opportunity to encounter an 'equally likely outcomes' situation.

Task 3 'Racing Cars': This game used four coloured cars (counters) and a four lane straight race track divided into eight squares between the start and finish lines. Four different spinners were prepared, with Red, Yellow, Blue and Green sectors structured in the following proportions: Spinner A 2:2:2:2, Spinner B 4:2:1:1, Spinner C 3:1:2:2 and Spinner D 3:1:4:0. The first game played was a 'warm-up' game using Spinner A. For the second game, the child was shown Spinner B and asked to choose the car most likely to win. For the third game the child was instructed to be the Red car, then asked to select the spinner (from all four) that would give them the best chance at winning. A series of 'hypothetical' race questions were then asked, which asked them to identify the spinner that provided equal chance for all cars, best chance for Yellow to win, make Red the least likely winner, and make it impossible for Green to win. For two of these choices, the children were questioned about their certainty of the predicted outcome. Reasons for every choice were sought.

Task 4 'Transfer': This task required the children to substitute a 'Bears in a Box' random generator for the spinners in the racing car game. First they were asked to construct a 'fair' game by placing some bears into the box. Then they were asked to construct a different 'fair' game, followed by one in which it would be impossible for Green to win. Finally, they were shown Spinner B and asked to work out how many bears to put into the box to make the game 'work the same way' that the spinner would. Each child was asked to explain their strategy for completing this task.

Task 5 'Proportions': This task was only completed by the children from the second school. In each of four games, the child was presented with two jars, each containing a different combination of Red and Yellow bears. They were asked to choose the jar that gave them the better chance of picking out a red bear, or to declare that both jars gave the same chance. To assist comparisons the bears were set out in lines of Red and Yellow before being placed into the jars. After choosing a jar and explaining their reasons, the children were asked to 'test' their choice by making five draws (with replacement). If two or more Red bears were drawn the child won a small chocolate.

Sample

The sample was drawn from one school in Sydney's south-western metropolitan area and one country school in the south of NSW. K-6 teachers were asked to nominate four children from each grade who were representative of the range of mathematical abilities and who would not be unduly uncomfortable in an interview situation. While a few children were clearly considered to be either 'above average' or 'below average', the vast majority could be considered to be of 'average' mathematical ability. For various reasons, such as malfunction of equipment, only fifty interview recordings were useable for analysis.

Table 1. Sample Sizes

Age Group (years)	Number
5/6	12
7/8	13
9/10	14
11/12	11

Total Sample = 50 48% Male 52% Female

Results

Although several layers of analysis and interpretation are planned only the preliminary results are reported in this paper. This preliminary analysis confines itself primarily to 'correct' and 'incorrect' responses from the children. In most questions the children were asked to make a choice, (for example, that blue was the most likely outcome) and were 'correct' if the mathematically correct choice was made. A reason for making the choice was also sought and this was deemed to be 'correct' if it drew appropriate information from the sample space (eg. There are more blue bears in the box).

Figure 1 shows the percentage of correct choices and reasons across all tasks except the Transfer Task where reasons were not relevant. It can be seen that there is a general trend of being able to make a correct choice but not give a successful explanation. This trend is repeated in the results for each task. The graph also reveals the clear development of probabilistic judgement with age, but with little difference in performance between the 9/10 years age group and the 11/12 years age group. Again this trend is repeated in the data for each task.

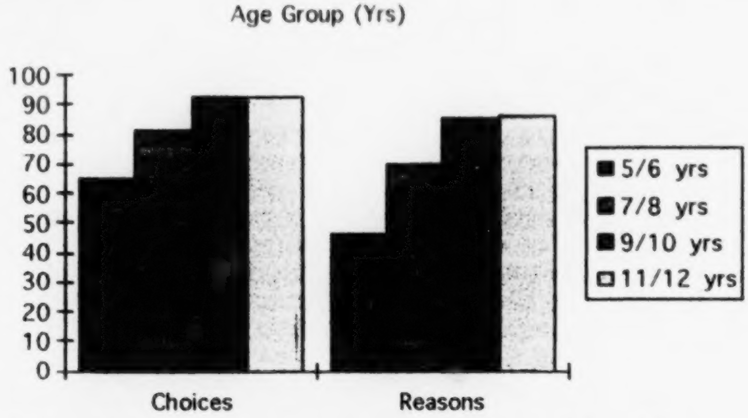


Figure 2: Combined percentage correct across Tasks 1, 2,3 and 5

When the percentages of correct choices and reasons are examined (see Figure 3) it becomes obvious that the Racing Car Task was somewhat easier for the children than the other tasks. It is also apparent that the children found it quite difficult to correctly explain their reasons for choosing a sample space in the Proportions Task. The most common error in the Proportions Task was to only compare the target colour (red) in each jar, and to ignore the amounts of yellows. In other words, the children tended not to use ratio and proportion in their reasoning, and when they did they had difficulty in explaining their thinking.

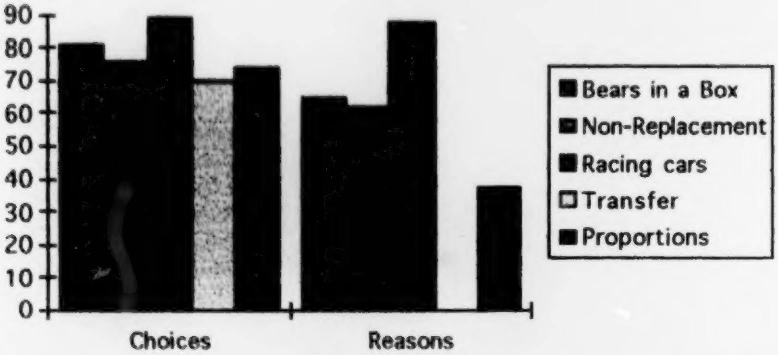


Figure 3: Summary of correct responses and reasons for each task. (Note: reasons were not a relevant feature of the 'Transfer Task')

Task 1 'Bears in a Box': In this task the children (with the exception of the 5/6 year olds) made very confident predictions about the outcome of the first draw, backed up by equally confident reasons. However, the predictions for the third and fifth draws showed a decline. For example, 9/10 years: 100%, 78.6%, 78.6%. This suggests that the children's thinking was inappropriately influenced by the frequency data provided by the outcomes of the previous draws. This is supported by the changes in the reasons given for choices.

83% of the 5/6 year olds said they expected the next six draws to produce a different sequence, but surprisingly only 63.6% of the 11/12 year olds made this prediction. Most children, (including 75% of the 5/6 yrs) understood the box was shaken to maintain the randomness of draws.

Non-replacement task: Table 2 shows the typical increase of appropriate probabilistic thinking with age. Of note in this set of results is the fact that the youngest age group failed to recognise the equally likely situation that arose when there was only one of each colour left in the box. Most of these children were able to state that one of each colour remained in the box, but insisted on naming one as being more likely to be drawn.

Table 2. Non-replacement: Percentages correct of total possible correct

Age Group (Years)	Correct Choices	Correct Reasons	Recognised Equally Likely
5/6	60%	30%	0%
7/8	73.1%	57.7%	50%
9/10	83.3%	75%	60%
11/12	86.4%	81.8%	75%

Racing Cars Task : All age groups were very successful in identifying the best sample space (spinner) to use to produce a specified winner of a game (eg. 5/6 years 91.7% and 11/12 years 100%). With the exception of the 5/6 years group (66.7% correct choice, and 58.3% correct reasons) there was 100% correct responses for identifying the most likely result of the game by looking at a biased spinner. The same number of correct reasons were supplied as choices for most questions. Again the 5/6 year olds had difficulty in recognising the 'equally likely' (58.3% correct) sample space. They also had difficulty in identifying the 'impossible for green to win' (41.7% correct) sample space.

When asked how certain they were about the specified colour winning the game with the specified spinner, 88.9% of 11/12 year olds were able to explain that because of the element of chance any colour represented on the spinner could win the game. However, when discussing the spinner they had chosen to try to make a colour lose the race only 50% applied the same type of reasoning. None of the 5/6 year olds acknowledged the element of chance when responding to these questions.

Transfer task: All age groups had little difficulty in constructing 'equally likely' sample spaces, with many children stating that the number of bears placed into the box was irrelevant as long as each colour was represented in equal amounts. However, the children found constructing an 'impossable for green' sample space more difficult (5/6 yrs 50%, 7/8 yrs 75%, 9/10 yrs 92.3%. 11/12 yrs 80% of responses correct). Although almost half the children managed to construct a sample space with the coloured bears that resembled the biased Spinner B (4:2:1:1), only 21.3% of the responses were mathematically correct. Analysis of the strategies used by the children to determine the numbers of bears revealed four distinct categories: non-comparison, measurement, ordering and fractional thinking. All of the correct responses, except one, arose from fractional thinking.

Proportions: In general the children (sample of 23) found this the hardest task, with 73.3% of choices being correct and 37.6% of the reasons correct. The children performed best in jar choice where comparing the target colours was an appropriate strategy (1 red, 4 yellows vs 3 red, 4 yellows), but most were unable to give a complete explanation (ie. note that the yellows were equal in both jars). The children had particular difficulty in recognising the proportionally equal sample spaces (1 red, 4 yellow vs 4 red, 8 yellow).

Symmetrical Sample Spaces: During the interview the children were given five opportunities to identify sample spaces where each outcome was equally represented. In general, this presented little difficulty in the Racing Car and Transfer tasks, but considerable difficulty in the Non-replacement and proportions tasks. For example, the 7/8 years olds responses were 100% correct in the Transfer Task, but only 33.3% in the Proportions task.

Discussion

One way of discussing these preliminary results is to view the interview tasks in terms of whether the random generator presented the sample spaces in a spatial model (sectors on spinners) or in a numerical model (discrete items, bears, in containers). This can be combined with whether the questions required consideration of a single sample space or the comparison of two or more sample spaces.

Figure 4. Relationship between type of random generator and type of probability judgement.

	Single sample space	Comparing sample spaces
Spatial model random generator	Racing cars	Racing cars
	Transfer	
Numerical model random generator	Bears in a box Non-replacement	Proportions

The matrix shows that each cell contains at least one of the interview tasks, with the Transfer Task occupying a unique bridging position between the spatial model and the numerical model. The results showed that the children found the Racing Car task easiest, suggesting that spatial model random generators are easier for them to interpret regardless of whether they are required to work with one sample space or to make comparisons between sample spaces. The children managed the Bears in a Box and Non-replacement tasks reasonably well suggesting that numerical models are not difficult to deal with when only a single sample space needs to be considered. The children found the Proportions Task a little more difficult, particularly considering they were usually unable to explain their choices appropriately. This task demanded that attention be paid to number in the comparison of two sample spaces. Similarly, the Transfer Task demanded attention to number and, similarly the children found this more difficult than the other tasks. This is consistent with the conceptual framework's position that children may be able to make sound probability estimates, but have difficulty with quantification of probabilities.

In terms of Piagetian theory, the spatial model random generator tasks allow probability to be dealt with spatio-temporally, while the numerical models would be dealt with logico-arithmetically. When comparing colours within spinners and comparing different spinners, children are able to make simple area comparisons, or as Hoemann and Ross (1982) say, make 'magnitude estimations'. A similar strategy can be applied when looking at numerical models by noticing the size of colour groups without dealing in precise numbers. It should also be noted that some children used numbers, particular fractions, to help explain their choices of spinners. A closer analysis of explanations and language may reveal some patterns in the voluntary use of numbers. It is interesting that the children found it very difficult to recognise equal probability in numerical models, but quite easy in spatial models.

There is obviously a strong link between age and the development of probabilistic thinking, though the details of this link to various aspects of tasks and reasoning are not yet clear. The inconsistency of responses in the 5 to 8 years age range highlights the tentative nature of emerging understandings. The SOLO Taxonomy (Biggs & Collis, 1982) may be useful for exploring this area. If the youngest children lacked sufficient working memory (Brainerd, 1981) to make successful probability estimates, then a decline in performance between the Bears in a Box task and the Non-replacement task would be expected, because the children had to remember what colours were left the box. However this was not the case.

There is obviously much yet to analyse, particularly in the reasons and explanations given by the children and in the patterns of their thinking. Currently, another, more diverse proportions task is being used in interviews at a western Sydney school with the purpose of further exploring numerical reasoning when comparing sample spaces.

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The beginnings of a phenomenology of writing-to-learn.

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The notion of 'Writing-to-learn' brings into conjunction an understanding of language and of learning. It is suggested that a particular view of language has unwittingly determined a view of the learner. By turning to a dialogical view of language a new view of the learner is presented.

Framing the question

In its narrowest sense writing-to-learn names an American adaptation of the British writing across the curriculum movement of the sixties. It is a loose collection of classroom practices which involve students externalising their thinking, being reflective, and monitoring their learning through the act of writing. For the most part the practice has emerged independently of any particular epistemology or theory of learning. Though given the time of its emergence - the late 70's through the early 80's - and the American context of its origins, it is not surprising that the pragmatic tradition of Dewey (philosophically) and Bruner (educationally) give a constructivist rhythm to American reports of writing-to-learn practice. In the Australian context, from which the present work emerges, the last 10 years have seen the adoption of the writing to learn movement expanded with the accretion of theoretical assumptions from Vygotsky (psychological) and Halliday (social linguistics). These assumptions give a social constructivist flavour to Australian reports of writing-to-learn practice.

However writing-to-learn also signifies a far broader theoretical context, namely what is the relationship of language and thought? This theme is well understood in education through the work of Piaget and Vygotsky and their debate about the nature if inner speech. What is less well understood is how that debate is now problematised in modern philosophy. With the linguistic turn of the twentieth century and from the perspectives of analytic, critical, and phenomenological philosophy, the link between language and some inner world has been broken. To say what I mean isn't founded on some inner intention rather it is a manifestation of the resources for saying available within the culture. What becomes problematised when language is viewed this way is the subject itself, the 'who' in the question 'who's speaking?'. The "I" can no longer be taken as referring to a 'something', rather it is taken as a move in a language game which is manifest always in a particular speech act.

A unifying feature of all the literature on writing to learn is its concern with language in action. This concern distinguishes the project of understanding writing-to-learn from a straight linguistic project. That is writing to learn takes up the instances of saying (What Saussure called parole) rather than the structure of the said (langue in Saussure's terminology). As Ricoeur has noted language in its human guise is always a saying something to someone. To ask "who says what to whom" locates the task of understanding writing-to-learn as a task of interpretation (hermeneutics). That is to say the reading of the texts of the writing-to-learn-mathematics journals becomes the object of study rather than the writer's of the journals. This switch of focus is a shift from a scientific discourse to a discourse in the humanities. In respect of a text based study this shift has two important consequences: firstly, language can't be externalised as system; and, secondly, there is no

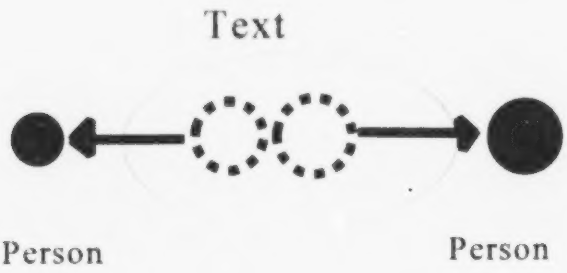
Archimedean point from which language can be talked about. In particular and in respect of the study of the journal texts what this means is that the notion of the competent reader, which is the device by which discourse analysis comes to trade as an objective or scientific discipline, must be abandoned. There can be no single right reading of a text and the object of the study of a text is the reading of it. A reading of the literature reveals a uniformity in appealing to the notion of language as action, it equal reveals a commitment to the prejudice that there is a transcendent source of action and that reason can somehow look through text to this source. What these prejudices do by creating an 'inner' world of language which is taken as informing the acts of expression in language is to cast learning as a predicate of some underlying 'thing'. It is exactly this essence ontology which underpins the 'inside/outside' split of Cartesian dualism. This split has been rethought from the outside by Wittgenstein's severing of the connection between words and things, and from the inside by Heidegger's reformulation of Being. This fundamental rethinking of language, thought, and being a learner are yet to be properly integrated into programs of educational theory and research.

The main point of discontinuity is apropos the construction of interiority and betweeness in respect of presence in the written text. When I read a text 'who' am I understanding and what is to be taken as being referred to by the text? These are fundamentally questions of the subject and of language, so, as preparation for stating a phenomenology of writing-to-learn, it will be useful to sketch the nature of the discontinuities between the extant corpus of writing-to-learn literature and the present study in respect of the key moments of language and presence and the nature of language as action.

Language and presence.

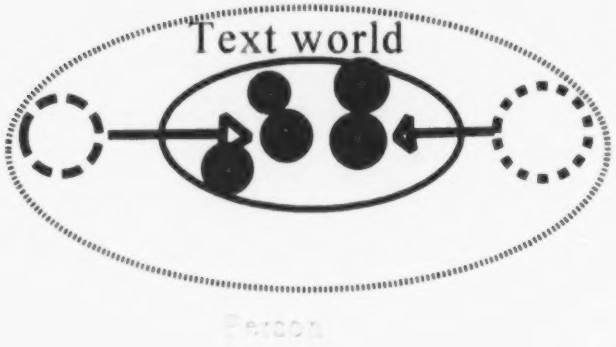
Modern Anglo-American and Continental philosophy has formulated language, as the choice to say or not to say, the distinctive ground of human presence. The strategy of this linguistic turn is to rethink the Being of being human. The following two figures summarize opposed views of the relationship of language to presence. Figure 1 typifies what might be called the natural attitude to language: Language is a medium through which two people meet. What is represented in language refers directly back to the intentions and

Figure 1



self presence of one to the other. The assumption is that what is being expressed is a true reflection of some prior synthesis of experience that is taken to be the subject making the utterance. Figure 2 on the other hand portrays a far more complex picture. The expressions in language are taken as the site of the subject, and indeed of multiple subjectivities. Where in the first model speaker and spoken are taken as continuous in the second model speaker and spoken are at a distance from each other. For the most part we are blind to this distance in the face to face of spoken language but it begins to reveal itself in written language. In written language author and reader are not present to each other except by an intervening act of interpretation. Further once something has been said in writing its meaning is removed from the intention of the author - it is free to be read and reread with ever differing meanings. Writing merely highlights what is in fact true for all language: the saying and the said can never be identical. To really appreciate the force of this view is to realise that "speaker" and "hearer" are language structures and are not necessarily mapped onto biological organisms. It is possible to speak to yourself, to hear what I just said, for marks in the sand to tell us something, to speak to and be understood by animals, etc. Though the biological organism is not structurally part of language what is, is an other to that biological organism. Put most boldly: without the other that constitutes the being of language there isn't consciousness.

Figure 2



As I mentioned above language inevitable marks a distance in the presence of one to another. What is important about recognising or not recognising the distancing played on self presence by language, as in the differences between model 2 and model 1, is that it determines the validity or otherwise of the questions: who is the subject? and, how is it constructed as an object of scientific inquiry? In model 1 the text is a window onto the mind of the speaking subject. In model 2 language makes the subject opaque and in principle inexpressible. While language is seen as a window onto a self this self can be made an object of study by analogy with other things in the world. This is indeed the path followed by psychology, even to the level of the depth psychology of Freud and Jung where what is mediated by language is taken as an empirical reality. Once language is seen as in fact hiding the self then it can no longer be studied as a thing in the world but must be approached as a phenomenon of human meaning or, in Heidegger's term, being-in-the-world. These two options have a great deal to do with method and what, how, and why objects become objects of study. In particular they demarcate a scientific project from a project in the humanities.

The analytic dimensions of language.

There are many dimensions along which language might be analysed. Four key dimensions that seem to underlie much of the work I have examined are: function, aboutness, experience of self and other, and role or voice.

Figure 3 sketches, in four dimensions, what might be called the universe of language discourse. Each dimension represents themes that have emerged as significant in reading of the writing-to-learn literature. This schematisation is a caricature of the analytic complexity of language, but is useful in that it acts as a mnemonic for the range of ideas that have to be kept together when looking at a text. Of the four descriptive dimensions it is the fourth, Agency/agent, that is the most important for reformulating the view of the learner. In fact it is the addition of this axis that enables the move from a naive description to a phenomenological one.

DESCRIPTION - EXPLANATION: This axis is an icon for the social functions of language. It is intended to resonate with themes developed by Halliday, and work on genre. Or it can be interpreted in a more homely way as the distinction between showing and telling.

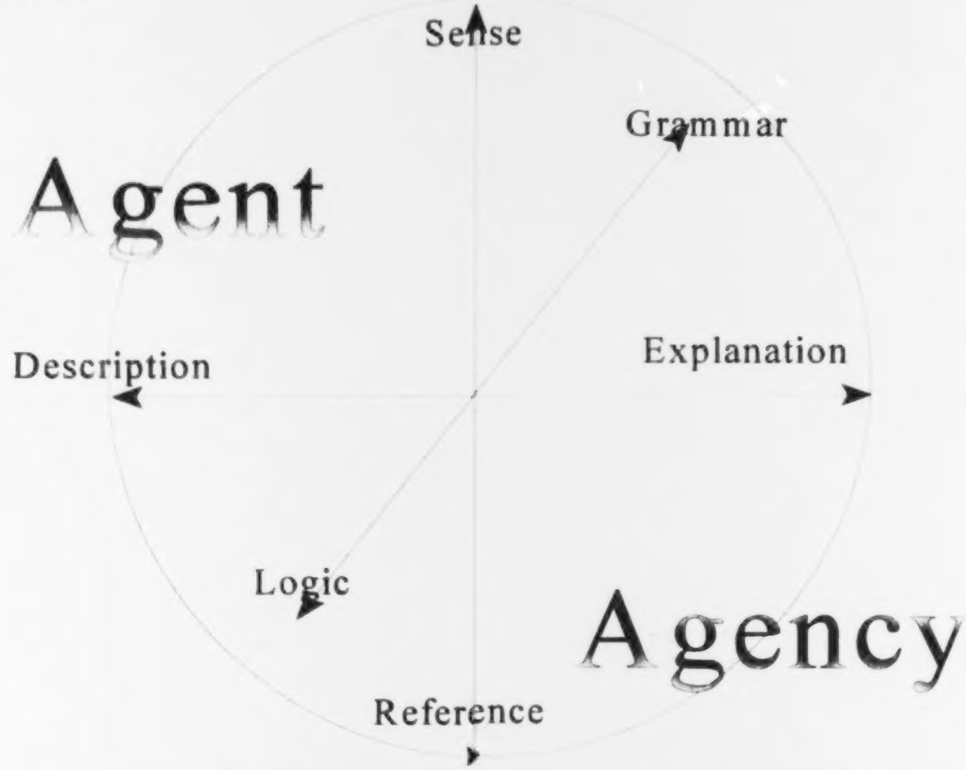
REFERENCE - SENSE: This axis symbolises the question of representation in discourse, as dealt with in discourse analysis and philosophy. A homely analogy for this distinction would be the distinction between touchable things and thinkable things. It is as Ricoeur typifies it the relation of 'what is said' to the 'about what'.

LOGIC - GRAMMAR: This axis introduces a philosophical dimension. In an earlier time 'logic' was used to refer to the relation between things, and 'grammar' to the relation between symbols. It is in this sense that I want to use the terms now. This is a very useful axis to have for our discussion of the universe of discourse, because this axis more than the others is related to that all pervading experience of the world, of cause/effect, and time passing. It is in this sense that Wittgenstein talks about 'forms of life' as grammar. Our experience is often taken as the basic logic from which we move towards constructing grammars.

AGENT - AGENCY: what would be the fourth axis in a four dimensional figure is represented here by the enclosing circle. This axis is to be understood as orthogonal to the other three, and it is represented as the fourth dimension because it is typically not a consideration in any naive descriptions of language but is prominent in any phenomenological description. It is by making this dimension of language analytically central to the study of the writing-to-learn journals that the present thesis begins to diverge from previous attempts to understand writing-to-learn. I will develop this distinction more carefully in the next section.

The above schematisation of the space of language gives us a useful shorthand for distinguishing domains of language use. I want to suggest that the vast majority of discourse happens in the space delimited by DESCRIPTION, REFERENCE, and LOGIC. Because the vast majority of discourse is defined within this space, I will call it Natural Language Discourse. It is the space of the market place, where people talk about themselves and the world around them in the informality of a shared culture. We give each other directions on how to do and how to find. All the sophistication of language operates, and we write expressively and transactionally. It is here that language serves to locate us within the hurly-burly of life as lived, and it is undoubtedly from fulfilling this function that language has the marked forms that it does. (In linguistics the term 'marked' refers to the common or unmodified syntactic forms.) Natural language is unproblematic because it is language which is ready-to-hand.

Figure 3



In addition to serving the pragmatic demands of life, language also serves the demands of imagination and reason. From the domain of natural language discourse, language is stretched in one direction to become an art form, and in another direction to express and communicate scientific theories. As art, language is located in the space to the left of the plane marked by REFERENCE, LOGIC, and SENSE not occupied by Natural language discourse. Another differentiation of language happens as it serves the needs of scientific inquiry. It is the demand of explaining that moderates modes of reference, and the demand of theorising that makes language oscillate between the demands of logic and grammar, and, sense and reference.

As I have emphasised the three dimensions sense/reference, grammar/logic, and description/explanation, come alive in a phenomenological sense when the agent/agency axis is added. The speaker and the spoken, the writer and the written are separable, the product no longer defines the process. The reason for this is that the product, for instance the text as a completed act, can be understood as a text only because of the existence of writing in the lifeworld. That is to say the agency resides in the culture. For an act to be considered to be an act and not a random movement it must make sense. It is the requirement of making sense that ties an act, not to individual minds but to the agency of the lifeworld in which the act is embedded. If we take the term 'event' to be the composite of agent and agency then the event of the spoken or written word is partitioned into a speaker and hearer, or a writer and reader, in accord with the sense they make in a particular culture, ie to the degree that the culture has a voice available for them to occupy. That an author is taken to be saying something (the act of speaking) is due to the agency of language held in common. Agent and agency define the space of speaker and audience. Both the self and the other can now be seen as structurally part of the discourse, a role within it.

A Phenomenology of writing to learn.

I take learning to be about meanings and meaning making and in this sense it is about language understood in its broadest sense. The paradigm for this process is wrongly taken to be natural language. All the literature on writing to learn stays within the confines of the twin western prejudices: there is a transcendent source; and, reason can 'see' below the surface. These prejudices give rise to the view that learning is a predicate: a developmental property of some underlying 'thing'. Such a view is implicit in the grammar of natural language. To breakout of these prejudices requires that we see language dialogically, ie as existing only in the space between biological entities. Where this notion of dialogue (intertextuality) is most fully developed is in the work of Bakhtin. Bakhtin develops his argument as follows:

> Knowledge of the thing and knowledge of the person. They must be depicted as boundaries: the thing, pure and dead, is nothing but externality; it exists only for the other (the knowing subject) who can reveal it entirely, down into its deepest recesses.....The second boundary is the thought of the person in presence of the person itself, dialogue, interrogation, prayer. Two boundaries of thought and praxis or two types of relations (the thing, the person). The deeper the person, that is the closer we edge to the personal boundary, the less applicable any generalising method; generalisation and formalisation efface the boundaries between genius and mediocrity ... Our *thought* and our *praxis* (not the technical one, but the *moral*, i.e. the set of our responsible acts) take place between two boundaries: the relation to the thing, and the relation to the person. *Thingification* and *personification*. (Todorov, p.18)

The knowledge which is *personification* is reduced to *thingification* by either of two false reductions of the Text: the text is reduced to its materiality (the structuralist enterprise); or, it can be reduced to a sequence of psychic states that exist in speaker and hearer ('I mean what I say' and 'I know what you mean'). For Bakhtin both these reductions share essentially the same defect, "they try to find the whole in the part". Text is the visible person and Text is an 'I and other' and not an 'I' and 'other'. For Bakhtin "this personalism is semantic and not psychological"(Todorov, p. 19).

There are events that, in principle, cannot unfold on the plane of a single

> and unified consciousness, but presuppose two consciousnesses that do not fuse; they are events whose essential and constitutive element is the relation of a consciousness to another consciousness, precisely because it is other. *Such are all events that are creatively productive, innovative, unique, and irreversible.* (Bakhtin quoted by Todorov, p. 100, emphasis added.)

The events that Bakhtin is talking about are all those events that are rightly spoken of as mine, eg my birth, my death, my becoming an adult in so far as it is valorised by a particular chronological age in a particular society, or simply my being a speaker when I speak creatively, but are not experienceable from within. For Bakhtin these events constitute being a person and there is more to being a person than there is to being conscious. 'Only the other is in possession of the values of the being of a given person'(Todorov, p. 98). The excess of being a person over being conscious is the structural excess of language itself - that is language as I am born into it. On this understanding of language it neither swallows me up nor condemns me to solitary alienation but rather in the structural excess of any dialogue gives the freedom to 'be spoken' or to 'speak out'. The 'structural excess' that Bakhtin refers to arises from the nature of words as communication. For a word to be a word it must simultaneously be said and be understood, without understanding a word is only a sound. The way Bakhtin expresses this is to say that the word is essentially dialogical. The dialogue is not between individuals but is structurally part of language. It is in this sense that the model represented in figure 2 locates the substantial interaction between people is in the world of text rather than in the 'real world' of things. So the 'I', 'we', or 'you', as words uttered by the speaker, are structurally part of language and don't refer back to the author-creator.

To be or not be a person is an ethical issue and from this it follows that language is essentially ethical because any saying is a proclamation of the world for the other. Personhood relies on being able to be a topic in discourse. Schooling is a discourse and within this discourse language offers students ways of being a person. 'Self' and 'other' is a false dichotomy where as 'self and other' is precisely person. Saying something to someone, apart from the content of what is said, is a confirmation and continuance of my dasein, it is what Heidegger calls being-with. For this to be the case it makes no difference whether in this discourse event I am speaker, hearer, or spoken about. Being a person is a matter of having become through a history of being with one another. In respect of this being and becoming a person there is a continuing lived tension which Heidegger notes as follows:

> Everyday Being-with-one-another maintains itself between the two extremes of positive solicitude - that which leaps in and dominates, and that which leaps forth and liberates [vorspringend-befreienden]. It brings numerous mixed forms of maturity; to describe these and classify them would take us beyond the limits of this investigation. (BT p. 159)

There is no doubt that the classroom is for the students an everyday-being-with-one-another. In this learning is a 'form of maturity'. What 'leaps in' or 'leaps forth' is the word. Advertising is a good example of a word event that 'leaps in' as language is used to grasp the person and reduce their saying to slogans. Such domination is overcome by the possibility of dialogue.

In the classroom there are two senses of dialogue to be considered. There is the dialogue of the classroom, and there is the dialogue of the individual with themselves - inner speech. In each case the speaking subject is other than the uniqueness of the being-there (dasein) of the person, yet it is these dialogues that constitute the becoming of dasein. What ever text makes this dialogue concrete in history is fundamental to a 'Hermeneutics of the "I am". This is true whether the concern of the dialogue is identity or thought.

Each 'I' recognised above is embodied by the way it enters language through the use of the tokens of language. To understand the 'I am' is to understand both its domination by language and its struggle to be liberated. To simplify the discussion of the forms of maturity that arise from the tension between being formed by language and forming within language I will introduce the names: intending-becoming and attending-becoming. The goal of intending-becoming is the uniqueness of the individual, it is signified when the 'I am' is projected into language so that the articulations have the force of insights and new possibilities. Attending-becoming has no goal of its own. For attending-becoming the articulations of language come from the commonness with the other (In Nietzsche's terms it is the language of the herd), whether from memory, tradition, culture. In attending-becoming the person is being defined, while in intending becoming the person is defining. In attending becoming we are mere points of intersection of the social roles and functions we share with others. As Guignon (1993) puts it:

> As placeholders in the public world, we become caught up in the turbulence of mindless busy-ness, yet we are at the same time tranquillized, complacently assured that everything has already been worked out and that nothing really calls for a decision. (p. 30)

On the other hand 'being called to decide' opens onto intending becoming as authentic being. To intend is a mature form of becoming. Phenomenologically writing to learn mathematics can be a 'leaping in' of the word of schooling or a 'leaping forth' as a word of learning, it is the dialogue of these words that is meant by attending or intending becoming. While the dominant metaphor of schooling is to attend it is difficult for students to become learners.

The communication that a writing-to-learn text is, is a communication about becoming a learner. The hermeneutical task is to recover this conversation given that:

> ...if we rid our minds of the idea that our language is the translation or cipher of an original text, we shall see that the idea of complete expression is nonsensical, and that all language is indirect or allusive - that is, if you wish, silence. (Merleau-Ponty, 1964)

The person is never expressed but is the silent frame in which each dasein becomes in the human world.

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A Phenomenological Approach to Upper Primary Mathematics Teachers And Their Beliefs About Specific Teaching Behaviours

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This paper reports upon a study that examined the responses of NSW upper primary teachers of mathematics to a series of questions about typical teaching behaviours. A phenomenographical approach was used to conduct the research. The study uncovered a range of conceptions held by the teachers and some of the findings are presented. The results provide information which can assist teachers in their critical reflection upon current practice whilest increasing our understanding of how teachers understand and experience their world.

Almost all NSW primary teachers are required to teach mathematics, a powerful and adaptable mental tool. Quite an amount of research has been completed dealing with the teaching and learning of mathematics (Grouws, 1992), yet:

> After more than twenty years of effort to improve mathematical education, there appears to have been little improvement. The problem is widespread. Unless we can identify at least some of the reasons why mathematics is still a problem subject for many, there is no reason to suppose that future efforts will be more successful than those of the past.(Skemp, 1989, p. 30).

It is possible to argue that one of the reasons for mathematics retaining its reputation as a problem subject, is the performance of the teacher (Siemon, 1989). There has also been growing evidence to suggest that rather than trying to prescribe how teachers should behave, the problems of teaching and learning should be approached by developing techniques that help teachers critically reflect upon their current teaching and learning practices (Prosser, 1994, p. 39). A phenomenographical approach (Marton 1981, 1986) was chosen because it is a useful technique in providing data for this reflective process due to its sharper focus upon peoples' current ideas and upon how their world is conceptualised Phenomenography seeks to uncover the qualitatively different ways in which people experience or think about various phenomena. The analysis involves the formation of categories of description to reflect these conceptions. It is essentially a process of identification, construction, delimitation and refinement of the descriptions of the differing conceptions held by teachers towards the set of specific teaching behaviours.

Galbraith (1995) uses the term paradigm to mean a basic set of beliefs or assumptions which are used as a reference point for research and for guiding further activity. Cohen & Manion (1994) present a summary of the differing viewpoints that have existed for some time between two broad theoretical positions: the 'normative' paradigm adopting an objectivist (positivist) stance and the 'interpretative' paradigm. Patton (1980) presents another point of view with his 'paradigm of choices'. Presenting his debate from a pragmatic, empirical perspective, Patton sees the paradigms as the endpoints upon a continuum. All research is to be found somewhere between the two ideal ends. Nisbett (1980) refers to the choice between paradigms as a spectrum with one side being the 'agricultural model' with controlled experiments to improve the product (positivist) and the other side being the 'go and live there and see what it is like' model. Others such as Lather (1991) provide alternative paradigms. Irrespective of which argument is the more appealing, an important implication of these models is that a paradigm is not a monolithic entity, but contains a variety of possible choices. "One approach may appear radically phenomenological; another, mildly positivistic in style, tone, and formation" (Fetterman, 1988, p. 5). Thus while the phenomenographic methodology falls within the interpretive paradigm and shares common ground with phenomenology it also has a difference of perspective. Marton (1981) regards phenomenology as seeking to orient the researcher to the world by making statements about it. Whereas the perspective of phenomenography seeks to examine other peoples' ideas about the world and their experience of it. It aims to analyse and describe differences in conceptions of a certain phenomenon from the subject's perspective.

A thorough examination of the issues arising from the theoretical positioning of this study is beyond the scope of this paper. However what is crucial is the intentions and perspective of the researcher. The theory of intentionality which proposes that humans are always related to reality through the intentional character of the consciousness is at the heart of this study (Husserl, 1970, p.1900-01). Thus

> the categories of description produced within phenomenography are not externally related to reality, in contrast with the view within the positivistic research tradition. Instead, the categories of description are internally related or, more precisely, intentionally related to the statements expressed by the individuals investigated (Sandberg, in press , p. 10).

As the researcher, I cannot escape from being intentionally related to the research object and the categories of description are intentionally constructed through the researcher's interpretation. Thus the methodology raises a number of important issues such as "its perceived lack of validity, lack of predictive power, researcher bias and denial of the voice of the individual through categorisation" (Bowden & Walsh, 1994, p.2). There is not the time nor space to devote to these issues. However I will briefly add two often quoted arguments given in defense of the methodology.

The first is that phenomenography passes the test of Miller and Parlett (1974) who claim that one of the best guides to the validity of a work lies in the extent of its 'recognizable reality'. Because of its success with the test, phenomenography has spread widely to other groups and countries and the 'cross-pollination' among various research groups had an important formative influence on the emerging descriptions of student learning (Entwistle & Marton, 1984). More recently, it has been used to describe only the learning outcomes or subject's conceptions of specific phenomena (Marton, 1988). For example, studies have been undertaken in subject areas such as Physics (Walsh, Bowden & Dall' Alba, 1995), Chemistry (Lybeck, Marton, Stromdahl & Tullberg, 1988), Economics (Dahlgren, 1984), History (Hounsell, 1984), Mathematics (Marton & Neuman, 1990), Business Administration (Sandberg, 1991), and Literary Texts (Marton, Asplund Carlson & Halasz, 1992).

The second involves the issue of scientific procedures and is usually concerned with the question: 'will another researcher independently arrive at the same set of categories of description by studying the same data?' Marton would argue that finding the categories is a form of discovery and as such it does not have to be replicated. However once the categories have been found, it should be possible to reach a high degree of intersubjective agreement. "Structually it is a distinction similar to that between inventing an experiment and carrying it out. Nobody would require different researchers independently to invent the same experiment; once it is invented, it should be possible, however, for it to be carried out in different places by different researchers with similar results" (Marton, 1988, p. 183).

While these two points are an inadequate response to the many theoretical issues raised by other paradigms a careful consideration can be found in the work of Bowden & Walsh (1994). It should also be noted that some researchers reject the notion of making a defence and propose the creation of new understandings for the terms reliability and validity are needed, which will be different to those accepted by the positivists (Sandberg, in press).

THE STUDY

This paper reports briefly upon an experimental phase of a much larger study and involves the use of a questionnaire. This questionnaire contained six scenarios with spaces provided for the teachers to record their responses to a number of questions. The teachers were asked to read each scenario and to imagine behaving in the manner described by the scenario and to record their reasons for doing so. They were also asked for reasons why they might refrain from teaching in the manner described. There were further questions investigating salient referents and relevant demographic information which will not be included in this report.

Participants

The sample was randomly selected from across the population of primary teachers who worked at a NSW Department of School Education (DSE) primary school and who taught mathematics to years 5 and/or 6. The teachers that were approached, belonged to schools:

(i) that had an enrollment above 100 against their listing in the 1993 Directory of Government Schools in New South Wales (DSE, 1993). The purpose of this condition was to eliminate the very small schools. It was felt that these schools would have composite classes with too broad a range of abilities in the one class. That is, the class would have three or more year levels within the one class (years 4-6). It is argued that the teaching strategies employed by the teachers at these schools are likely to be different from those employed in larger schools. A question was included to monitor the levels within the classes of the teachers in the study. A disadvantage, however of this condition, is the reduction in the number of country schools in the sample.

(ii) that were not designated as a 'special' (SSP) school. Again it is argued that the teaching strategies employed by the teachers are different from those employed in larger schools.

There were 116 schools chosen and after obtaining approval from the Department of School Education, a letter was sent to the Principals of these schools seeking permission to approach teachers at their school and asking for a list of the teachers names. A total of 55 Principals gave their consent, and using the names provided from these schools, 150 teachers were sent a questionnaire and eighty-nine teachers completed and returned the questionnaire.

The analysis

A phenomenographical approach (Marton, 1981, 1986) was used to analyse the results of this questionnaire. This phase aimed to produce a set of descriptive categories which encapsulated the variety of conceptions from the data and thus indicated the differences in the ways the phenomenon is understood. The nature of these differences can be obtained by comparing the categories. The categories were formed by the researcher interpreting the data from his or her own perspective and constructing descriptions of the relationships of the subjects to the phenomenon. The methodology assumes that the data will be refined via an iterative process in order to construct a set of categories that are internally consistent and related to each other in a way that is satisfactory for later use. "There is an inevitable tension between being faithful to the data and at the same time creating, from the point of view of the researcher, a tidy construction that is useful for some further explanatory or educational purpose" (Walsh, 1994, p. 19). The possible problems of this phase are: the imposition of a logical framework upon the data which is not justified ; an over-reliance upon the framework of the researcher so that the interpretation of the data is skewed; and the making of adjustments to categories that are not supported by the data. To reduce these dangers, the data was analysed by a team of three who were familiar with the methodology but with two members who were not familiar with the purpose of the study. It is argued that this will reduce the chance of one view dominating the construction phase. The procedure consisted of distributing to each member, two weeks before the first meeting, a common set of copies of twenty responses randomly selected from the total sample. Trigwell (1994, p. 59) favours the use of between 15-20 scripts when initially analysising phenomenographic interview data. "At the lower end, about 15 are required to have a reasonable chance of getting the extremes of the range", with the problem of too much detail as you go beyond twenty. Two meetings were then held in order to produce a common set of categories for each scenario. This formation process involved an iterative procedure whereby the categories were refined upon the basis of evidence provided by the scripts. The set of transcripts were arranged into categories to exhibit a hierarchial structure whereby a particular conception was subsumed and extended in a higher category. This ordering of categories, however, is based upon the nature of the understanding from the perspective of the subject (e.g. mathematics teaching) and not imposed in advance upon the data by the researcher. During these meetings, a second unseen sample of five responses randomly chosen were also coded by each member according to the agreed category descriptions and any differences were discussed and the clarified. The meetings were taped and a summary of each meeting was submitted to the members to ascertain if they agreed that it was an accurate record and to make further observations and comments.

The iterative process consisted of a number of stages. The first stage was achieved by identifying and marking comments which were deemed relevant. What was deemed to be relevant went beyond the mere meaning of words and "in general, the interpretation had to be made in relation to the context within which that comment had been made" (Marton & Säljö, 1984, p. 38). These comments formed a pool of meaning for the following stages. The following stages required a change of focus away from the individual subjects and to the meanings embedded within the comments regardless of whether these different meanings came from the same individual or not. Hence the boundaries between the subjects were discarded. Categorising these comments became a process whereby a decision was made about the specific level of the comments in relation to the other transcripts. Also when a response was unclear, it was examined within the context of what was said in the rest of the transcript (Bowden, 1994, p. 11). Thus this reading for meaning strategy required the members of the team to think about what was being stated rather than just categorising the responses according to key words. It is a process of considering the similarities and differences. The similarities contribute the detail within each category and the differences help set the limits of each category in relation to the others. The characteristics of each category are described by using the language of the scripts that are classified against that category.

> The process involves the *reduction* of unimportant dissimilarities e.g. terminology or other superficial characteristics, and the integration and generalization of important similarities i.e. a specification of the core elements which make up the content and structure of a given category (Dahlgren, 1984, p.24).

The format of each category scale consists of a number of levels with the first receiving a score of zero for a blank or nonresponse. The next level received a score of one and was reserved for answers that were off task and were unable to be categorised. The decison to do this was based upon Biggs and Collis's (1982) work with the SOLO Taxonomy which uses a score of one for the prestructural responses. The rest of the scale is hierarchial and the steps are inclusive. Using the category coding guides, the researcher then coded the 89 survey responses. Again the reading for meaning strategy was adopted and as a check, the researcher examined the correlation between the twenty-five examples coded in the team meetings and the code allocated during the coding of the whole group.

RESULTS

"To provide a full description of the categories identified necessitates the presentation of the whole range of quotations covering the delimiting instances" (Entwistle, 1984, p. 17). This presents a difficulty and so I have chosen to present only a part of the results for one scenario and to present the data in the following manner:

(i) Firstly, the scenario is presented;

(ii) Next, a frequency distribution of the responses (Table 1) and the coding categories assigned *for not* teaching in the manner described by the scenario; and

(iii) The coding categories, descriptions and examples of the reasons *for not* using a prepared stencil described by the scenario (Table 2). Direct quotes from the survey are used.

Scenario description: You are teaching mathematics to your class at the usual time and place. You have recently finished teaching some mathematics to your class and decide they need some practice. You consider various options but decide to use a prepared stencil. (In this study a prepared stencil is considered to be a teacher or commercially produced sheet containing a set of questions with spaces provided for working and/or answers).

Table 1: Frequency and Percentage Distribution of the categories of reasons for not using a stencil (n = 89).

	Category Code	*Frequency*	*Percentage**
0.	Non response.	4	4.5
1.	Prestructural.	0	0
2.	Teacher administration.	13	14.6
3.	Curriculum reasons.	17	19.1
4.	Narrow medium.	23	25.8
5.	Doesn't cater for individuals.	21	23.6
6.	Language and reading difficulties.	8	9.0
7.	Cheating.	3	3.4
	Total	89	100

Table 2: Coding categories, descriptions and examples of the reasons for not using a prepared stencil

Code	Category Description
0.	Non response
1.	Prestructural
2.	Administration - lack of paper or time, not convenient, presentation factors e.g. "Children are not properly focused on topic. Waste of paper. Children take longer answering questions. When work is written on the blackboard they must try to keep up".
3.	Curriculum - doesn't cater for all topics or activities; it may not be relevant; or is used as a whole class tool. e.g. " some activities e.g. measurement, volume etc are not relevent to be dealt with by stencil - may not deal exactly with work covered". "activities might involve the use of concrete materials eg.volume, 3D shapes & nets, compassses".
4.	Too narrow a medium: limited to pen and paper; ignores alternative methods which are available. e.g. " If it has been decided that more practice is needed then it is assumed that not all students have adequately achieved the desired outcomes. Therefore a reteaching (either by inventing a new strategy or by

rehashing a previous lesson) of concepts is needed rather than student practising something they can't do".

5. Doesn't cater for all the the students - individual learning styles
e.g.
"May not be exactly appropriate, wrong level, too easy/hard. Not stimulating. If chn can do the process etc. why repeat the process 20 times. Just a time filler".
"Boring repetition not best practice. Commercial sheets cannot solve problems children may be experiencing".

6. Language and reading difficulties
e.g.
"Some ch'n have mathematical skills but are unable to reproduce them in written form because they don't have the "reading" skills that are often required.
Paper usage - we are constantly being asked to 'cut back' ".

7. Cheating: the students can't be trusted to be responsible for their learning.
e.g.
"Less opportunity to identify individual problems
no certainty that student has done work if done at home; doubles time taken if follow-up lesson required".

Discussion

During the initial consideration of each scenario it became apparent to the panel that there was a movement in the quality and quantity of the responses. In those questions considering the reasons for doing a particular behaviour there was a movement in the focus of responses going from teacher reasons only, to a combination of teacher and child reasons, and in some cases to student focused reasons. This movement seemingly progressed beyond an inward looking model with its emphasis upon the teacher and so concentrating upon administration and management concerns to a more outward looking model which considered factors which would enhance the student learning within the class. A similar movement was also observable in the responses to the quesions considering the reasons for not doing a particular behaviour. However this movement had a negative focus. From these observations it was possible to construct a continuum (Fig. 1) which provided an overview of the descriptions of the conceptions contained within the responses to all of the scenarios and the relationships within the categories used in the coding phase.

Fig. 1: Continuum of responses given to the scenarios and their relationship with the categories

Negative student learning concerns					Teacher centred classroom concerns					Positive student learning concerns	
<-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	-------	------->
6	5	4	3	2	1	0	1	2	3	4	5
	Usual categories										

CONCLUSION

The outcomes of this research study were a set of categories describing the range of the conceptions held by upper primary school teachers of mathematics towards a specific teaching behaviour contained within each scenario. While presenting only a very small selection of data, nevertheless it was possible via the continuum to provide a clear picture of the relationships existing between the conceptions held by this group of teachers. These conceptions are relational and peculiar to each individual teacher within a particular context. The categories although hierarchial in nature, do not represent a developmental progression. Thus, it is not argued that because there is a range of conceptions resulting from the study that it means that people have attained a particular conception by moving through the lower ones (such as Van Hiele's (1985) developmental hierarchy for geometrical thought). Similarly, it is not argued that in order to foster a higher level conception, one should address all the issues between the current and the desired conception. Nevertheless, "it would not be a misuse of the results to use the relations between conceptions to develop a mechanism to facilitate conceptual change" (Trigwell, 1994, p. 71). For although arising from a contextual understanding, the descriptions are really decontextualised and can be used in contexts other than the original one. Thus these complex categories of description can: be a useful tool in the task of understanding how other teachers think about and experience their world; and provide teachers with a resource to assist in a critical reflection of their current teaching and learning practices.

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Secondary School Students' Understanding Of Inequalities In A Linear Programming Task

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A two-part study of the difficulties experienced by senior secondary (Year 12) students in writing inequalities from graphical linear programming problem statements is reported. The first part of the study identifies possible difficulties and the second part evaluates a teaching programme designed to alleviate these difficulties. The teaching programme, based around a set of specific heuristics, is successful in alleviating some of these difficulties. The "pronumeral-as-object" notion and certain misconceptions of "constraint" prove resistant to remediation.

For many students, the last year of secondary education, Year 12 in Australia, is the pinnacle of their schooling. The widely-held perception in the general community of the value of mathematics in developing thinking skills and the fact that some form of Year 12 mathematics is pre-requisite for a range of tertiary courses lead a very substantial number of students to study some mathematics course in Year 12. By this stage, it is assumed, students possess an understanding of mathematical concepts and processes sufficient for them to cope with the demands of courses offered at this level.

As their teachers well know, however, not all students who present themselves for Year 12 courses have in fact achieved such mathematical understanding. Documented examples of common difficulties include: i) lack of understanding of algebraic symbols and their manipulation, ii) lack of understanding of relationships between variables, iii) difficulties in graphing relationships between variables, and iv) difficulties in interpreting worded statements or "problems".

Herscovics (1989) believed that learning difficulties were of two basic types:

1. The learner attempts to map new material onto an existing mental structure which is valid in another domain but inappropriate for the knowledge to be learned. Difficulties i) to iii) above might belong to this category.
2. The inherent structure of the new material might be such that the learner has no existing mental structure which would allow assimilation of the new material. Difficulty iv) above might be of this type.

Herscovics coined the term "*cognitive obstacle*" (1989, p. 61) to refer to either 1. the existing mental structure (or its attempted use) or 2. the structure of the new material.

This report focuses on the cognitive obstacles to the writing of inequalities from worded statements experienced by two groups of Year 12 students. The context is that of linear programming, one of the more demanding topics from the current Victorian Certificate of Education (VCE) Further Mathematics Units 3 and 4 course, regarded as the least difficult of the three VCE pre-tertiary mathematics subjects offered at this level. The material from which this report is drawn is a study by the author (White, 1995) of the cognitive obstacles experienced by students in learning to solve by graphical means linear programming problems involving two decision variables.

An example of the type of problem considered in this study is the following.

> A small toy factory produces models of cars and boats. There is sufficient plastic to produce 12 models a day. (At least) Five cars and three boats a day are ordered. The profit on cars is $1.00 and on boats is $1.50. ... Find the possibility which gives the maximum profit (Andrews, 1990, p. 210).

The students must first read this densely worded statement and interpret it. They must understand a number of inter-related concepts, both practical and mathematical, and be able to express these in various representations (verbal, symbolic, graphical). Some of the mathematical concepts are specific to linear programming and would be taught as part of a linear programming unit. Other concepts assumed by the syllabus to have been gained previously by the students would include the notions of "variable", "equality", "inequality" and "the co-ordinate plane". This study will examine students' understanding of inequalities via their performance on linear programming tasks.

Literature Review

Although there have been many research studies of cognitive obstacles related to the writing of equalities from worded statements, the area of writing inequalities seems to have received scant attention thus far. Research on the former which is relevant to this study is summarized below.

Obtaining an Equation from a Worded Description

Students commonly find it difficult to obtain an equation from a verbal description of a relationship between two or more variables. The classic case of this in the mathematics education literature is the "Student-Professor Problem", as follows.

> Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university". Use S for the number of students and P for the number of professors (Rosnick and Clement, 1980, p. 4).

Only 63% of first-year engineering students at a major United States university could answer this problem correctly. Two-thirds of the errors took the form of a "reversed equation" ("6S = P" instead of "S = 6P"). Rosnick and Clement attempted remediation through tutoring interviews but concluded that "the misconceptions students possess relating to variable and equation are deep seated and resistant to change" (p. 23).

The researchers attempted to determine the particular "misconceptions" held by the students, or, in the terminology of this study, the "cognitive obstacles" concerned. From interview protocols, two conceptual sources of reversal error were identified (Clement, Lochhead & Monk, 1981; Clement, 1982):

In *syntactic word order matching*, the student has the notion that a direct mapping of the key words of the problem statement into algebraic symbols will produce the required equation, thus obtaining "6S = P".

There are six times as many students as professors.
6 S = P

In *semantic static comparison*, the student was believed to have formed a mental picture of the problem. "There are six times as many students as professors" implies that the group of students is much larger than the group of professors. Some subjects of the study made this hypothetical mental picture concrete by means of a diagram.

S S S S S S P

These subjects then represented the relationship by the equation "6S = P", in which "6S" is used to show the larger group and "P" is used to show the smaller group. Other explanations of the reversal error have been offered, among them MacGregor and Stacey's (1993) theory of *intuitive cognitive models*. The authors studied the attempts of over 1000 Australian secondary school children at formulating linear equations involving two variables. Although the test items were designed to eliminate all causes of reversal error given previously in the literature, a high incidence of reversal error was observed. It was postulated that the reversed equations were attempts to represent on paper cognitive models of the relationship between variables. Because the models were based on the notion of comparison, rather than equality, they could not be translated directly into mathematical notation. Clement's (1982) static comparison model was regarded as a special case of such an intuitive cognitive model when concrete referents, e.g., "students" and "professors", were involved (MacGregor and Stacey, 1993, p. 229). The consequence of this hypothesis which is relevant to the present study is that relationships between variables ought to be paraphrased or re-organized in some fashion before an attempt is made to express them mathematically.

Difficulties in Problem Comprehension

De Corte and Somers (1981) obtained evidence that the difficulties experienced by sixth grade children in solving word problems were due mainly to their inability to master the comprehension phase, not the solution phase, of mathematical problem solving. MacGregor (1989, 1991) proposed that the difficulty of interpreting mathematical language stems from the fact that it demands an awareness of structure not normally required in natural-language processing. The importance of understanding problem structure was confirmed by Low and Over (1989), who found that the capacity of Year 10 students to identify necessary and sufficient data for solution of algebraic story problems accounted for 90% of variance in the solution rates of these problems.

Students' Understanding of Equalities and Inequalities

MacGregor and Stacey (1993) noted that some students, in attempting to represent relational statements mathematically, used inequality symbols when an "=" sign was required, so it might be concluded that students' understanding of both equalities and inequalities is likely to be deficient.

The discussion of possible difficulties related to the writing of inequalities and the learning of linear programming prompts the following questions.

1. What cognitive obstacles in the area of inequalities are possessed by the students of this teacher/ researcher?
2. In what manner and to what degree do these cognitive obstacles interfere with the learning of graphical linear programming?
3. How effectively does the teaching process deal with these cognitive obstacles?

Methodology

The general methodology used for the investigation was that of "action research", whose primary goal, the solution of a local problem in a local setting (Gay, 1992), is consistent with the stated purpose of the study. A suitable teaching programme was developed and evaluated by descriptive means, including analysis of teacher–student interactions in the actual learning situation and of students' responses to test items. A key feature of action research which was crucial to this study was the involvement of the researcher as teacher. A limitation of this research model is that no method of control typical of other research approaches was used, nor was it intended that the results ought to be generalizable to any other setting (Gay, 1992, p. 11). One complementary advantage of the action research model was the freedom of the teacher-researcher to respond to the perceived demands of the moment, in the desire to effect (possibly immediate) improvement in the understanding of his students.

The permission of the school Principal for the teacher to conduct this investigation was sought and obtained, on condition that the demands of the syllabus be met and that the procedure followed would be consistent with the normal teaching pattern. This reasonable condition placed restrictions on the time, teaching approaches and research methods available for the classroom part of the investigation, e.g., use of controlled comparison studies was clearly inappropriate. One teaching method considered by the teacher was that of group work but this was not adopted because the students had little experience of this in mathematical contexts and there was insufficient time to foster suitable dynamics. The permission of the students for the teacher to audiotape the lessons and use the transcripts for research was sought and obtained. In any reference to an individual student which follows, the name has been altered.

The subjects of the study were students in two classes of at least 20 taught by the author (1993 & 1994) at an inner Melbourne, Catholic, boys' senior secondary college. The study was in two parts: the first part, the "trial" unit of 1993, aimed to determine difficulties students had in learning linear programming; the second part, the teaching unit of 1994, was to evaluate a programme designed to alleviate these difficulties. The basic design followed in each unit was that of pre-test, teaching/learning and post-test. The teaching/learning time for each unit was 5 lessons of approximately 45-50 minutes.

The "Trial" Unit (1993)

The "trial" unit of 1993 was an exploratory research tool in which the researcher taught linear programming to a group of students similar to that which would be the subject of the 1994 teaching model. The purpose was to identify the cognitive obstacles to the learning of linear programming possessed by the students in order to formulate a suitable teaching program for this topic. Only those cognitive obstacles considered most relevant to the writing of inequality statements will be discussed here.

In the teaching/learning process, some students found it difficult to express mathematically inequality terms, e.g., "not more than". The transcripts (given in their entirety in White, 1995) showed that it took four students to attempt a mathematical description of the statement, "Angela can't buy more than 3 packets of crisps", before a correct answer, "c is less than or equal to 3," was obtained. In the post-test, some students experienced similar difficulties in trying to write mathematically, "at least".

Another cognitive obstacle to the writing of inequality statements was the notion that the pronumeral represented an object. When the class was considering the problem of expressing mathematically, "The number of cars is at least 5", and Josh mentioned "5c", the teacher reminded the students that "c" stood for "the number of cars" and asked what "5c" would represent. When Jim replied, "5 cars", the teacher repeated the meaning of "c". Jim insisted that "5c" was "still 5 cars". When the class returned to the original problem, Jack stated, "5 times c equals the number you're going to get".

The responses to one of the pre-test items suggested that half the class had a good intuitive understanding of "constraint", yet 14 out of 19 students answered incorrectly a post-test item which required them to express a constraint mathematically. Four incorrect responses could *not* be explained by a "pronumeral-as-object" conception.

Thus the trial unit suggested these (probably inter-related) cognitive obstacles to the writing of inequality statements.

Cognitive Obstacle 1: The lack of understanding of terms of inequality, e.g., "at least".
Cognitive Obstacle 2: The notion that algebraic letters represent concrete objects, e.g., "5c" means "5 cars".
Cognitive Obstacle 3: The difficulty of writing constraints in mathematical language.

The Teaching Unit (1994)

Choice and Justification of Strategies for Teaching

Following the advice of Bell (1991) and Sweller and Low (1992), the unit focused on several key examples, whose solution was to be developed through the medium of class discussion. A set of specific heuristics (Steps 1-8 in Table 1) provided a model for solution. It was hoped that these heuristics would help break the problem down into manageable steps and avoid or alleviate cognitive obstacles to linear programming.

Table 1: Summary of Steps in Solving Graphically a Linear Programming Problem

Step 1:	Locate the "decision variables".
Step 2:	Name the decision variables, representing each by a different letter (usually x or y).
Step 3:	Name the variable which must be maximized or minimized (e.g., profit, or cost) and express it in terms of x and y, the decision variables.
Step 4:	What constraints (restrictions) are imposed on each of the decision variables? State these in words using "The number of ...".
Step 5:	Express "The number of ..." constraints in mathematical language, using inequality symbols.
Step 6:	Using x and y axes, sketch the areas defined by the inequality statements. Hence find the "feasible region".
Step 7:	Find the co-ordinates of the vertices of the feasible region.
Step 8:	Find the solution to the problem by calculating the profit (or cost, etc.) for each of the vertices of the feasible region.

Some justification is given here for those steps which were intended to alleviate Cognitive Obstacles 1-3 to the writing of inequality statements.

Cognitive Obstacle 1: The student was asked to "translate" inequality terms, e.g., "no more than" or "at least", into "less than or equal to" or "greater than or equal to", each of which has a mathematical representation ("$\leq$" or "$\geq$") known – hopefully – to the student, who was to verify the accuracy of his "translation" by examining possible numerical values of the variable concerned. This formed part of Steps 4-5.

Cognitive Obstacle 2: Once the key or "decision" variables had been identified (Step 1), the student was to write a statement defining those variables: "Let the number of boats produced per day be x" (Step 2). This follows MacGregor (1986), who suggested using "The number of" statements to introduce the use of the literal symbol as representing a variable. The symbols for the decision variables were "x" and "y", not the initial letters of the objects themselves, in order to avoid confusion with common shorthand, "b = boats" or "c = cars", and because the students had some experience in using "x" and "y" as numbers and in graphing relations involving "x" and "y".

Cognitive Obstacle 3: Step 4 required the student to write a "The number of" statement, such as, "The number of boats produced per day is greater than or equal to 3", to express each constraint on the decision variables. This form would enable a "syntactic" translation (Step 5) to "$x \geq 3$". The re-structuring of constraint statements is consistent with MacGregor and Stacey's (1993) conclusion that relationships between variables need to be re-organized before being expressed mathematically.

Teaching/learning

Lesson 1 focused on the notion of "constraint" and its mathematical expression as an inequality. Consideration of a pre-test item which required an intuitive understanding of the idea of constraint led naturally to the testing of each alternative by the device of numerical comparison, a strategy readily understood by the students.

> If Optus Affirmative can produce no more than 12 litres of orange juice and no more than 20 litres of ginger beer per day, which of the following daily sales are possible?
> 1: 6 litres of orange juice and 12 litres of ginger beer ....
> T. Ah, Pietro, what would you say about number 1?
> Pietro It is possible because he can produce 6 litres of orange juice; he can produce 12 litres ...

The word "constraint" was introduced in the following fashion.

> T. ... Now a constraint is a restriction on something; ... with the orange juice, it says here you can produce no more than 12 litres. What's the least the person can sell?
> x. Nothing.
> T. Nothing. O.K. So the least for the orange juice is 0 litres. What's the most able to be sold for orange juice?
> x. 12. ...
> Justin That's the maximum.
> T. That's the maximum. Right, very good. So the minimum is 0 litres and the maximum is 12. Anywhere in between 0 and 12 is able to be sold ... this is the restriction on the orange juice.

This approach appeared to be successful. The teacher used the word "restriction" rather than "limit" because "limit", in general usage, refers to a maximum only. In linear programming problems, a minimum is often considered. Nor do constraints always give simple maxima or minima, e.g., "At least twice as many cars as boats must be made."

Lesson 3 introduced the students to Steps 4-8 (Table 1). The intention of the teacher was to re-write the statement, "At least three boats are ordered", in the form, "The number of boats is greater than or equal to three", thus permitting a syntactic translation to the inequation, "$x \geq 3$" (Step 5). Some students initially had difficulty expressing "The number of boats has a minimum of three" in the desired form.

T. Now how can we express that in language ... "greater than" or "less than" or "greater than or equal to" or "less than or equal to"? What can we say about the number of boats? Adrian?
Adrian Less than or equal to. ...
Riccardo Greater than. ...
Terry ... x is greater than or equal to.

The teacher then used the strategy of verifying the suitability of possible statements by examining different values for the number of boats, a strategy which appeared to be successful in helping the students to express the constraints on the decision variables.

T. Hold on, which — We have to decide which one it is. We have to make "at least three boats". So is 3 boats O.K.?
xx. Yes.
T. Is 4 boats O.K.?
Adrian Yes. Anything more than —
T. Right, so is 2 boats O.K.?
xx. No.
T. Why not?
Adrian Because 3 boats have to be bought by someone, by 3 people.
T. That's right. So we need to make at least that number.
Anthony So that's greater than.
T. Yes, Anthony. It's greater than or equal to 3. ...The number of boats is greater than or equal to 3.

The students then found it easy to write the inequality, "$x \geq 3$", having obtained the worded statement, "The number of boats is greater than or equal to 3".

The Post-test of the 1994 Unit

Responses to selected items on the post-test of the 1994 unit are examined in order to gain insight into students' notions of constraints and inequalities.

Item B1: "At the local milk bar, Cherry Ripes are \$1 and a dozen eggs cost \$2. If I have \$10 to spend, write down the constraint on my spending, using x = the number of Cherry Ripes bought and y = the number of dozens of eggs bought."

Only one response to Item B1 was completely correct. There were, however, common notions of "constraint". A correct notion was that a constraint involves an inequality: nine of the 21 responses involved at least one inequality. One incorrect notion was that a constraint is any permissible solution for the variables concerned: six of the 21 responses were of this type. A possible source of this in the teaching process was the testing of possibilities for x and y to verify that a chosen inequality was correct. However, the word "constraint" was used only in relation to "inequality" or "restriction" or "limit", each of which appears to contradict the previous notion. Another incorrect notion was that a constraint is the maximum value of the variable: there were two such responses. A possible explanation for the notion that an equality, as in, "$4x + 3y = 10$", describes a constraint is that some students attach their own meanings to mathematical symbols. If this response meant "4 Cherry Ripes and 3 dozen eggs cost \$10", it would make sense, and, although it would not a be constraint, it would give a possible combination of x and y based on the constraint in the problem statement.

Item B2: "If x = the number of Cherry Ripes I buy, and I need to buy at least two Cherry Ripes, write down this last piece of information in mathematical language."

Item B2 (six correct answers from 21) was better answered than Item B1. The response "$x = \geq 2$" [sic!] showed confusion about the meaning of the signs for equality and inequality. The response "$x \leq 2$" used the wrong inequality sign. Whether this was due to the students' not understanding "at least" or to their not knowing the correct symbol for "greater than" cannot be known. The response "$x = 2$, $y = 4$" seemed to equate "at least two" with "equal to two". This response and "$x = 10$" maximized the total cost, suggesting that their authors confused the notions of "constraint" and "maximizing cost". The responses "$2x \leq 10$" and "$2x + y \leq 10$" seemed to involve incorrect notions of "x", possibly as the object "Cherry Ripes" or its cost. Thus incorrect answers to the same question could have their source in different cognitive obstacles.

Item B8, Step 4: "What constraints (restrictions) are imposed on each of the decision variables? State these in words using 'The number of ...'."

Three of the 21 students correctly identified the constraints and expressed in them in the desired format, "The number of ... ". Two of the these three scored the highest two marks on Item B8 (a complete linear programming problem). The post-test of 1993 gave evidence that understanding the problem constraints and expressing them in mathematical language was for most students a stumbling block. The 1994 results were similar. Some students understood the constraints but did not format them correctly; others identified the constraints but merely paraphrased them. Some did not attempt Step 4. In the teaching/learning process, this was perhaps the most difficult step, so these results were no surprise. They emphasize the persistence of Cognitive Obstacle 3.

Item B8, Step 5: "Express 'The number of ...' constraints in mathematical language, using inequality symbols."

The success rate for Step 5 of B8 was similar to that for Step 4. Each of the five students who achieved substantial success in Step 4 was able to write at least one correct inequation for Step 5. This suggests that writing "The number of... " statements to represent the problem did not hinder, and probably helped, the process of expressing these constraints mathematically. The fact that two of the four students who were successful in Step 5 either did not complete Step 4 or merely paraphrased the problem constraints prior to Step 5 demonstrates that success in Step 4 was not essential for success in Step 5. In other words, a re-structured, written version of the problem constraints may assist the student to write these constraints mathematically but it is not a pre-requisite. It cannot be concluded on this evidence, however, that the student does not enter a process by which an intermediate set of statements is constructed mentally.

Discussion

The evidence of the post-test and transcripts was that the students as a group seemed to have few difficulties understanding inequality terms, e.g., "at least" (Cognitive Obstacle 1). How much this was due to the taught technique of checking inequality statements using numerical values for variables cannot be determined.

Evidence was obtained that students who held Cognitive Obstacle 2 (the notion that literal symbols represent objects) found it difficult to write correct inequality statements. This cognitive obstacle was more obvious in the post-test responses than during the teaching/learning process. The pattern of results in the post-tests of 1994 and 1993 suggested a positive association between the ability to regard variables as numbers and success in solving the type of linear programming problems encountered.

The importance to the solution of the linear programming problem of overcoming the persistent Cognitive Obstacle 3 (difficulty in expressing constraints mathematically) was demonstrated by the post-test results. As an explanation of the existence of this obstacle for some students, these misconceptions of "constraint" were identified.

- A constraint is any permissible solution (or set of solutions) for a variable.
- A constraint is the maximum possible value of a variable.
- A constraint is expressed mathematically as an equality statement.

Students who wrote a "The number of" statement as a means of re-structuring the problem constraints were successful in formulating the inequations representing these constraints. The heuristic of using "The number of" statements is therefore not rejected. However, since few students were able to complete this step, it is suggested that some further techniques are necessary to help students interpret and represent the key features of constraint statements and other parts of linear programming problems.

Implications

It is suggested that the teaching approach adopted is worthwhile pursuing. It is noted particularly that those students who followed the set of heuristics (Steps 1 to 8) achieved a high degree of success in the solution of the linear programming problems encountered. The perhaps disappointing results of the other students might be ascribed more to their lack of confidence in tackling individual steps, rather than the use of the set of heuristics in the first place. Observing the tendency of heuristic teaching methods to lead to equivocal results, Schoenfeld (1985) offered this explanation: "Although heuristic strategies can serve as guides to relatively unfamiliar domains, they do not replace subject matter knowledge or compensate easily for its absence" (p. 73).

It is recommended that students receive more opportunities and guidance in developing skills in problem comprehension and representation, as well as in problem solution. This aim ought to lie at the heart of the school mathematics curriculum.

Further investigation, especially via interviews of individual students, of the cognitive obstacles discovered, might yield insights into students' cognitive structures and therefore the best means of alleviating these obstacles. Controlled comparison studies of the effectiveness of different teaching strategies, including Steps 1 to 8 from this study, could provide useful data to assist teachers. Given that one of the students' major difficulties in writing inequalities and in linear programming overall is likely to be problem comprehension, research into the effectiveness of different means of enhancing students' ability to represent and re-structure problems would be valuable.

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